

p -adic Solenoids and the Three-Gap Theorem

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1 Introduction

The p -adic integers \mathbb{Z}_p can be defined as the inverse limit of a consistent system of residue classes modulo powers of p . That is, we can define \mathbb{Z}_p as the inverse limit of the projective system formed by $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 0$. However, we are not limited to $\mathbb{Z}/p^n\mathbb{Z}$ in construction. The p -adic solenoid is constructed in the same way as the p -adic integers, except we form a projective system with $\mathbb{R}/p^n\mathbb{Z}$ instead.

2 p -adic Solenoid

The homomorphisms

$$\varphi_n : \mathbb{R}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{R}/p^n\mathbb{Z}, \quad x \bmod p^{n+1}\mathbb{Z} \mapsto x \bmod p^n\mathbb{Z}$$

create a projective system $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)_{n \geq 0}$

Definition 1. The p -adic solenoid is the projective limit $\mathbb{S}_p = \varprojlim \mathbb{R}/p^n\mathbb{Z}$ of the projective system $(\mathbb{R}/p^n\mathbb{Z}, \varphi_n)$.

Definition 2. The p -adic solenoid has canonical projections

$$\psi_n : \mathbb{S}_p \rightarrow \mathbb{R}/p^n\mathbb{Z}$$

by definition.

These are continuous and surjective homomorphisms.

Proposition 1. *The p -adic solenoid \mathbb{S}_p has a unique cyclic subgroup C_m of order m for all integers $m \geq 1$.*

Proof. Let C_m^n be the cyclic subgroup of order m of the circle $\mathbb{R}/p^n\mathbb{Z}$. φ_n has a kernel of order p , which is prime to m , so it induces isomorphisms $C_m^{n+1} \rightarrow C_m^n$. Taking the inverse limit of this system produces the cyclic subgroup of order m $C_m \subset \mathbb{S}_p$.

Now, we need to show that this group is unique. Let $\sigma : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{S}_p$ be a homomorphism. The homomorphism $\psi_n \circ \sigma : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{R}/p^n\mathbb{Z}$ has an image in the unique cyclic subgroup C_m^n of $\mathbb{R}/p^n\mathbb{Z}$. Thus, σ has an image in C_m , so C_m must be unique. \square

We can project the cyclic subgroup C_m to the circle. $\psi_0(C_m) = m^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$. Since $\psi_0^{-1}(m^{-1}\mathbb{Z}/\mathbb{Z}) = C_m \times \mathbb{Z}_p$, C_m is the maximal finite subgroup of $\psi_0^{-1}(m^{-1}\mathbb{Z}/\mathbb{Z})$

Proposition 2. *The p -adic solenoid \mathbb{S}_p has no p -torsion. In other words, \mathbb{S}_p contains no element of order p .*

Proof. Let $\sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{S}_p$ be a homomorphism from a cyclic group of order p into the p -adic solenoid. We wish to show that $\varphi_n \circ \psi_{n+1} \circ \sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{S}_p \rightarrow \mathbb{R}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{R}/p^n\mathbb{Z}$ is always trivial. Notice that $\psi_{n+1} \circ \sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{S}_p \rightarrow \mathbb{R}/p^{n+1}\mathbb{Z}$ will have an image in the unique cyclic subgroup of order p in $\mathbb{R}/p^{n+1}\mathbb{Z}$, which is $\mathbb{R}/p^n\mathbb{Z}$. This is the kernel of φ_n , and by extension $\psi_n \circ \sigma = \varphi_n \circ \psi_{n+1} \circ \sigma$. Therefore, there is no element of order p in \mathbb{S}_p . \square

By Proposition 2, there are also no elements of order np^k for any $n, k \geq 1$ in \mathbb{S}_p .

Lemma 1. *Let E be the inverse limit of the topological projective system $(E_n, \varphi_n)_{n \geq 0}$, and let ψ_n be the projective map from E to E_n . For any $A \subset E$, the closure of A is given by*

$$\overline{A} = \bigcap_{n \geq 0} \psi_n^{-1}(\overline{\psi_n(A)}).$$

Proof. Clearly, A is contained in this intersection. In addition, this intersection is closed, so \overline{A} is contained within it.

