LOCAL FUNCTION FIELDS

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1. Introduction

The p-adic numbers $\mathbb{Q}_p$ are characterized by three properties: they are complete with respect to a discrete valuation, their valuation ring has a unique prime ideal which is principal, and their residue field is finite. Oddly enough, the field of formal Laurent series in characteristic p has exactly these characteristics. This papers goal is to show that in fact these fields and their extensions are the only fields with these properties, and just how similar they are forced to be because of these properties.

We begin in Section 2 by introducing $\mathbb{F}_p((t))$, the field of formal Laurent series over $\mathbb{F}_p$, and showing how it exists in analogy with $\mathbb{Q}_p$. Section 3 develops the general theory of local fields, capping off with Hensel's lemma and Teichmüller representatives. In section 4 we discuss extensions of local fields and prove the degree formula $[L : K] = ef$ for finite extensions. Section 5 proves the classification theorem: every local field is isomorphic to either a finite extension of $\mathbb{Q}_p$ or to $\mathbb{F}_{p^n}((t))$.

2. Function fields

To begin with we introduce the field of characteristic p Laurent series. This has a nice intuitive description, but we will see that it can be constructed in a way analogous to the p-adics.

Definition 2.1. For a prime p define $\mathbb{F}_p((t))$ as the field with elements of the form

$$\sum_{k=-\infty}^{\infty} a_n t^n$$

where there are only finitely many $a_n \neq 0$ for $n < 0$ and each $a_n \in \mathbb{F}_p$. Addition is defined component wise and multiplication is defined by

$$\sum_{n\in\mathbb{Z}} a_n t^n \times \sum_{n\in\mathbb{Z}} b_n t^n = \sum_{n\in\mathbb{Z}} t^n \sum_{k\in Z} a_k b_{n-k}$$

The ring of formal power series $\mathbb{F}_p[[t]]$ is then defined to be the subring of $\mathbb{F}_p((t))$ consisting of those Laurent series with $a_n = 0$ for all $n < 0$.

The beginning of our analogy with $\mathbb{Q}_p$ is the observation that $\mathbb{F}_p[[t]]$ is a local ring with a unique prime ideal which happens to be principal.

Proposition 2.2. *The ring $\mathbb{F}_p[[t]]$ is a local ring with unique non-zero prime ideal (t).*

Proof. Suppose $f = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{F}_p[[t]]$ and $f \notin (t)$. Then $a_0 \neq 0$ and so we may define b_k recursively by taking $b_0 = \frac{1}{a_0}$ and defining

$$b_k = -\frac{1}{a_0} \sum_{n=1}^{k} a_n b_{k-n}.$$

Then if we take $g = \sum_{k=0}^{\infty} b_k t^k$ we have

$$fg = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{\infty} t^n \left(a_0 b_n + \sum_{k=1}^{n} a_k b_{n-k} \right) = 1 + \sum_{n=1}^{\infty} t^n \left(-\sum_{k=1}^{n} a_k b_{n-k} + \sum_{k=1}^{n} a_k b_{n-k} \right) = 1$$

Therefore $g = \frac{1}{f}$ so f is a unit. Now if $f \in (t)$ then f has $a_0 = 0$ so for any $g \in \mathbb{F}_p[[t]]$ we have $fg \in (t)$ has first coefficient 0 and is thus not equal to 1. This shows that (t) is maximal. Further, since every non-zero $f \in \mathbb{F}[[t]]$ has a least non-zero coefficient a_m, it follows that $f = t^m u$ where u is a unit. Therefore if $\mathfrak{p}$ is a non-zero prime ideal then there is some $f \in \mathfrak{p}$ and thus some $t^n \in \mathfrak{p}$. Since prime ideals are radical it follows that $t \in \mathfrak{p}$ and so $(t) = \mathfrak{p}$. Therefore $\mathbb{F}_p[[t]]$ is a local ring with unique prime ideal (t). ■

Corollary 2.3. *The field $\mathbb{F}_p((t))$ is the field of fractions of $\mathbb{F}_p[[t]]$.*

Proof. The fraction field of an integral domain R is defined to be the initial ring extension F which is a field. Note that $\mathbb{F}_p((t)) \cong \mathbb{F}_p[[t]][t^{-1}]$. Since any field $F \supset \mathbb{F}_p[[t]]$ must have t invertible it follows that $\mathbb{F}_p((t)) \subset F$. Therefore $\mathbb{F}_p((t))$ is the field of fractions of $\mathbb{F}_p[[t]]$. ■

This already gives us the first glimpse of an analogy between $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$. Namely, they are both fields with a distinguished subring which has unique prime ideal. Now to complete the analogy we would need some sort of valuation on $\mathbb{F}_p((t))$. Luckily we have such a valuation.

