Don't Bring a Square Cake to the Triangle Party

A Proof of Monsky's Theorem

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1. The Triangle Party

This is a story of an ill-fated triangle party, where the arrival of a square-shaped cake caused chaos among the party participants.

This is also a proof of Monsky's Theorem.

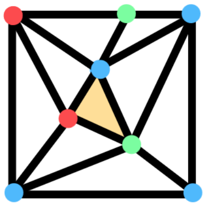
Claim (Monsky's Theorem) [2]: A square cannot be triangulated into an odd number of triangles with equal area.

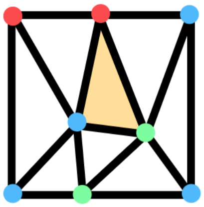
Let's go back to the triangle party, to see where the trouble started.

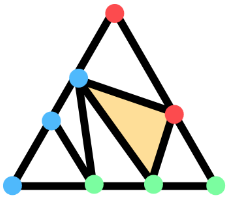
2. Cool Triangle Drawings (Sperner's Lemma)

Consider a triangulation of a polygon P whose points we color either red, blue, or green. A *rainbow triangle* is a triangle within the triangulation where all three points are colored differently. If a triangulation of a polygon has a rainbow triangle, then the triangulation is a *cool triangle drawing.*

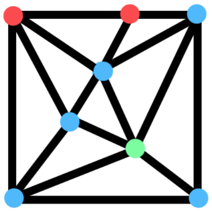
Here are some examples of cool triangle drawings, with one rainbow triangle highlighted in each drawing:

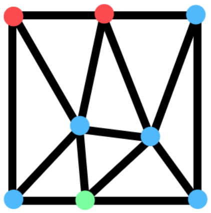


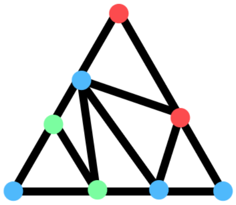




Here are some examples of uncool triangle drawings:







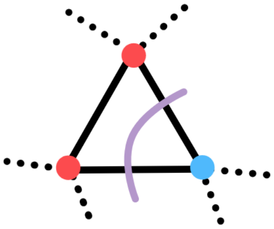
One more definition: we call line segments on the border of P with one vertex colored red and the other vertex colored blue *red-blue borders*.

Claim (Sperner's Lemma): In a tri-colored triangulation of polygon P, if there are an odd number of red-blue borders, there exists a rainbow triangle in P (and thus the triangulation of P is a cool triangle drawing).

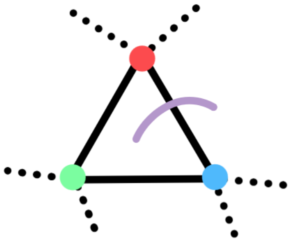
Proof: Assume we have an odd number of red-blue borders in our triangulation of polygon P. We can draw a *red-blue path* on our triangulation starting from outside our polygon, that only crosses through lines whose vertices are colored red and blue.

Whenever we cross a red-blue line segment, three things can happen:

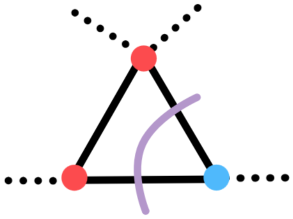
1. The triangle we entered has two vertices colored red, and one vertex colored blue, or vice versa. In this case, we have exactly one way out of the triangle, which we take to continue our red-blue path.



2. We enter the rainbow triangle. Our path ends here, since there are no red-blue lines to cross.

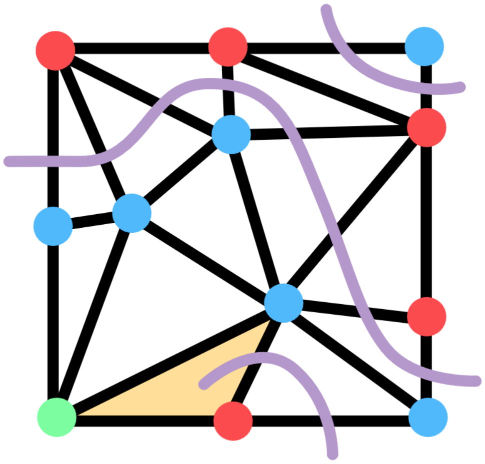


3. We exit the triangulation through a red-blue border, and our path ends here. This red-blue path crosses two red-blue borders.