Now, let b be an element in the intersection, and let V be a neighborhood of b . Without loss of generality, assume that $V = \psi_n^{-1}(U_n)$ for some open set $U_n \subset E_n$. $\psi_n(b) \in U_n$. By assumption, $b \in \psi_n^{-1}(\overline{\psi_n(A)})$, so $\psi_n(b) \in \overline{\psi_n(A)}$. This means that the open set U_n must meet $\overline{\psi_n(A)}$. Thus, there is some $a \in A$ such that $\psi_n(a) \in U_n$. In other words, $a \in A \cap \psi_n^{-1}(U_n) = A \cap V$. Since the neighborhood of b meets A , $b \in \overline{A}$. \square

Corollary 1. *A subset A of a topological projective limit is dense if and only if all of its projections $\psi_n(A)$ are dense in their respective codomains.*

Theorem 1. *There is a dense subset of \mathbb{S}_p equivalent to \mathbb{R} .*

Proof. Let f_n be the projection map from \mathbb{R} to $\mathbb{R}/p^n\mathbb{Z}$. Notice that $f_n = \varphi_n \circ f_{n+1}$. Since the f_n s are compatible with the transition maps of the p -adic solenoid, we can create a map $f : \mathbb{R} \rightarrow \mathbb{S}_p$ such that $f_n = \psi_n \circ f : \mathbb{R} \rightarrow \mathbb{S}_p \rightarrow \mathbb{R}/p^n\mathbb{Z}$ for all n .

Let $x \in \mathbb{R} \setminus \{0\}$. Whenever $p^n > x$, $0 \neq f_n(x) \in \mathbb{R}/p^n\mathbb{Z}$, so $0 \neq f(x) \in \mathbb{S}_p$. Thus, f is injective.

By Corollary 1, the image of f is dense if and only if the image of f_n is dense for all n . The image of f_n is $\mathbb{R}/p^n\mathbb{Z}$, so it is trivially dense. Therefore, this subset is dense. \square

Theorem 2. *There is a dense subset of \mathbb{S}_p equivalent to \mathbb{Q}_p .*

Proof. Consider the subgroups $H_k = \psi_0^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) \subset \mathbb{S}_p$ for $k \geq 0$. Notice that $\ker \psi_0 = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$, so $H_0 = \psi_0^{-1}(\mathbb{Z}/\mathbb{Z}) = \ker \psi_0 = \mathbb{Z}_p$. For $k \geq 1$, we have $H_k = \lim_{\leftarrow n} p^{-k}\mathbb{Z}/p^n\mathbb{Z} \cong p^{-k}\mathbb{Z}_p$. Thus, we have

$$\mathbb{Q}_p \cong \psi_0^{-1}(\mathbb{Z}[1/p]/\mathbb{Z}) = \bigcup \psi_0^{-1}(p^{-k}\mathbb{Z}/\mathbb{Z}) = \bigcup H_k \subset \mathbb{S}_p$$

To show density, we once again apply Corollary 1. $\psi_n(\mathbb{Q}_p) = \mathbb{Z}[1/p]/p^n\mathbb{Z} \subset \mathbb{R}/p^n\mathbb{Z}$. $\psi_n(\mathbb{Q}_p)$ is clearly dense in $\mathbb{R}/p^n\mathbb{Z}$, so \mathbb{Q}_p is dense in \mathbb{S}_p . \square

We define a sequence of continuous homomorphisms

$$f_n : \mathbb{R} \times \mathbb{Q}_p \rightarrow \mathbb{R}/p^n\mathbb{Z}, \left(t, \sum_{\nu \leq i \leq \infty} a_i p^i \right) \mapsto t + \sum_{i < n} a_i p^i \pmod{p^n\mathbb{Z}}.$$

These homomorphisms are compatible with the transition maps of \mathbb{S}_p . Therefore, we can create a continuous homomorphism

$$f : \mathbb{R} \times \mathbb{Q}_p, (t, x) \mapsto t + x$$

such that $\psi_n \circ f = f_n$.