Definition 2.4. For $0 \neq f \in \mathbb{F}_p((t))$ define $v_t(f)$ to be the least $n \in \mathbb{Z}$ such that $a_n \neq 0$. Define $|f|_t = p^{-v_t(f)}$.

Lemma 2.5. *This valuation satisfies the following properties*

- *For any $f, g \in \mathbb{F}_p((t))$ we have $v_t(fg) = v_t(f) + v_t(g)$.*
- *We have $v_t(x+y) \geq \min(v_t(x), v_t(y))$ With equality whenever $v_t(x) \neq v_t(y)$.*

In particular defining $d(f,g) := |f-g|_t$ gives an ultrametric.

Proof. Suppose f has coefficients a_n and g coefficients b_n. For the first property, note that for $k < v_t(f) + v_t(g)$ the contributions to that coefficient of fg will come in the from $a_n b_m$ where $n + m = k$. But then $\min(n, m) \leq k$ and so one of either a_n or a_m is zero. Further the only contribution to the $v_t(f) + v_t(g)$ coefficient is $a_{v_t(f)} b_{v_t(g)}$ which is non-zero because both $a_{v_t(f)}, b_{v_t(g)}$ are non-zero. Thus $v_t(fg) = v_t(f) + v_t(g)$.

For the second property the inequality is easy. Since addition is coefficient by coefficient the least non-zero coefficient of $f+g$ clearly can't be less then the least non-zero coefficient of f and g. Thus $v_t(x+y) \geq \min(v_t(x), v_t(y))$. Now if $v_t(f) \neq v_t(g)$ assume without loss of generality that $v_t(f) < v_t(g)$. Then the $v_t(f)$th coefficient of $f+g$ is $a_{v_t(f)} + b_{v_t(f)} = a_{v_t(f)} \neq 0$ so we have $v_t(x+y) \geq \min(v_t(x), v_t(y))$. ■

One should view v_t in analogy with v_p. Just as how $\mathbb{Z}_p$ is distinguished as the set of x such that $v_p(x) \geq 0$, the subring $\mathbb{F}_p[[t]]$ can be identified as the set of x with $v_t(x) \geq 0$.

Lemma 2.6. *The set of x such that $v_t(x) \geq 0$ is exactly $\mathbb{F}_p[[t]]$. The set of x such that $v_t(x) > 0$ is exactly (t).*

Proof. If $v_t(x) \geq 0$, then its first non-zero coefficient a_n must have $n \geq 0$. In other words $a_k = 0$ for all $k < 0$, this is exactly the condition an element must satisfy to be contained in $\mathbb{F}_p[[t]]$. If $v_t(x) > 0$ then the least non-zero coefficient must be a_n where $n > 0$. Thus we may write such an x as

$$\sum_{n=1}^{\infty} a_n t^n = t \sum_{n=0}^{\infty} a_{n+1} t^n.$$

Since $\sum_{n=0}^{\infty} a_{n+1} t^n \in \mathbb{F}_p[[t]]$ it follows that $x \in (t)$. Conversely if $x \in (t)$ then we may write

$$x = t \sum_{n=0}^{\infty} a_n t^n = \sum_{n=1}^{\infty} a_{n-1} t^n.$$

Which has no non-zero coefficients a_k with $k \leq 0$. Therefore $v_t(x) > 0$. ■

The final thing to show, that would finish the analogy would be to show that $\mathbb{F}_p((t))$ is complete with respect to the absolute value $|\cdot|_t$.

Theorem 2.7. *The field $\mathbb{F}_p((t))$ is complete with respect to the absolute value $|\cdot|_t$.*

Proof. Suppose (x_n) is a cauchy sequence in $\mathbb{F}_p((t))$. Then for any $k \in \mathbb{Z}$ we have some N such that for all $i, j > N$

$$|x_i - x_j|_t < p^{-k}$$

In other words

$$v_t(x_i - x_j) > k.$$

So, all coefficients of $x_i - x_j$ which are $\leq k$ are zero, that is to say, x_i and x_j agree on all coefficients less than or equal to k. Since k was arbitrary, we see that any given coefficient of the x_n eventually stabalizes to a constant value. Call this value a_k and define

$$x = \sum_{k \in \mathbb{Z}} a_k t^k$$

Then for any $\varepsilon > 0$ choose k with $p^{-k} < \varepsilon$. Choose N such that all x_i, $i > n$ agree on their coefficients $\leq k$, then $|x_i - x|_t < p^{-k} < \varepsilon$. So $x_i \to x$. ■

Remark 2.8. We have now established that $\mathbb{F}_p((t))$ shares the three major features of $\mathbb{Q}_p$: a discrete valuation, a valuation ring with finite residue field, and completeness. The differences are also interesting, most notably that $\mathbb{F}_p((t))$ has characteristic p, whereas $\mathbb{Q}_p$ has characteristic 0. In the next section we ask what can be proved about a field satisfying these properties, without assuming anything further about its characteristic.

