The p-Adic Solenoid

Jules Gardner

Spring 2026

Abstract

Initially, we will review the p-adic numbers, as well as group theory and topology. We will define abelian groups and automorphism groups, and go over both the basic intuition of topology, and the specifics of open sets. Then we will define the inverse limit and solenoids. The inverse limit is the "limit" of a sequence of mappings between groups or topological spaces. A solenoid is a structure obtained from by taking the inverse limit of a series of topological spaces $\ldots, \mathbb{R}/a_2\mathbb{Z}, \mathbb{R}/a_1\mathbb{Z}, \mathbb{R}/a_0\mathbb{Z}$, mapped between by a certain function. Examples include the 2-solenoid, in which $a_n = 2^n$ for all n. But we can generalize to other sequences, such as the 3-solenoid, in which $a_n = 3^n$ for all n. The p-adic solenoid, with $a_n = p^n$, is one pattern of directly generalizing the n-solenoid, making the 2-solenoid also the 2-adic solenoid. This is useful because the p-adic numbers are obtained from the inverse limit $\ldots, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^1\mathbb{Z}, \mathbb{Z}/p^0\mathbb{Z}$. This makes sense; to obtain a p-adic number, we have a sequence of longer and longer base-p expansions. For example, a 7-adic ending in $\ldots 2135$ comes from $\ldots, 2135, 135, 35, 5$, where each element of the sequence is in a corresponding set in the sequence $\ldots, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^1\mathbb{Z}, \mathbb{Z}/p^0\mathbb{Z}$.

1 Introduction

Definition 1. *The p-adic number system, denoted $\mathbb{Q}_p$, for a prime p is the completion of $\mathbb{Q}$ under a metric in which the absolute value of $n \neq 0$ is the reciprocal of the p part of the prime factorization of n and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$. We also define the p-adic valuation $v_p(x)$ to be the number of factors of p that are in x, and $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$.*

A p-adic number can be represented as an infinite string of base-p digits extending forever to the left, and operations on these representations work similarly to normal integers. Now, these representations are simply Cauchy sequences of integers. For example, a 7-adic ending in $\ldots 2135$ comes from $5, 35, 135, 2135, \ldots$. Since the ending digits stabilize, it turns out that the integer sequence has to converge.

The p-adic numbers are a field under the addition and multiplication operations defined as

$$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n + b_n)$$

and

$$\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} a_n b_n.$$

Now, we see that these operations are well-defined due to the p-adic metric satisfying the requirements of an absolute value, and what's more, $\mathbb{Q}_p$ turns out to be a field.

A p-adic number can be represented as an infinite string of base-p digits extending forever to the left, and operations on these representations work similarly to normal integers. Now, these representations are simply Cauchy sequences of integers. For example, a 7-adic ending in $\ldots 2135$ comes from $5, 35, 135, 2135, \ldots$. Since the ending digits stabilize, it turns out that the integer sequence has to converge.

The p-adic numbers are a field under the addition and multiplication operations defined as

$$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n + b_n)$$

and

$$\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = \lim_{n\to\infty} a_n b_n.$$

Now, we see that these operations are well-defined due to the p-adic metric satisfying the requirements of an absolute value, and what's more, $\mathbb{Q}_p$ turns out to be a field.

Definition 2. *A topology on a set X is a set $\mathcal{T}$ of subsets of X that follows the following axioms:*

1. $\{\varnothing, X\} \subset \mathcal{T}$
2. *If* $\{X_\alpha | \alpha \in A\} \subseteq \mathcal{T}$*, then* $\bigcup_{\alpha\in A} X_\alpha \in \mathcal{T}$
3. *If* $Y, Z \in \mathcal{T}$*, then* $Y \cap Z \in \mathcal{T}$

Now, there are two "trivial" topologies present on every set X:

- The coarsest topology $\{\varnothing, X\}$, which is commonly referred to as *the* trivial topology.
- The finest topology $\mathcal{P}(x)$, also known as the power set, which contains all subsets of X.

2 Inverse Limits

An inverse limit of a sequence of mappings $\cdots \xrightarrow{f_3} F_3 \xrightarrow{f_2} F_2 \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0$ is defined as the entity F with a mapping $F \xrightarrow{q_n} F_n$ for all n, such that $f_n \circ q_{n+1} = q_n$. More specifically, it is the set of all sequences such that each element of the sequence maps to the next.

Definition 3. *A solenoid is the inverse limit of a series of topological spaces* $\ldots \xrightarrow{f_2} \mathbb{R}/a_2\mathbb{Z} \xrightarrow{f_1} \mathbb{R}/a_1\mathbb{Z}, \mathbb{R}/a_0\mathbb{Z}$*. These topological spaces can be visualized as circles of circumference* a_n*.*

Take the 2-solenoid. We have the sequence

$$\ldots, \mathbb{R}/2^2\mathbb{Z}, \mathbb{R}/2^1\mathbb{Z}, \mathbb{R}/2^0\mathbb{Z}$$

We have the map $\mathbb{R}/2^{n+1}\mathbb{Z} \xrightarrow{f_n} \mathbb{R}/2^n\mathbb{Z}$ given by taking the input value and evaluating modulo 2^n. This means the map is surjective but not injective.