Lemma 2.

$$\ker f = \{(a, -a) : a \in \mathbb{Z}[1/p]\} \subset \mathbb{R} \times \mathbb{Q}_p$$

In addition, this is a discrete subgroup of $\mathbb{R} \times \mathbb{Q}_p$. We denote this group Γ_p .

Proof. Suppose that $f(t, x) = 0$. $f_0(t, x) = \psi_0 \circ f(t, x) = 0 \in \mathbb{R}/\mathbb{Z}$, so $t + \sum_{i < 0} a_i p^i \in \mathbb{Z}$. Therefore,

$$t = - \sum_{i < 0} a_i p^i \in \mathbb{Z}[1/p].$$

In general, for $n \geq 0$, if $f_n(t, x) = 0$, then

$$t + \sum_{i < n} a_i p^i \in p^n\mathbb{Z}.$$

Therefore, the p -adic expansion of t is $-\lim_{n \rightarrow \infty} \sum_{i < n} a_i p^i = -x$. Clearly, $\Gamma_p \subseteq \ker f$, so $\Gamma_p = \ker f$.

Now, we need to show that Γ_p is discrete. To do this, we will find a neighborhood of $(0, 0)$ in $\mathbb{R} \times \mathbb{Q}_p$ such that the only element of Γ_p contained within it is $(0, 0)$. Consider the open set $(-1, 1) \times \mathbb{Z}_p \subset \mathbb{R} \times \mathbb{Q}_p$. If $(a, -a) \in \Gamma_p \cap (-1, 1) \times \mathbb{Z}_p$, then the p -adic expansion of $a \in \mathbb{Z}[1/p]$ must be $\sum_{i \geq 0} a_i p^i$ for some sequence a_i . Therefore, $a \in \mathbb{Z}_p$. However, $\mathbb{Z}_p \cap \mathbb{Z}[1/p] = \mathbb{Z}$, so $\Gamma_p \cap (-1, 1) \times \mathbb{Z}_p = \{(0, 0)\}$. \square

Theorem 3. The sum homomorphism $f : \mathbb{R} \times \mathbb{Q}_p \rightarrow \mathbb{S}_p$ furnishes an isomorphism $f' : (\mathbb{R} \times \mathbb{Q}_p)/\Gamma_p \cong \mathbb{S}_p$ algebraically and topologically.

Proof. Let $x = \sum_{\nu_p(x) \leq i} x_i p^i \in \mathbb{Q}_p$. We define the integral and fractional parts as follows:

$$[x] = \sum_{i \geq 0} x_i p^i \in \mathbb{Z}_p$$

$$\langle x \rangle = \sum_{i < 0} x_i p^i \in \mathbb{Z}[1/p]$$

Since every f_n is surjective, the image of f is dense. Let $(t, x) \in \mathbb{R} \times \mathbb{Q}_p$. $f(t, x) = f(t + \langle x \rangle, x - \langle x \rangle) = f(s, [x])$ where $s \in \mathbb{R}$ and $[x] \in \mathbb{Z}_p$. Furthermore, $f(s, [x]) = f(s - [s], [x] + [s])$, where $s - [s] \in [0, 1)$ and $[x] + [s] \in \mathbb{Z}_p$. This means the image of f , $f(\mathbb{R} \times \mathbb{Q}_p)$, is equal to $f([0, 1) \times \mathbb{Z}_p)$. Therefore, the image of f is compact and closed. Hence, f is surjective and f' is bijective. These equalities also show that the Hausdorff quotient is the image of the compact set $[0, 1] \times \mathbb{Z}_p$, and is therefore compact itself. Since f' is a continuous bijection between two compact spaces, it is a homeomorphism. \square

Corollary 2. *The p -adic solenoid \mathbb{S}_p is topologically and algebraically isomorphic to the quotient of $\mathbb{R} \times \mathbb{Z}_p$ by the discrete subgroup $\Delta_{\mathbb{Z}} = \{(-m, m) : m \in \mathbb{Z}\}$.*