3. Discrete valuation rings and local fields

It turns out that the property of having a unique prime ideal which is principal is essentially equivalent to having a valuation.

Definition 3.1. A ring A is a discrete valuation ring (DVR) if it is a PID with exactly one prime ideal (π) (equivalently if it is a local PID that isn't a field), any generator π of this ideal is reffered to as a uniformizer. The field $A/(\pi)$ is then referred to as the residue field we will denote it by k. The field of fractions of A will be denoted by K.

Lemma 3.2. *If A is a DVR every element $x \in A$ can be written uniquely as $\pi^n u$ where u is a unit and n is a non-negative integer.*

Proof. If x is a unit, the claim is completely trivial. Now note that because A is a PID it is also a UFD, so we may write $x = wp_1p_2\cdots p_n$ where w is a unit and each p_k is prime. Now for each p_k the ideal (p_k) is prime, since there is only one prime ideal of A we have $(p_k) = (\pi)$ and so $p_k = u_k\pi$ where u_k is some unit. Since the product of units is a unit it follows that $x = u\pi^n$ where u is some unit.

As for uniqueness, suppose $x = u\pi^n = w\pi^m$ where either $n \neq m$ or $w \neq u$. If $n \neq m$ then assume $n < m$. We have

$$\pi^n(u - w\pi^{m-n}) = 0.$$

Since u is a unit and $w\pi^{m-n}$ is not, $(u - w\pi^{m-n}) \neq 0$. And clearly $\pi^n \neq 0$. Since A is a PID it has no zero-divisors so we have reached a contradiction. Similarly if $u \neq w$ and $n = m$ then we have

$$\pi^n(u - w) = 0$$

Since neither π^n nor $u - w$ are zero we reach the same contradiction. ■

Corollary 3.3. *Consider A naturally as a subring of its fraction field K. Then any $x \in K$ can be written uniquely as $\pi^n u$ where $n \in \mathbb{Z}$ and $u \in A$ is a unit in A.*

Proof. Write $x = \frac{a}{b}$ with $a, b \in A$. Then we may write $a = s\pi^n$ and $b = w\pi^m$ with s, w units in A. Since $u := s/w$ is also a unit in A we have

$$x = \frac{a}{b} = \frac{s\pi^n}{w\pi^m} = u\pi^{n-m}.$$

Uniqueness follows identically as in the proof of 3.2. ■

From this unique factorization we obtain our valuation.

Definition 3.4. For a DVR A with fraction field K define $v_\pi : K \to \mathbb{Z} \cup \{\infty\}$ by taking $v_\pi(u\pi^n) = n$ and $v_\pi(0) = \infty$. Define $|x|_\pi := \exp(-v_\pi(x))$.

Lemma 3.5. *The function v_π satisfies the following*

- *For any $x \in K$, $v_\pi(x) = \infty$ iff $x = 0$.*
- *We have*

  $$v_\pi(xy) = v_\pi(x) + v_\pi(y)$$

- *For any $x, y \in K$ we have*

  $$v_\pi(x + y) \geq \min(v_\pi(x), v_\pi(y))$$

  with equality when $v_\pi(x) = v_\pi(y)$.

Proof. The first claim is immediate from the definition fo v_π. Now write $x = u\pi^n$ and $y = v\pi^m$. Then

$$v_\pi(xy) = v_\pi(uw\pi^{n+m}) = n + m = v_\pi(x) + v_\pi(y)$$

Assume without loss of generality that $n \leq m$ so tjat $\min(v_\pi(x), v_\pi(y)) = v_\pi(x)$. Then $x + y = x(1 + \pi^{m-n}wu^{-1})$. If $v_\pi(x) \neq v_\pi(y)$ then $1 + \pi^{m-n}wu^{-1}$ is 1 plus a multiple of π and thus not a multiple of π. Hence

$$v_\pi(x + y) = v_\pi(x) + v_\pi(1 + \pi^{m-n}wu^{-1}) = v_\pi(x)$$

Now if $v_\pi(x) = v_\pi(y)$ then $1 + \pi^{m-n}wu^{-1} = 1 + wu^{-1}$. This is a sum of elements in A and thus an element of A. As established in 3.2 all elements of A have non-negative valuation so

$$v_\pi(x + y) = v_\pi(x) + v_\pi(1 + \pi^{m-n}wu^{-1}) \geq v_\pi(x)$$

■

Now we have seen how to obtain a valuation starting with a DVR. But can we obtain a DVR from a valuation? The answer is yes, first though we should properly define what a valuation is.