One might imagine that the 2-solenoid is an infinitely large circle folded over onto itself infinitely many times so that it becomes a small circumference-1 circle.

The 2-solenoid can be generalized to, for example, the 3-solenoid, in which 2 is replaced with 3. In fact, we can replace 2 with any prime, forming the p-adic solenoid. Since 2 is a prime, the 2-solenoid is also the 2-adic solenoid Σ_2.

There is a specific reason why we may wish to include "adic" in the name. This is because a p-solenoid is analogous to the p-adic integers: the p-adic integers are obtained from the inverse limit $\ldots, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^1\mathbb{Z}, \mathbb{Z}/p^0\mathbb{Z}$. This makes intuitive sense; to obtain a p-adic number, we have a sequence of longer and longer base-p expansions. For example, a 7-adic ending in $\ldots 2135$ comes from $\ldots, 2135, 135, 35, 5$, where each element of the sequence is in a corresponding set in the sequence

$$\ldots, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^1\mathbb{Z}, \mathbb{Z}/p^0\mathbb{Z}$$

. To formalize, we write $f_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ given by reducing modulo p^n, and a map from $f_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ by reducing modulo p^n.

Sidenote 1. *By definition the p-adic solenoid lends itself to these homomorphisms.*

3 Topology Of The p-Adic Solenoid

The p-adic solenoid is a topological structure, more specifically an infinitely long loop folded over onto itself infinitely many times until it becomes a circumference-1 circle. Sort of like a infinite coil, hence the term solenoid, suggesting similarity to a coil of wire.

Now, the p-adic solenoid can also be considered as a sequence of elements of the circles, such that each element of the sequence maps to the next one. Similar to the 7-adic $\ldots 2135$, which is the sequence $\ldots, 2135, 135, 5$, where, for example, 2135 maps to 135 by reducing modulo 7^3.

The p-adic solenoid is, notably, connected but not path connected: There are no two disjoint nonempty open sets U, V such that $U \cap V$ is the entire p-adic solenoid. However, two different points can be in a sense "infinitely far apart," making it impossible for a continuous path between them to exist.

4 p-Multiplication

Consider multiplying by p in $\mathbb{R}/\mathbb{Z}$. As we repeat the operation, we can look at this another way, saying that get elements of the sets $\mathbb{R}/p^n\mathbb{Z}$, mapped between by the mapping we already saw. This means that each element of the p-adic solenoid represents an entire chain of events leading up to one number. As the p-adic solenoid coils around forever, each point ends up getting placed on top of some point, and the many points that map there represent the many pasts that could have led there.

This idea has an analogue in $\mathbb{Q}_p$, but it is not as nice: the elements of the sequence

$$\ldots, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^1\mathbb{Z}, \mathbb{Z}/p^0\mathbb{Z}$$

are not isomorphic to one another. However, if there is a sequence of homomorphisms $\ldots, f_2, f_1, f_0$ on F, and $f \circ f_n$ is the identity function, then each element of the inverse limit represents an entire chain of events leading up to one member of the object.

5 The Circle-Doubling Map

Denote $\mathbb{T}$ to be the circumference-1 circle as a group, under angle addition. More specifically $\mathbb{R}/\mathbb{Z}$. The Circle-Doubling Map is the map $T_2(t) = 2t \pmod 1$ from $\mathbb{T}$ to itself. To make the map invertible, define

$$X_2 = \left\{ x \in \mathbb{T}^{\mathbb{Z}} \middle| x_{k+1} = T_2(x_k) \right\}$$

. Now, X_2 is essentially the 2-adic solenoid: It stores an entire timeline of the division-by-p operation, just as the 2-adic numbers do. And it turns out that this can be extended to other p-adic solenoids:

$$T_p(t) = pt \pmod 1$$

$$X_p = \left\{ x \in \mathbb{T}^{\mathbb{Z}} \middle| x_{k+1} = T_p(x_k) \right\}$$

We construct the following bijection from X_p to Σ_p: Take $x \in X_p$. Each element of the p-adic solenoid Σ_p is a sequence of elements of each $\mathbb{R}/p^n\mathbb{Z}$ that map to each other with T_p. And so we let $n \geq 0$ and take $x_n p^n$ to be in $\mathbb{R}/p^n\mathbb{Z}$, giving the bijection. Apart from being one more way of visualizing the p-adic solenoid, X_p allows some interesting results:

