INTRODUCTION TO THE GRUNWALD-WANG THEOREM

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1. Introduction

The Grunwald–Wang theorem is a local-to-global theorem in algebraic number theory. In its original form, it concerns the following question: if we prescribe cyclic extensions of finitely many local fields, can we find one global cyclic extension whose completions realize the prescribed local extensions?

The full answer requires class field theory. In this paper, we focus on a related and more concrete formulation involving n-th powers. The guiding question is simple: if $a \in K^\times$ is an n-th power in almost every completion of a number field K, must a already be an n-th power in K?

At first, one might expect the answer to be yes. Over $\mathbb{Q}$, for instance, this asks whether a rational number that looks like an n-th power in almost every p-adic field must already be an n-th power rationally. The Grunwald–Wang theorem says that this expectation is almost correct, but there is a special obstruction involving powers of 2. The simplest example is $16 = 2^4$. Although 16 is not an eighth power in $\mathbb{Q}$, it is an eighth power in $\mathbb{Q}_p$ for every odd prime p. Thus local eighth-power behavior away from 2 does not force global eighth-power behavior. This paper follows the power formulation of the Grunwald–Wang theorem. We first introduce completions and the local-to-global question for powers. Then we prove the example of 16, state the power form of the Grunwald–Wang theorem, and explain why powers of 2 are exceptional. Finally, we connect this power formulation back to the original problem about cyclic extensions.

2. Local Fields and the Local-to-Global Question

A number field is a finite extension of $\mathbb{Q}$. Examples include $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, and cyclotomic fields such as $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity.

Definition 2.1. Let K be a number field. A place v of K is an equivalence class of absolute values on K. The completion of K at v, denoted K_v, is the field obtained by completing K with respect to an absolute value representing v.

For $K = \mathbb{Q}$, there is one archimedean completion, namely $\mathbb{R}$, and one nonarchimedean completion $\mathbb{Q}_p$ for each prime p. Thus the completions of $\mathbb{Q}$ are $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \ldots$. These are the local fields attached to $\mathbb{Q}$.

A local-to-global question asks whether information over all or almost all completions K_v determines information over the original field K. The local-to-global question for powers is the following.

Question 2.1.1. Let K be a number field, let $n \geq 1$, and let $a \in K^\times$. Suppose that a is an n-th power in K_v for all but finitely many places v of K. Must a be an n-th power in K?

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Equivalently, we ask whether the existence of local solutions to $x^n = a$ in almost all completions K_v forces the existence of a global solution in K.

For squares, one might expect good local-to-global behavior because of quadratic reciprocity. But for higher powers the situation is subtler. The Grunwald–Wang theorem describes exactly when this local-to-global principle for powers fails.

3. THE EXAMPLE OF 16

The simplest counterexample to the naive local-to-global principle is $K = \mathbb{Q}$, $n = 8$, and $a = 16$. We will prove that 16 is an eighth power in $\mathbb{Q}_p$ for every odd prime p, but it is not an eighth power in $\mathbb{Q}$.

First, 16 is not an eighth power in $\mathbb{Q}$. If $x^8 = 16 = 2^4$ for some $x \in \mathbb{Q}^\times$, then the exponent of 2 in the prime factorization of x^8 would have to be divisible by 8. But the exponent of 2 in 16 is 4. Hence $16 \notin (\mathbb{Q}^\times)^8$.

Now we prove the local statement. We start with a simple observation.

Lemma 3.1. *For every odd prime p, at least one of -1, 2, and -2 is a quadratic residue modulo p.*

Proof. Using Legendre symbols, we have

$$\left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right).$$

If either $\left(\frac{-1}{p}\right) = 1$ or $\left(\frac{2}{p}\right) = 1$, then we are done. If both are -1, then their product is 1, so $\left(\frac{-2}{p}\right) = 1$. Therefore at least one of -1, 2, and -2 is a square modulo p. ☐

Proposition 3.2. For every odd prime p, 16 is an eighth power modulo p.