Proof. The restriction of the sum homomorphism $f : \mathbb{R} \times \mathbb{Q}_p \rightarrow \mathbb{S}_p$ to $\mathbb{R} \times \mathbb{Z}_p$ is surjective, so there is a topological and algebraic isomorphism $f' : \mathbb{R} \times \mathbb{Z}_p / \ker f' \rightarrow \mathbb{S}_p$. $\ker f' = (\ker f) \cap (\mathbb{R} \times \mathbb{Z}_p) = \{(m, -m) : m \in \mathbb{Z}\}$, so $(\mathbb{R} \times \mathbb{Z}_p) / \Delta_{\mathbb{Z}} \cong \mathbb{S}_p$. \square

3 Topology of the p -adic Solenoid

Theorem 4. *The p -adic solenoid \mathbb{S}_p is connected.*

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{S}_p$ be the homomorphism given in Theorem 2. $f(\mathbb{R})$ is a connected and dense subset of \mathbb{S}_p . Any topological space with a connected dense subset is necessarily connected itself, so \mathbb{S}_p is connected. \square

Theorem 5. *The p -adic solenoid \mathbb{S}_p is compact.*

Proof. As it turns out, the projective limit of compact and nonempty spaces will be always compact and nonempty, although we will only prove compactness. Let (K_n, φ_n) be a projective system where each K_n is compact. By Tychonoff's theorem, the product of the K_n s is compact, and the projective limit K of the system is a closed subset of this product, so K is itself compact. Since \mathbb{S}_p satisfies the premise, it is compact. \square

Lemma 3. *Let $a > 1$ be an integer, $\varphi : \mathbb{R}/a\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ the canonical projection, and C a connected subset of $\mathbb{R}/a\mathbb{Z}$ containing two distinct points s, t such that $\varphi(s) = \varphi(t)$. $\varphi(C) = \mathbb{R}/\mathbb{Z}$.*

Proof. If $C = \mathbb{R}/a\mathbb{Z}$, this is obviously true, so assume $C \neq \mathbb{R}/a\mathbb{Z}$. Choose some $P \in (\mathbb{R}/a\mathbb{Z}) \setminus C$. Consider the homeomorphism f representing the stereographic projection of the circle $\mathbb{R}/a\mathbb{Z}$ from P onto a line \mathbb{R} . The image $f(C)$ of C is a connected subset of the real line, and it contains the images of two points congruent mod \mathbb{Z} . A connected set on the real line is always an interval, so $f(C)$ contains an interval between these two points. Therefore, $f(C)$ contains an arc I of the circle $\mathbb{R}/a\mathbb{Z}$ with image $\varphi(I) = \mathbb{R}/\mathbb{Z}$. \square

Theorem 6. *The p -adic solenoid \mathbb{S}_p is indecomposable.*

Proof. Let A, B be two compact connected subsets covering \mathbb{S}_p . We need to show that either $A = \mathbb{S}_p$ or $B = \mathbb{S}_p$. This is clearly true if $A = \mathbb{S}_p$, so assume $A \neq \mathbb{S}_p$. Clearly, $B \neq \emptyset$. By 1, $A \neq \mathbb{S}_p$ implies that $\psi_n(A) \neq \mathbb{R}/p^n\mathbb{Z}$ for some $n = n_0$, and by extension, all $n \geq n_0$. Choose some $n \geq n_0$ and some $b \in B$. $\varphi_n^{-1}(b) \subset \mathbb{R}/p^{n+1}\mathbb{Z}$ has cardinality $p \geq 2$, and the restriction of φ_n to the connected set $C = \varphi_n^{-1}(\psi_n(B)) = \psi_{n+1}(B)$ is not injective. By Lemma 3, $\psi_n(B) = \varphi_n(\psi_{n+1}(B)) = \varphi_n(C) = \mathbb{R}/p^n\mathbb{Z}$, which is a contradiction. Therefore, $B = \mathbb{S}_p$, so \mathbb{S}_p is indecomposable. \square

4 Adeles

The three-gap theorem is a result of Diophantine approximation related to the action of placing points around a circle. This theorem has applications to the study of plant growth since plants often

grow leaves or petals in a spiral pattern. It also has applications in music theory when applied to the chromatic circle, allowing for the study of existent scales and the creation of non-standard scales.