The diagonal embedding $\delta(r) = (r, r)$ embeds $\mathbb{Z}\left[\frac{1}{2}\right]$ as a discrete subgroup of $\mathbb{R} \times \mathbb{Q}_2$. Moreover,

$$X_2 \cong \mathbb{R} \times \mathbb{Q}_2/\delta\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \cong \mathbb{R} \times \mathbb{Z}_2/\delta(\mathbb{Z}).$$

Extending to general p, we get

$$X_p \cong \mathbb{R} \times \mathbb{Q}_p/\delta\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cong \mathbb{R} \times \mathbb{Z}_p/\delta(\mathbb{Z}).$$

6 Pontryagin Duality

The Pontryagin dual of a locally compact abelian topological group is the group of continuous homomorphisms to the circle of circumference 1.

We use the term "dual" because the operation of taking the Pontryagin dual is an involution; that is, the dual of the dual is the original group. Some other well-known involutions are reciprocation of real numbers, negation of real numbers, and reversing a finite sequence.

The Pontryagin dual of the p-adic integers is known as the Prüfer p-Group. The Pontryagin dual of the p-adic solenoid Σ_p is the group $\mathbb{Z}\left[\frac{1}{p}\right]$, formed by asserting $\frac{1}{p}$ as a member of $\mathbb{Z}$. This can be thought of alternatively as the group of rationals whose denominators are powers of p.

It makes sense that these two are duals: These $\mathbb{Z}\left[\frac{1}{p}\right]$ elements are rationals, and they correspond to rotating the 2-solenoid so that a point on the "circumference" moves a certain distance.

Theorem 1. *We can prove the Pontryagin dual of the p-adic solenoid Σ_p is the group $\mathbb{Z}\left[\frac{1}{p}\right]$ by using the fact that the dual of the inverse limit is the inverse limit of the duals. So the p-adic solenoid's dual is the inverse limit of $\mathbb{Z}$ (the dual of $\mathbb{T}$) with the $\times p$ mapping. Taking this inverse limit, we get the final result $\mathbb{Z}\left[\frac{1}{p}\right]$.*

7 Dynamical Systems

Definition 4. *A dynamical system $H \curvearrowright \mathcal{X}$ is a group action of a semigroup H on a topological space $\mathcal{X}$. For example, take the action of $\mathbb{T}$ on $\mathbb{R}^2$, with the action, of course, being rotation.*

> Classically, $\mathcal{X}$ is the phase space, the semigroup H represents time, and the action describes time evolution of the system. Accordingly, the acting semigroup is typically a subsemigroup of the additive group of real numbers (e.g., the semigroup of non-negative reals, the group of integers, or the semigroup of natural numbers).[6]

However, this does not have to be the case for the temporal interpretation to work, as our example showed: Our example describes a point rotating around the origin.

Consider the dynamical system $\mathbb{R} \curvearrowright \Sigma_p$, in which we rotate the point on the Σ_p space, around the infinite loop. This is a dynamical system, with the semigroup being $\mathbb{R}$.

Definition 5. *The central topic of topological dynamics is the study of topological properties of the orbits $Hx = \{hx | h \in H\}$ of a dynamical system $H \curvearrowright \mathcal{X}$.*

In our example, orbits are circles around the origin. The name suggests a path followed by an object over time, though they need not be circles, ellipses, or even loops at all: In the translation system $\mathbb{Z} \curvearrowright \mathbb{Z}$, the entire system is one orbit. In the horizontal translation system $\mathbb{Z} \curvearrowright \mathbb{Z}^2$, the orbits are horizontal lines. In our example $\mathbb{Z} \curvearrowright \Sigma_p$, the orbits are infinite lines. These orbits are fundamental to the structure of Σ_p, but Σ_p is not simply a collection of infinite lines. Instead, each line is connected to another. This can be thought of as the set $(0, \infty)$ on one of the lines not being open, but it is when put together with $(-\infty, 0)$ on the next line. This is why Σ_p is connected but not path-connected, an important yet unintuitive phenomenon.

8 Conclusion

The p-adic solenoid is a useful class of object, with applications in many fields of study, ranging from p-adic mathematics to topology to dynamics. Its construction helps exemplify the inverse limit. As a topological space, it helps show why connectedness does not imply path connectedness. As part of a dynamical system, it shows how the orbits of a system can reveal concepts about the structure of a space. It helps explain Pontryagin duals, and stores entire histories of systems.

References

[1] Simon Rubenstein-Salzedo *p-adic Analysis*.

[2] Maxim Gilula *Fundamentals of Mathematics*

[3] Paul Garrett *The 2-solenoid*

[4] Paul Garrett *Adele groups, p-adic groups, solenoids*

[5] Manfred Einsiedler and Thomas Ward *Ergodic Theory with a View Towards Number Theory*

[6] Volodymyr Nekrashevych *Groups and Topological Dynamics*