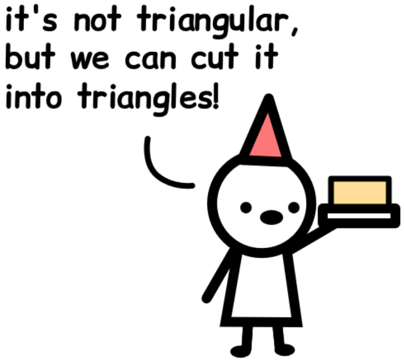


If we draw all possible red-blue paths starting outside the polygon in our triangulation, each red-blue border will be crossed. Suppose there was no rainbow triangle in our triangulation. Then, our paths would cross an even number of red-blue borders, since each path would start and end by crossing a red-blue border. However, since there are an odd number of red-blue borders, there must be at least one rainbow triangle somewhere within our triangulation. ▲

Here's a triangulation with the red-blue paths drawn:



3. Frosting the Cake, 2-adically



The cake is a square, but that's okay! Our fearless partygoers will just cut the square into triangles—that's a quest for later. Also, conveniently, the cake's frostable surface C is $[0,1] \times [0,1]$! With our incredibly-precise piping bags, we can frost our cake (and the rest of the $\mathbb{R}^2$ plane, which will be helpful later) based on the 2-adic absolute value of the coordinates.

We'll frost

- all points $P_1 = \{(x,y) : |x|_2 < 1, |y|_2 < 1\}$ red,
- all points $P_2 = \{(x,y) : |x|_2 \geq 1, |x|_2 \geq |y|_2\}$ blue,
- all points $P_3 = \{(x,y) : |y|_2 \geq 1, |y|_2 > |x|_2\}$ green.

"But wait," you say. "This only frosts points whose coordinates are in $\mathbb{Q}$! What if we cut the cake irrationally?"

I do not understand enough abstract algebra to prove this, but with the power of magic (also known as Chevalley's theorem on extensions of valuations [1]), it is possible to extend the 2-adic absolute value to $\mathbb{R}$ in a natural way.

Here's an interesting property about the frosting we just made:

Claim: Let $(x_1, y_1) \in P_1, (x_2, y_2) \in P_2, (x_3, y_3) \in P_3$. Then, we have that $(x_1 + x_2, y_1 + y_2) \in P_2$, and $(x_1 + x_3, y_1 + y_3) \in P_3$.

Proof: We know that d_2 is an ultrametric, so based on the ultrametric triangle inequality, we have for $x, y \in \mathbb{Q}_2$,

$$|x+y|_2 = \max(|x|_2, |y|_2)$$

unless $|x|_2 = |y|_2$.

Then, since $|x_1|_2 \neq |x_2|_2$, we have that

$$|x_1 + x_2|_2 = \max(|x|_1, |x|_2) = |x_2|_2 \geq 1,$$

and that

$$|y_1 + y_2|_2 \le \max(|y_1|_2, |y_2|_2) \le |x_2|_2 = |x_1 + x_2|_2.$$

Thus, $(x_1 + x_2, y_1 + y_2) \in P_2$.

Similarly, we have $|y_1|_2 \ne |y_3|_2$, so

$$|y_1 + y_3|_2 = \max(|y_1|_2, |y_3|_2) = |y_3|_2 \ge 1,$$

and

$$|x_1 + x_3|_2 \le \max(|x_1|_2, |x_3|_2) < |y_3|_2 = |y_1 + y_3|_2.$$

This means that $(x_1 + x_3, y_1 + y_3) \in P_3$.

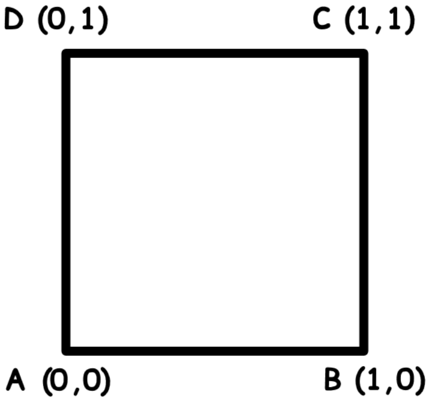
What this claim means is that the blue and green points are translation-invariant with respect to red points: we can move around rainbow triangles, and, as long as the red point gets translated to another red point, we'll still maintain our rainbow triangle. This is a surprise tool that'll help us later. ▲

Great, we've frosted our cake!

4. A Cool Triangle Drawing in Disguise

Claim: No matter how we cut our cake, we will create a cool triangle drawing with the colors of our frosting.