Proof. By the lemma, one of -1, 2, and -2 is a square modulo p.

If $2 \equiv r^2 \pmod{p}$, then $16 = 2^4 \equiv r^8 \pmod{p}$. If $-2 \equiv r^2 \pmod{p}$, then $16 = (-2)^4 \equiv r^8 \pmod{p}$.

It remains to handle the case where $-1 \equiv r^2 \pmod{p}$. Then $(1 + r)^2 = 1 + 2r + r^2 = 2r$. Therefore $(1 + r)^4 = (2r)^2 = 4r^2 = -4$, and so $(1 + r)^8 = 16$. Thus 16 is an eighth power modulo p in all cases. ☐

Finally, Hensel's lemma lifts these congruence solutions to p-adic solutions.

Proposition 3.3. For every odd prime p, the equation $x^8 = 16$ has a solution in $\mathbb{Q}_p$.

Proof. Let $f(x) = x^8 - 16$. By the previous proposition, there exists $a \in \mathbb{F}_p^\times$ such that $a^8 \equiv 16 \pmod{p}$. Since p is odd and $a \neq 0$, we have $f'(a) = 8a^7 \not\equiv 0 \pmod{p}$. By Hensel's lemma, the solution modulo p lifts to a solution in $\mathbb{Q}_p$. Therefore 16 is an eighth power in $\mathbb{Q}_p$. ☐

Combining these results, 16 is an eighth power in $\mathbb{Q}_p$ for every odd prime p, but 16 is not an eighth power in $\mathbb{Q}$.

4. THE POWER FORMULATION

The example of 16 suggests that we should measure the difference between being an n-th power locally and being an n-th power globally.

Definition 4.1. Let K be a number field, let S be a finite set of places of K, and let $n \geq 1$. Define

$$P(n,S) = \{a \in K^\times : a \in (K_v^\times)^n \text{ for every } v \notin S\}.$$

Thus $P(n,S)$ is the set of elements of $K^\times$ that become n-th powers in every completion outside S.

There is always an inclusion $(K^\times)^n \subseteq P(n,S)$, because a global n-th power is automatically an n-th power in every completion. The local-to-global question asks whether this inclusion is an equality.

The example of 16 says that when $K = \mathbb{Q}$, $n = 8$, and $S = \{2, \infty\}$, we have $16 \in P(8,S)$, but $16 \notin (\mathbb{Q}^\times)^8$. Therefore $P(8,S) \neq (\mathbb{Q}^\times)^8$.

The power formulation of the Grunwald–Wang theorem describes exactly when this failure can occur.

Theorem 4.2 (Grunwald–Wang theorem, power form)**.** *Let K be a number field, let S be a finite set of places of K, and let $n \geq 1$. Then, outside Wang's special case, one has*

$$P(n,S) = (K^\times)^n.$$

In Wang's special case, there is exactly one additional obstruction class. Equivalently, $P(n,S)/(K^\times)^n$ has order 2.

The exact special case is technical in general. It involves the 2-power part of n, the behavior of 2-power roots of unity over K, and the places of K lying above 2. For this paper, the most important point is that the obstruction only occurs in a special 2-power situation.

Over $\mathbb{Q}$, the statement becomes much simpler.

Corollary 4.3 (The case $K = \mathbb{Q}$)**.** Let $a \in \mathbb{Q}^\times$ and let $n \geq 1$. Then a is an n-th power in $\mathbb{Q}_p$ for all but finitely many primes p if and only if either $a \in (\mathbb{Q}^\times)^n$, or $8 \mid n$ and there exists $b \in \mathbb{Q}^\times$ such that

$$a = 2^{n/2} b^n.$$

This corollary explains why 16 is the first example. Taking $n = 8$, we have $2^{n/2} = 2^4 = 16$. Thus 16 represents the exceptional class predicted by the theorem.