Theorem 7 (Classical Three-Gap Theorem). *Place N points around a circle such that each point is some angle θ from the last. The gaps between adjacent points will have at most 3 different sizes for fixed N and θ .*

This section will show an extension of the three-gap theorem to be true in p -adic solenoids. The proofs of all of the required theorems are too long to show here, so only important definitions and an outline of the proof will be included. The full proof can be seen in [Das22]. To generalize this to the p -adic solenoid, we will first need to make a generalization of the solenoid.

Definition 3. Let $(G_i)_{i \in I}$ be a family of locally compact groups indexed by a set I , and let S be a finite subset of I . For each $i \in I \setminus S$, let U_i be an open compact subset of G_i . We define the *restricted product* to be the subset of $\prod_{i \in I} G_i$ s which consists of all elements $(g_i)_{i \in I}$ such that $g_i \in U_i$ for all but finitely many i . We define the open sets to be of the form $\prod_{i \in I} A_i$ where A_i is open in G_i for all i and $A_i = U_i$ for all but finitely many i .

Definition 4. Let \mathcal{P} be the set of all primes. The *topological ring of adèles* \mathbb{A} over \mathbb{Q} is the restricted product of $(\mathbb{Q}_p)_{p \in \mathcal{P} \cup \{\infty\}}$ where \mathbb{Q}_∞ represents \mathbb{R} and we choose the open compact subset \mathbb{Z}_p for each \mathbb{Q}_p . Note that we treat \mathbb{R} and each \mathbb{Q}_p as an abelian group under addition.

In other words, \mathbb{A} is the collections of elements $\alpha = (\alpha_\infty, \alpha_{p_1}, \alpha_{p_2}, \dots) \in \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p$ such that $\alpha_p \in \mathbb{Z}_p$ for all but finitely many primes p . We call the quotient group \mathbb{A}/\mathbb{Q} the *Adelic Torus*.

Definition 5. Let $\mathcal{Q} = \{p_1, p_2, \dots\}$ be a non-empty subset of \mathcal{P} . We define $\mathbb{A}_{\mathcal{Q}}$ to be the projection of \mathbb{A} onto only the parts indexed by $\{\infty\} \cup \mathcal{Q}$.

The metric we use on $\mathbb{A}_{\mathcal{Q}}$ is defined as follows:

$$|\alpha - \beta| = \begin{cases} \max(|\alpha_\infty - \beta_\infty|_\infty, \max_{p \in \mathcal{Q}} |\alpha_p - \beta_p|_p) & |\mathcal{Q}| < \infty \\ \max(|\alpha_\infty - \beta_\infty|_\infty, \max_{p \in \mathcal{Q}} \frac{|\alpha_p - \beta_p|_p}{p}) & |\mathcal{Q}| = \infty \end{cases}$$

This produces the same topology as the restricted product.

We can embed the additive group $\Gamma_{\mathcal{Q}} = \mathbb{Z}[1/p_1, 1/p_2, \dots]$ into $\mathbb{A}_{\mathcal{Q}}$ via the injective homomorphism $\gamma \mapsto \gamma = (\gamma, \gamma, \gamma, \dots)$. We define $X_{\mathcal{Q}}$ to be the quotient $\mathbb{A}_{\mathcal{Q}}/\Gamma_{\mathcal{Q}}$. Notice that $X_{\{p\}}$ is the same as the p -adic solenoid S_p in terms of addition, and $X_{\{p\}}$ is only defined under addition.

Remark 1. If we wish, we may generalize the multiplication of \mathbb{S}_p to $X_{\mathcal{Q}}$. To multiply $a, b \in X_{\mathcal{Q}}$, we do a piecewise multiplication, and then then we multiply the \mathbb{R} component by -1 . This swaps the properties of positive and negative numbers, allowing multiplication to behave as in \mathbb{S}_p . While $X_{\mathcal{Q}}$ under this definition of multiplication is a valid extension of the concept of \mathbb{S}_p , this is not the typical extension. More often, the extension is defined as $\lim_{\leftarrow} \mathbb{R}/(\prod_{i=1}^n a_n)\mathbb{Z}$ for some positive integer sequence a_n . This generalization is simply referred to as a solenoid.