Definition 3.6. A (discrete non-archimadean) valuation v on a field K is a function $K \to \mathbb{Z} \cup \{\infty\}$ satisfying

- For any $x \in K$, $v(x) = \infty$ iff $x = 0$.
- We have
$$v(xy) = v(x) + v(y)$$
- For any $x, y \in K$ we have
$$v(x+y) \geq \min(v(x), v(y))$$
with equality when $v(x) = v(y)$.

We also assume that v is surjective. This excludes the trivial valuation taking $v(x) = 0$ for all $x \neq 0$.

In particular, if we define $|x|_v := \exp(-v(x))$ and then $d(x, y) := |x - y|_v$ the function d is an ultrametric.

We call a field K equipped with a valuation v a valued field. A valued field is said to be complete if it is complete under the metric induced by v.

Lemma 3.7. *If K is a valued field then the set $\mathcal{O}_K$ of x such that $v(x) \geq 0$ is a ring. Further, it is a local ring with unique maximal ideal $\mathfrak{m}_K$ given by those elements with $v(x) > 0$.*

Proof. If x, y have $v(x), v(y) \geq 0$ then $v(x+y) \geq \min(v(x), v(y)) \geq 0$ and $v(xy) = v(x) + v(y) \geq 0$. Therefore $\mathcal{O}_K$ is a ring.

To show that $\mathfrak{m}_K$ is the unique maximal ideal of $\mathcal{O}_K$ we show that $u \in \mathcal{O}_K$ is a unit iff it is not contained in $\mathfrak{m}$. First if u is not contained in $\mathfrak{m}$ then $v(u) = 0$. Now consider its inverse w in K. We have $0 = v(1) = v(uw) = v(u) + v(w) = v(w)$ and so by definition $w \in \mathcal{O}_K$ meaning u is a unit.

Conversely if u is a unit of $\mathcal{O}_k$ and contained in $\mathfrak{m}_K$ then $v(u) > 0$. If w is the inverse of u then we have $0 = v(u) + v(w)$ and hence $v(w) = -v(u) < 0$ so by definition $w \notin \mathcal{O}_K$. But this contradicts our assumption that u is a unit. Therefore u is a unit of $\mathcal{O}_K$ iff it is not contained in $\mathfrak{m}_K$ so $\mathfrak{m}_K$ is the unique maximal ideal of $\mathcal{O}_K$. ■

Definition 3.8. For a valued field K we call the set $\mathcal{O}_K$ of $x \in K$ such that $v(x) \geq 0$ the valuation ring. We call the field $k_K = \mathcal{O}_K/\mathfrak{m}_K$ the residue field.

We now see our main connection. Namely that $\mathcal{O}_K$ is a DVR.

Theorem 3.9. *For a valued field K The valuation ring $\mathcal{O}_K$ is a DVR.*

Proof. Note first that the units of $\mathcal{O}_K$ are exactly those $x \in K$ such that $v(x) = 0$. Now choose some $\pi \in K$ such that $v(\pi) = 1$. Then for any $x \in O_K$ the element $\frac{x}{\pi^{v(x)}} \in K$ has valuation
$$v(\frac{x}{\pi^{v(x)}}) = v(x) - v(x)v(\pi) = v(x) - v(x) = 0.$$
Therefore $\frac{x}{\pi^{v(x)}} \in \mathcal{O}_K$ and is a unit so every element of $\mathcal{O}_K$ can be written as $u\pi^{v(x)}$ where u is a unit. In particular every $x \in \mathfrak{m}$ is a multiple of π. Therefore $\mathfrak{m} \subset (\pi)$. But since $v(\pi) > 0$ we also have $(\pi) \subset \mathfrak{m}$ so $\mathfrak{m} = (\pi)$. Now if $\mathfrak{p} \subset \mathcal{O}_K$ is a non zero prime we have some non-zero $x \in \mathfrak{p}$ and thus $\pi^n \in \mathfrak{p}$. Since primes are radical it follows that $\pi \in \mathfrak{p}$ and

thus that $\mathfrak{p} = (\pi)$. Therefore $\mathcal{O}_K$ has a unique prime ideal and that prime is principal. So $\mathcal{O}_K$ is a DVR. ∎

Lemma 3.10. *If $x \in \mathfrak{m}_K^n$ then $v(x) \geq n$.*

Proof. Pick a uniformizer π. Note that $v(\pi) = 1$ and $\mathfrak{m}_K = (\pi)$ so $\mathfrak{m}_K^n = (\pi^n)$. Therefore if $x \in \mathfrak{m}_K^n$ then $x = \pi^n y$ for some $y \in \mathcal{O}_K$ so

$$v(x) = v(\pi^n) + v(y) \geq v(\pi^n) = n.$$

∎

The final relevant characteristic we will prescribe is a finite residue field.