Proof: Consider a triangulation of the cake $C = [0,1] \times [0,1]$ that we've colored 2-adically according to the rules in the section above. We'll show that there are an odd number of red-blue borders, which, when we apply Sperner's Lemma, means that there's a rainbow triangle somewhere within our triangulation.



There can be no red-blue borders on $\overline{AD}$, since $x = 0$ means $|x|_2 = 0$: there are no points colored blue because $|x|_2 \not\ge 1$.

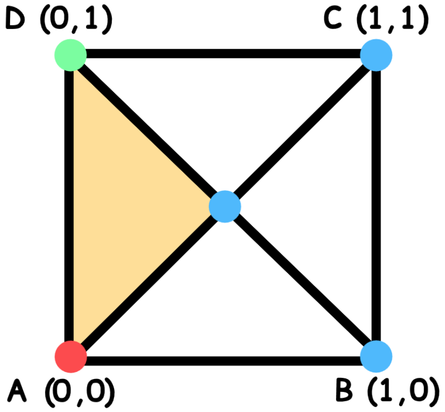
There can be no red-blue borders on $\overline{CD}$, since $y = 1$ means $|y|_2 = 1$, so there are no points colored red.

Similarly, there can be no red-blue borders on $\overline{BC}$, since $x = 1$ means that $|x|_2 = 1$, so no points are colored red.

Thus, red-blue borders can only appear on $\overline{AB}$.

Since point A is colored red, and point B is colored blue, there will be an odd number of red-blue borders in our triangulation. Thus, by applying Sperner's Lemma, we can find a rainbow triangle in C, and our cake is officially cool! ▲

Here's an example triangulation of our cake, with the rainbow triangle highlighted:



5. The Issue

You knew this was coming.

Claim: Let $\triangle T$ be a rainbow triangle in $\mathbb{R}^2$. Then, we have that

$$|\text{area of } \triangle T|_2 \geq 2.$$

Proof: We can first make our lives much easier by translating the rainbow triangle so that the red point of $\triangle T$ is at the origin, which is another red point.

Our translated triangle has the same area, and has coordinates $(0, 0)$, $(x_2, y_2) \in P_2$, and $(x_3, y_3) \in P_3$. Then, we can use the formula for the area of a triangle given its coordinates to get that

$$\text{area of } \triangle T = \frac{x_2y_3 - x_3y_2}{2}.$$

We know that since $(x_2, y_2) \in P_2$, and $(x_3, y_3) \in P_3$, we have $|x_2|_2 \geq |y_2|_2$ and $|y_3|_2 > |x_3|_2$. This means that $|x_2y_3|_2 > |x_3y_2|_2$, so

$$|x_2y_3 - x_3y_2|_2 = \max(|x_2y_3|_2, |-x_3y_2|_2) = |x_2y_3|_2.$$

Thus, we have that

$$|\text{area of } \triangle T|_2 = \left|\frac{1}{2}\right|_2 \cdot |x_2y_3 - x_3y_2|_2 = \left|\frac{1}{2}\right|_2 \cdot |x_2y_3|_2 = 2 \cdot |x_2y_3|_2 \geq 2,$$

where the last inequality is true because we know that $|x_2|_2 \geq 1$ and $|y_3|_2 \geq 1$. ▲

And this is all to say that...

Claim: Our cake C cannot be triangulated into an odd number of triangles with equal area.

Proof: Let's say we split our cake into k triangles with equal area. Then, since we've made a cool triangle drawing, we know there is a rainbow triangle $\triangle T$ somewhere with $|\text{area of } \triangle T|_2 \geq 2$.

If all triangles have equal area, and the area of C is 1, we have that

$$k \cdot [\text{area of } \triangle T] = 1.$$

Since our 2-adic absolute value is multiplicative, this implies that

$$|k|_2 \cdot |\text{area of } \triangle T|_2 = |1|_2 = 1,$$

so $|k|_2 < 1$. Thus, we must have that k is divisible by 2, and so k cannot be odd. ▲

The party was destined to fail, as no cut exists to fairly give everyone a triangle slice. And with that, we conclude the tragedy of the triangle party.

References

[1] Antonio J Engler and Alexander Prestel. *Valued fields*. Springer, 2005.

[2] Paul Monsky. “On dividing a square into triangles”. In: *The American Mathematical Monthly* 77.2 (1970), pp. 161–164.