5. Why the Prime 2 Is Special

The exceptional case in the Grunwald–Wang theorem comes from the behavior of 2-power roots of unity. For odd powers, the local-to-global principle behaves more cleanly. The first obstruction over $\mathbb{Q}$ appears at the level of eighth roots of unity.

Let ζ_8 be a primitive eighth root of unity. Then $\zeta_8 = e^{2\pi i/8} = e^{\pi i/4}$. The field $\mathbb{Q}(\zeta_8)$ has a special structure.

Proposition 5.1.

$$\mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2}).$$

Proof. We may write $\zeta_8 = e^{\pi i/4} = (1+i)/\sqrt{2}$. Since $i \in \mathbb{Q}(i,\sqrt{2})$ and $\sqrt{2} \in \mathbb{Q}(i,\sqrt{2})$, this shows that $\zeta_8 \in \mathbb{Q}(i,\sqrt{2})$. Hence $\mathbb{Q}(\zeta_8) \subseteq \mathbb{Q}(i,\sqrt{2})$.

For the reverse inclusion, note that $\zeta_8^2 = e^{\pi i/2} = i$, so $i \in \mathbb{Q}(\zeta_8)$. Also,

$$\zeta_8 + \zeta_8^{-1} = e^{\pi i/4} + e^{-\pi i/4} = 2\cos(\pi/4) = \sqrt{2}.$$

Thus $\sqrt{2} \in \mathbb{Q}(\zeta_8)$. Therefore $\mathbb{Q}(i, \sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$, and so $\mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2})$. □

This identity explains why the number 8 is the first place where the obstruction appears over $\mathbb{Q}$. The field $\mathbb{Q}(\zeta_8)$ contains three quadratic subfields:

$$\mathbb{Q}(i), \qquad \mathbb{Q}(\sqrt{2}), \qquad \mathbb{Q}(\sqrt{-2}).$$

The field $\mathbb{Q}(\zeta_8)$ has degree 4 over $\mathbb{Q}$. Its automorphisms are determined by where they send ζ_8, and ζ_8 can be sent to any primitive eighth root of unity:

$$\zeta_8, \quad \zeta_8^3, \quad \zeta_8^5, \quad \zeta_8^7.$$

Each of the three nontrivial automorphisms has order 2. This means that the Galois group is not cyclic of order 4; instead, it has three different subgroups of order 2. These correspond to the three quadratic subfields

$$\mathbb{Q}(i), \qquad \mathbb{Q}(\sqrt{2}), \qquad \mathbb{Q}(\sqrt{-2}).$$

This is the first place where the 2-power cyclotomic fields display the extra quadratic structure that appears in Wang's exceptional case.

6. CONNECTION TO THE ORIGINAL GRUNWALD PROBLEM

The power formulation is not the original form of the Grunwald–Wang theorem. The original Grunwald problem concerns cyclic field extensions. Suppose K is a number field and S is a finite set of places of K. For each $v \in S$, suppose we are given a cyclic extension E_v/K_v. The Grunwald problem asks whether there exists a cyclic extension L/K such that, for each $v \in S$, there is a place w of L above v with $L_w \cong E_v$ as extensions of K_v. Grunwald originally claimed that this was always possible with the expected degree. Wang found that this claim was false in a special 2-power case. The corrected statement is now called the Grunwald–Wang theorem.

To see why powers are related to cyclic extensions, we recall the basic idea of Kummer theory. Let μ_n denote the group of n-th roots of unity. Suppose that K contains μ_n. If $a \in K^\times$ and $\alpha^n = a$, then the roots of $x^n - a$ are $\zeta\alpha$, where $\zeta \in \mu_n$. Therefore any K-automorphism of $K(\alpha)$ must send α to $\zeta\alpha$ for some $\zeta \in \mu_n$. This is the source of the connection between adjoining n-th roots and cyclic extensions.