Definition 3.11. A local field is a complete valued field K with finite residue field k_K.

We can now formulate a general version of Hensel's lemma.

Theorem 3.12 (Hensels Lemma)**.** *Suppose K is a complete valued field and suppose $f \in \mathcal{O}_K[x]$. Then if f has a simple root $\bar{a}$ in k_K (recall that a root is simple if $f'(\bar{a}) \neq 0$) there is a unique $a \in \mathcal{O}_K$ such that $a \equiv \bar{a} \bmod \mathfrak{m}_K$ and $f(a) = 0$.*

Proof. We construct a sequence $(a_n)_{n \geq 0}$ in $\mathcal{O}_K$ satisfying

$$f(a_n) \equiv 0 \pmod{\mathfrak{m}_K^{n+1}}, \qquad a_n \equiv a_{n-1} \pmod{\mathfrak{m}_K^n}.$$

For the base case, let $a_0 \in \mathcal{O}_K$ be any lift of $\bar{a}$. Since $\bar{a}$ is a root of $\bar{f}$, we have $f(a_0) \equiv 0 \pmod{\mathfrak{m}_K}$.

For the inductive step, suppose a_n has been constructed with $f(a_n) \in \mathfrak{m}_K^{n+1}$. Set

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}.$$

We first verify this is well-defined, i.e. that $f'(a_n) \in \mathcal{O}_K^\times$. Since $\bar{a}$ is a simple root of $\bar{f}$, we have $\bar{f}'(\bar{a}) \neq 0$, so $f'(a_0) \notin \mathfrak{m}_K$. Since $a_n \equiv a_0 \pmod{\mathfrak{m}_K}$ by the inductive hypothesis, we get $f'(a_n) \equiv f'(a_0) \pmod{\mathfrak{m}_K}$, hence $f'(a_n) \in \mathcal{O}_K^\times$ as required.

Setting $h = -f(a_n)/f'(a_n)$, note that $h \in \mathfrak{m}_K^{n+1}$ since $f(a_n) \in \mathfrak{m}_K^{n+1}$ and $f'(a_n)$ is a unit. In particular $a_{n+1} = a_n + h \equiv a_n \pmod{\mathfrak{m}_K^{n+1}}$, so the congruence condition holds. For the valuation condition, we use the algebraic identity

$$f(a_n + h) = f(a_n) + hf'(a_n) + h^2 \cdot g(a_n, h)$$

for some $g \in \mathcal{O}_K[x, y]$, which holds purely algebraically in $\mathcal{O}_K[x]$ regardless of the characteristic. Substituting $h = -f(a_n)/f'(a_n)$ the first two terms cancel, giving

$$f(a_{n+1}) = h^2 \cdot g(a_n, h).$$

Since $h \in \mathfrak{m}_K^{n+1}$ we have $h^2 \in \mathfrak{m}_K^{2n+2} \subseteq \mathfrak{m}_K^{n+2}$, and $g(a_n, h) \in \mathcal{O}_K$, so $f(a_{n+1}) \in \mathfrak{m}_K^{n+2}$ as required.

We now show convergence. Since $a_{n+1} - a_n = h \in \mathfrak{m}_K^{n+1}$, the sequence (a_n) is Cauchy. By completeness of K it converges to some $a \in \mathcal{O}_K$, and by continuity of f we have $f(a) = \lim f(a_n) = 0$. By construction $a \equiv a_0 \equiv \bar{a} \pmod{\mathfrak{m}_K}$.

Finally we show uniqueness. Suppose $b \in \mathcal{O}_K$ also satisfies $f(b) = 0$ and $b \equiv \bar{a} \pmod{\mathfrak{m}_K}$. Then $a - b \in \mathfrak{m}_K$, and expanding f around a gives

$$0 = f(b) = f(a) + (b-a)f'(a) + (b-a)^2 \cdot g(a, b-a) = (b-a)\big[f'(a) + (b-a) \cdot g(a, b-a)\big].$$

Since $f'(a) \equiv \bar{f}'(\bar{a}) \not\equiv 0 \pmod{\mathfrak{m}_K}$ and $(b-a) \in \mathfrak{m}_K$, the bracket is a unit, hence $b - a = 0$. ■

As a corollary we get teichmuller representatives for local fields in general.