On $X_{\mathcal{Q}}$, we define the metric as follows:

$$|\alpha - \beta| = \min_{\gamma \in \Gamma_{\mathcal{Q}}} (|\alpha - \beta - \gamma|)$$

This induces the quotient topology. For $\alpha \in \mathbb{A}_{\mathcal{Q}}$ and some $N \in \mathbb{N}$, we define

$$S_N(\alpha) = \{\xi_n = n\alpha + \Gamma_{\mathcal{Q}} : 1 \leq n \leq N\} \subset X_{\mathcal{Q}}.$$

This is the equivalent of placing points around the circle in the classical three-gap theorem. We also define the following for $1 \leq n \leq N$:

$$\delta_{n,N} = \min_{1 \leq m \leq N} (|\xi_m - \xi_n|)$$

This is the adelic analogue for the gaps between points on the circle. We use the function $g_N(\alpha) = |\{\delta_{n,N}(\alpha) : 1 \leq n \leq N\}|$ to count the gaps. This gives us the adelic three-gap theorem:

Theorem 8 (Adelic Three-Gap Theorem). *Let \mathcal{Q} be a non-empty set of prime numbers. For any $\alpha \in X_{\mathcal{Q}}$ and $N \in \mathbb{N}$, $g_N(\alpha) \leq 3$. In addition, there are $\alpha \in X_{\mathcal{Q}}$ and $N \in \mathbb{N}$ such that $g_N(\alpha) = 3$.*

Let $G = \mathrm{SL}(2, \mathbb{A}_{\mathcal{Q}})$ and let $\Gamma = \mathrm{SL}(2, \Gamma_{\mathcal{Q}})$. We make the following definitions to convert this problem into lattices. For $t \in \mathbb{Q}$:

$$A_t(\alpha) = \begin{bmatrix} \mathbf{1} & \alpha \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{t}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{t}^{-1} & t\alpha \\ \mathbf{0} & \mathbf{t} \end{bmatrix} \in G$$

For $M \in G$, $t \in (0, 1)$, and $z \in \mathbb{N}$:

$$Q(M, t, z) = \left\{ (\mathbf{u}, \mathbf{v}) \in \Gamma_{\mathcal{Q}}^2 M : \mathbf{v} \neq \mathbf{0}, -t < u_{\infty} < 1 - t, |u_p|_p \leq \left| \frac{2}{z} \right|_p \text{ for all } p \in \mathcal{Q} \right\}$$

$$F(M, t, z) = \min_{(\mathbf{u}, \mathbf{v}) \in Q(M, t, z)} (|\mathbf{v}|)$$

For $M \in G$ and $z \in \mathbb{N}$:

$$\mathcal{G}(M, z) = |\{F(M, t, z) : 0 < t < 1\}|$$

For $M \in G$ and $N \in \mathbb{N}$:

$$\mathcal{G}_N(M) = \left| \left\{ F \left(M, \frac{n}{N + 1/2}, 2N + 1 \right) : 1 \leq n \leq N \right\} \right|.$$

We may worry that F is not well-defined due to a non-existence of a minimum for some parameters or due to emptiness of $Q(M, t, z)$ for some parameters. However, it turns out that F is well-defined and nonzero. This result follows from the uniform discreteness of $\Gamma_{\mathcal{Q}}^2 M$, the discreteness of the p -adic absolute value, and an adelic analogue of Minkowski's convex body theorem.

Through these various functions, we are able to express g_N as $g_N(\alpha) = \mathcal{G}_N(A_{N+1/2}(\alpha)) \leq \mathcal{G}(A_{N+1/2}(\alpha), 2N + 1)$. We are able to show that, for any $M \in G$ and $z \in \mathbb{N}$, $\mathcal{G}(M, z) \leq 3$, so $g_N(\alpha) \leq 3$. In addition, we are able to show that there are some α, N for every \mathcal{Q} such that $g_N(\alpha) = 3$. The cases for this are as follows:

- $\mathcal{Q} = \{2\}$: $\alpha = (\alpha_{\infty}, \alpha_2) = (351/100, 1)$ and $N = 52$. Direct computation shows that $\delta_{1,N}(\alpha) = |51\alpha| = \frac{1}{100}$, $\delta_{2,N} = |35\alpha| = \frac{3}{20}$, and $\delta_{18,N}(\alpha) = |16\alpha| = \frac{4}{25}$, so $g_N(\alpha) = 3$.