For example, suppose $a \in K^\times$, $\mu_n \subset K$, and $x^n - a$ is irreducible over K. Let $L = K(\alpha)$, where $\alpha^n = a$. Since K contains all n-th roots of unity, all roots of $x^n - a$ lie in L, namely

$$\alpha, \zeta\alpha, \zeta^2\alpha, \ldots, \zeta^{n-1}\alpha.$$

Thus L/K is Galois. For each $\zeta \in \mu_n$, the rule $\alpha \mapsto \zeta\alpha$ defines a K-automorphism of L. These automorphisms form a cyclic group, so L/K is a cyclic extension of degree n. More generally, Kummer theory says that when $\mu_n \subset K$, cyclic extensions of exponent dividing n are controlled by classes in $K^\times/(K^\times)^n$.

The quotient $K^\times/(K^\times)^n$ appears because adjoining an n-th root only depends on the class of the element modulo n-th powers.

Lemma 6.1. *Let K be a field, let $n \geq 1$, and let $a, b \in K^\times$. Then*

$$K(\sqrt[n]{a}) = K(\sqrt[n]{ab^n}).$$

Proof. Let $\alpha = \sqrt[n]{a}$. Then $\sqrt[n]{ab^n} = b\alpha$. Since $b \in K^\times$, adjoining α is the same as adjoining $b\alpha$. Therefore $K(\sqrt[n]{a}) = K(\sqrt[n]{ab^n})$. □

Thus, in the presence of the relevant roots of unity, a class $[a] \in K^\times/(K^\times)^n$ gives rise to a cyclic radical extension $K(\sqrt[n]{a})/K$, at least under the usual irreducibility assumptions. The same construction works locally: a class $[a_v] \in K_v^\times/(K_v^\times)^n$ gives a local radical extension $K_v(\sqrt[n]{a_v})/K_v$. Therefore a global class $[a] \in K^\times/(K^\times)^n$ produces compatible local classes in $K_v^\times/(K_v^\times)^n$, and hence compatible local cyclic extensions.

This explains the relation between the two formulations. The original Grunwald problem asks whether prescribed local cyclic extensions can be patched into one global cyclic extension. The power formulation asks whether local n-th power information comes from one global n-th power class. Kummer theory shows that, when roots of unity are present, these are two versions of the same idea: cyclic radical extensions are governed by power classes. In full generality, when the necessary roots of unity are not present, the correct connection is made through class field theory. In that language, local cyclic extensions correspond to local characters, global cyclic extensions correspond to global characters, and the Grunwald problem becomes the question of whether local characters can be patched into one global character.

This is why the example of 16 is important. It shows the obstruction in the power formulation: $16 \in P(8, \{2, \infty\})$, but $16 \notin (\mathbb{Q}^\times)^8$. Through Kummer theory and class field theory, this same 2-power obstruction is responsible for the failure of Grunwald's original local-to-global theorem for cyclic extensions.

Here is the full statement of the theorem.

Theorem 6.2 (Grunwald–Wang theorem [3])**.** *Let K be a number field, and let P be a finite set of places of K. For each $v \in P$, let*

$$E_v/K_v$$

be a cyclic extension of degree n_v. Let

$$n = \operatorname{lcm}_{v \in P}(n_v).$$

Then, except in Wang's special case, there exists a cyclic extension L/K of degree n such that, for every $v \in P$, there exists a place w of L lying above v with

$$L_w \cong E_v$$

as extensions of K_v.

Wang's special case is the exceptional case caused by the 2-power part of n. More precisely, it can occur only when $8 \mid n$ and the prescribed local data force a special obstruction coming from the behavior of 2-power roots of unity. In this case, the conclusion above may fail. The failure is measured by a group of order 2: there is exactly one possible obstruction to patching the given local cyclic extensions into a single global cyclic *extension. Thus, outside this special* 2*-power case, every finite collection of prescribed local cyclic extensions can be realized as the completions of one global cyclic extension.*

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