Corollary 3.13. *Suppose K is a local field with residue field $k_K \cong \mathbb{F}_{p^n}$. Then for each $0 \neq \bar{a} \in k_K$ there exists a unique lift $[\bar{a}] \in \mathcal{O}_K$ such that $[\bar{a}]$ is a root fo $x^{p^n-1} - 1$. These lifts, along with zeros are known as teichmuller representatives.*

Proof. If $0 \neq \bar{a} \in k_K \cong \mathbb{F}_{p^n}$ then by the structure theorem for finite fields $\bar{a}^{p^n} - 1 = 0$. Now the derivative of $x^{p^n-1} - 1$ is $(p^n - 1)x^{p^n-2}$ which is non-zero for $x \neq 0$ so by hensels lemma $\bar{a}$ lifts uniquely to a root $[\bar{a}]$ of $x^{p^n-1} - 1$ in $\mathcal{O}_K$. ■

This allows us to write any element of a local field as something looking like a Laurent series in π with coefficients in the teichmuller representatives.

Theorem 3.14. *If $x \in K$ we may write x uniquely as*

$$\sum_{n=k}^{\infty} \pi^n [a_n]$$

for some $k \in \mathbb{Z}$ possibly dependent on n.

Proof. Every x can be written uniquely as $u\pi^n$ where $u \in O_K$ is a unit. So it is sufficient to show that units can be written uniquely of the form

$$\sum_{n=0}^{\infty} [a_n]\pi^n.$$

For any unit w let $\bar{w}$ be the reduction to the residue field and $[\bar{w}]$ the corresponding teichmuller lift. We construct a sequence of coefficients $[a_n]$ inductively such that $u - \sum_{n=0}^{N} [a_n]\pi^n \in (\pi^{N+1})$. The base case is trivial by taking $[a_0] = [\bar{u}]$. Now suppose we have constructed the firsyt N coefficients, then

$$u - \sum_{n=0}^{N} [a_n]\pi^n = \pi^{N+1} u_N$$

for some unit u_N. Taking $[a_n] = [\bar{u}_{N+1}]$ we have

$$u - \sum_{n=0}^{N+1} [a_n]\pi^n = \left(u - \sum_{n=0}^{N} [a_n]\pi^n \right) - [\bar{u}_{N+1}]\pi^{N+1} = \pi^{N+1}(u_{N+1} - [\bar{u}_{N+1}])$$

Since $u_{N+1} \equiv [\bar{u}_{N+1}] \bmod (\pi)$ we thus have $(u_{N+1} - [\bar{u}_{N+1}]) \in (\pi)$ and thus $\pi^{N+1}(u_{N+1} - [\bar{u}_{N+1}]) \in (\pi^{N+2})$. Therefore

$$u - \sum_{n=0}^{N+1} [a_n]\pi^n \in (\pi^{N+2})$$

so our inductive construction is complete. By construction $v(u - \sum_{n=0}^{N} [a_n]\pi^n) \to \infty$ and so $\sum_{n=0}^{N} [a_n]\pi^n \to u$. ■

4. Extensions of local fields

Let K be a local field and L/K a finite extension. Our first goal is to extend the valuation v from K to L. To do so we define the valuation in terms of the minimal polynomial

Definition 4.1. Let $x \in L$ have minimal polynomial $p(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0 \in K[t]$ over K. Define

$$v_L(x) = \frac{1}{d}v(a_0), \qquad v_L(0) = \infty.$$

Remark 4.2. For $x \in K$ the minimal polynomial is $t - x$, so $a_0 = -x$ and $v_L(x) = v(-x) = v(x)$. Thus v_L extends v. Note also that v_L takes values in $\frac{1}{[L:K]}\mathbb{Z}$ rather than $\mathbb{Z}$, so it is not yet normalized.

Theorem 4.3. *The function v_L is the unique valuation on L extending v, further L is complete with respect to this valuation and O_L is a free O_K module of degree $[L : K]$.*

Proof. These proofs require the machinery of integrality, the norm, and trace, all of which are slightly beyond the scope of this paper. See [1] for a complete treatement. ∎

Corollary 4.4. *L is a local field.*

Proof. We need only establish that k_L is finite. To do so note that because v_L extends v we have $\mathcal{O}_K \cap \mathfrak{m}_L = \mathfrak{m}_K$. Then we have a natural inclusion

$$k_K = \mathcal{O}_K/\mathfrak{m}_K \cong \mathcal{O}_K/(\mathcal{O}_K \cap \mathfrak{m}_L) \subset \mathcal{O}_L/\mathfrak{m}_L = k_L.$$

So k_L is an extension of k_K. Now since O_L is a finitely generated O_K module it follows that k_L is a finitely generated k_K vector space. So k_L is a finite extension of k_K which is finite and thus k_L is finite. ∎

Having established that v_L is a valuation on L, we now normalize it. Since v_L extends v and v is surjective onto $\mathbb{Z}$, the image $v_L(L^\times)$ is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, hence of the form $\frac{1}{e}\mathbb{Z}$ for a unique positive integer e.