- $\mathcal{Q} = \{3\}$: $\alpha = (\alpha_\infty, \alpha_3) = (16/5, 1)$ and $N = 5$. By direct computation, $\delta_{1,N}(\alpha) = |4\alpha| = \frac{1}{5}$, $\delta_{2,N}(\alpha) = |3\alpha| = \frac{3}{5}$, and $\delta_{3,N}(\alpha) = |\alpha| = \frac{4}{5}$, so $g_N(\alpha) = 3$.
- $\mathcal{Q} = \{p_1, p_2, \dots, p_k\}$ such that $p_1 < p_2 < \dots < p_k$ $p_1 p_2 \dots p_k \geq 5$: $\alpha = (1/4 p_1 p_2 \dots p_k, -1, \dots, -1)$ and $N = p_1 p_2 \dots p_k + 1$. Applying a theorem shows that $\delta_{1,N}(\alpha) = \max(\frac{1}{4}, \frac{1}{p_1}) \leq \frac{1}{2}$, $\delta_{2,N}(\alpha) = \frac{3}{4} + \frac{1}{4 p_1 p_2 \dots p_k}$, and $d_{3,N}(\alpha) = 1$, so $g_N(\alpha) = 3$.
- $|\mathcal{Q}| = \infty$ and the smallest prime in \mathcal{Q} is 3: Let $\alpha_\infty = 1/9$, $\alpha_3 = 1$, and every other α_p equal 0. Let $N = 11$. We find that $\delta_{1,N}(\alpha) = |10\alpha|$, which equals either $\frac{1}{5}$ or $\frac{1}{9}$, $\delta_{2,N}(\alpha) = |7\alpha| = \frac{2}{9}$, and $\delta_{5,N}(\alpha) = |\alpha| = \frac{1}{3}$, so $g_N(\alpha) = 3$.
- $\{2, 3\} \subset \mathcal{Q}$: Let $\alpha_\infty = 27/50$, $\alpha_2 = -1$, and every other α_p equal 0. Let $N = 6$. $\delta_{1,N}(\alpha) = |5\alpha| = \frac{3}{10}$, $\delta_{2,N}(\alpha) = |4\alpha| = \frac{1}{3}$, and $\delta_{3,N}(\alpha) = |\alpha| = \frac{23}{50}$, so $g_N(\alpha) = 3$.
- \mathcal{Q} contains 2, but not 3: Let $\alpha_\infty = 8/49$, $\alpha_2 = -1$, $\alpha_5 = 3$ if $5 \in \mathcal{Q}$, and every other α_p equal 0. Let $N = 8$. We have $\delta_{1,N}(\alpha) = |7\alpha| = \frac{1}{7}$, $\delta_{2,N} = |5\alpha| = \frac{1}{4}$, and $\delta_{4,N}(\alpha) = |2\alpha| = \frac{16}{49}$, so $g_N(\alpha) = 3$.
- The smallest prime q in \mathcal{Q} is greater than or equal to 5: $\alpha_\infty = \frac{q-1}{q(q-2)}$, $\alpha_q = -1$, and every other α_p is 0. This creates the gaps $\delta_{1,N}(\alpha) = |(q-1)\alpha| = \max(\frac{1}{q(q-2)}, \max_{p \in \mathcal{Q} \setminus \{q\}}(\frac{1}{p})) < \frac{1}{q}$, $\delta_{2,N}(\alpha) = |(q-2)\alpha| = \frac{1}{q}$, and $\delta_{3,N}(\alpha) = |\alpha| = \frac{1}{q} + \frac{1}{q(q-2)}$, so $g_N(\alpha) = 3$.

5 Conclusion

The p -adic solenoid, which is initially created as a generalization of a construction of \mathbb{Z}_p , is related to both \mathbb{R} and \mathbb{Q}_p . It is compact, connected, and indecomposable. In addition, it can be considered a generalization of the adelic torus $X_{\{p\}}$, and it satisfies a modified version of the three-gap theorem.

References

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