Definition 4.5. The ramification index of L/K is the unique positive integer $e = e(L/K)$ such that $v_L(L^\times) = \frac{1}{e}\mathbb{Z}$. The residue degree is $f = f(L/K) = [k_L : k_K]$. The normalized valuation on L is $w_L = e \cdot v_L$, which is surjective onto $\mathbb{Z}$.

Remark 4.6. The normalized valuation w_L satisfies $w_L|_K = e \cdot v$, so it does not extend v itself but rather scales it by e. A uniformizer of L with respect to w_L is any $\pi_L \in O_L$ with $w_L(\pi_L) = 1$, or equivalently $v_L(\pi_L) = \frac{1}{e}$. The original uniformizer π_K of K satisfies $w_L(\pi_K) = e$.

Theorem 4.7 (degree formula)**.** *If L is a local field and an extension of K another local field such that f is finite, then $[L : K] = ef$.*

Proof. Let π_L be a uniformizer of L, then π_L^e is a uniformizer of K. Choose elements $x_1, \dots, x_f \in \mathcal{O}_L$ whose reductions $\bar{x}_1, \dots, \bar{x}_f$ form a k_K-basis for k_L. We claim that

$$\mathcal{B} = \{x_i\pi_L^j : 1 \le i \le f,\ 0 \le j \le e-1\}$$

is an $\mathcal{O}_K$-basis for $\mathcal{O}_L$. Since $\mathcal{O}_L$ is free of rank $[L : K]$ over $\mathcal{O}_K$ this gives the formula.

We first show that every $y \in \mathcal{O}_L$ can be written as $\sum_{i,j} a_{ij}x_i\pi_L^j$ with $a_{ij} \in \mathcal{O}_K$. We construct the coefficients by successive approximation.

Since $\bar{x}_1, \ldots, \bar{x}_f$ span k_L over k_K, we may write $\bar{y} = \sum_i \bar{c}_i^{(0)} \bar{x}_i$ with $\bar{c}_i^{(0)} \in k_K$. Lift each $\bar{c}_i^{(0)}$ to $c_i^{(0)} \in \mathcal{O}_K$. Then $y - \sum_i c_i^{(0)} x_i \in \mathfrak{m}_L$, so we may write

$$y - \sum_i c_i^{(0)} x_i = \pi_L y_1$$

for some $y_1 \in \mathcal{O}_L$. Applying the same argument to y_1, we find $c_i^{(1)} \in \mathcal{O}_K$ such that $y_1 - \sum_i c_i^{(1)} x_i = \pi_L y_2$, and so on. After e steps we have

$$y - \sum_{j=0}^{e-1} \pi_L^j \sum_i c_i^{(j)} x_i = \pi_L^e y_e.$$

Since $w_L(\pi_K) = e$ we have $\pi_L^e = u\pi_K$ for some unit $u \in \mathcal{O}_L^\times$, so $\pi_L^e y_e \in \pi_K \mathcal{O}_L$. Repeating the entire process with y_e in place of y and extracting successive powers of π_K, we obtain for each $n \geq 0$ an approximation

$$y - \sum_{j=0}^{e-1} \pi_L^j \sum_i \left(\sum_{k=0}^{n} c_i^{(j,k)} \pi_K^k \right) x_i \in \mathfrak{m}_L^{en+e}$$

where $c_i^{(j,k)} \in \mathcal{O}_K$. Setting $a_{ij} = \sum_{k=0}^\infty c_i^{(j,k)} \pi_K^k \in \mathcal{O}_K$ (convergent by completeness of K), we obtain $y = \sum_{i,j} a_{ij} x_i \pi_L^j$. This shows that $\mathcal{B}$ spans $\mathcal{O}_L$.

Suppose $\sum_{i,j} a_{ij} x_i \pi_L^j = 0$ with $a_{ij} \in \mathcal{O}_K$ not all zero. Since v is discrete we may divide through by a power of π_K to assume some a_{ij} is a unit. Let j_0 be the smallest index j for which some a_{ij} is a unit. Then

$$\sum_{i,j} a_{ij} x_i \pi_L^j = \pi_L^{j_0} \left(\sum_i a_{ij_0} x_i + \pi_L(\cdots) \right) = 0.$$

Since $\pi_L \neq 0$ and $\mathcal{O}_L$ is an integral domain we may divide by $\pi_L^{j_0}$, and then reducing modulo $\mathfrak{m}_L$ gives

$$\sum_i \bar{a}_{ij_0} \bar{x}_i = 0 \quad \text{in } k_L,$$

with some $\bar{a}_{ij_0} \neq 0$ since a_{ij_0} was a unit. This contradicts the k_K-linear independence of $\bar{x}_1, \ldots, \bar{x}_f$ in k_L.

We have thus shown that $\mathcal{B}$ is a basis and therefore that $[L : K] = ef$. ■

Corollary 4.8. *If L/K is an extension such that both L and K are local fields with v_L extending v_K then $[L : K]$ is finite.*

Proof. Since k_L and k_K are finite, $f = [k_L : k_K]$ is finite and $e = v(\pi_K)$ is finite by definition. So $[L : K] = ef$ is finite. ■

5. Classification of local fields

We finally arrive at our ending theorem. That any local field K, that is any complete discretely valued field with a finite residue field, is isomorphic to either $\mathbb{F}_{p^n}((t))$ or a finite extension of $\mathbb{Q}_p$. This shows that the only real analogs of $\mathbb{Q}_p$ in positive characteristic are local function fields. We first prove the characteristic zero case.

Theorem 5.1. *If K is a characteristic 0 local field it is a finite extension of $\mathbb{Q}_p$.*

Proof. Since K is characteristic 0, $\mathbb{Q} \subset K$. Restricting $|\cdot|_v$ to $\mathbb{Q}$ gives a non-archimadean absolute value on $\mathbb{Q}$. But by Ostrawaski's theorem this must be the p-adic absolute value for some p. Since K is complete it follows that the closure of $\mathbb{Q}$ in K is $\mathbb{Q}_p$ for some p. Therefore K is an extension of $\mathbb{Q}_p$. Since K is a local field it then follows that K is a finite extension of $\mathbb{Q}_p$. ∎

Now for the characteristic $p > 0$ case we first prove the following lemma.

Lemma 5.2. *If K is a local field of characteristic p. Its residue field must also be characteristic p.*

Proof. Since K is characteristic p we have a ring homomorphism (field extension) $\mathbb{F}_p \to K$ mapping $n \mapsto n$. Since $n \in \mathcal{O}_K$ we have a ring homomorphism $\mathbb{F}_p \to \mathcal{O}_K$, composing with the projection $\mathcal{O}_K \to k_K$ gives a field extension $\mathbb{F}_p \to k_K$. Therefore k_K is characteristic p. ∎

Lemma 5.3. *If K is characteristic p with residue field $\mathbb{F}_{p^n}$ then its teichmuller representatives form a subfield isomorphic to $\mathbb{F}_{p^n}$.*

Proof. The teichmuller representatives are exactly the roots of $x^{p^n} - x$ in K. But in characteristic p the set of these roots is the field $\mathbb{F}_{p^n}$. Therefore the teichmuller representatives form a subfield isomorphic to $\mathbb{F}_{p^n}$. ∎

Theorem 5.4. *If K is a characteristic $p > 0$ local field it is isomorphic to $\mathbb{F}_{p^n}((t))$.*

Proof. Let π be a uniformizer of K. The elements of K are formal Laurent series

$$\sum_{n=-\infty}^{\infty} a_n \pi^n$$

where a_n are teichmuller representatives. But the teichmuller representatives form a subfield isomorphic to $\mathbb{F}_{p^n}$. So the function $f : K \to \mathbb{F}_{p^n}((t))$ given by

$$\sum_{n=-\infty}^{\infty} a_n \pi^n \mapsto \sum_{n=-\infty}^{\infty} a_n t^n$$

is a field isomorphism. Therefore $K \cong \mathbb{F}_{p^n}((t))$. ∎

6. Conclusion

The classification theorem shows that the list of local fields is remarkably short. In characteristic 0, Ostrowski's theorem forces K to contain $\mathbb{Q}_p$, and the degree formula then gives $[K : \mathbb{Q}_p] = ef < \infty$. In characteristic p, the Teichmüller expansion constructs an explicit isomorphism $K \cong \mathbb{F}_{p^n}((t))$. We see then that the analogy between $\mathbb{Q}_p$ and local function fields is forced by its basic characteristics.

This paper has omitted several natural next steps. A fuller treatment would include the theory of unramified and totally ramified extensions, the different and discriminant, and the higher ramification groups. These connect the elementary theory developed here to local class field theory, which describes the abelian extensions of local fields in terms of $K^\times$ and is one of the most important results in algebraic number theory.

References

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