Cantor Sets

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May 2026

1 Introduction

The usual Cantor set is one of the basic examples of a space that is compact, totally disconnected, and still has no isolated points. The construction begins with the interval $[0, 1]$. At the first step, we remove the open middle third $(1/3, 2/3)$. At the next step, we remove the open middle third from each of the two remaining intervals, and we continue this process forever.

The Cantor set is interesting because it is made by removing many intervals, but there are still many points left. In fact, the Cantor set is uncountable, even though it does not contain any interval. This makes it a useful example because it shows that different ideas of size can behave differently.

Another useful feature of the Cantor set is that it repeats its own structure. At every stage, the remaining set is made of smaller copies of the previous stage. Because of this, the Cantor set can be described geometrically, but it can also be described using number expansions. The usual Cantor set is related to base 3 expansions. Later, we will define similar sets that are related to base $2p - 1$ expansions.

This paper will explain the connection and then relate these Cantor-type sets to the p-adic integers. The main idea is that both objects canbe described by infinite sequences of digits.

2 Cantor Sets and Generalized Cantor Sets

We first define the usual Cantor set. The construction starts with the whole interval $[0, 1]$. Then, at each step, we remove the open middle third from every interval that is still left. The Cantor set is the set of points that are never removed.

Definition 2.1 (Cantor Set)**.** A Cantor Set is the set obtained after starting with the set $[0, 1]$ and successive removing the middle third interval from each of the remaining intervals. We can recursively define this by setting $C_0 = [0, 1]$ and C_n is the set remaining after removing the open middle third of each interval in C_{n-1}. Then

$$C = \bigcap_{n=0}^{\infty} C_n$$

Equivalently, the construction can be described recursively in a more algebraic way. At each step, we keep a scaled copy of the previous stage on the left and another scaled copy on the right.

Definition 2.2. Let $C_0 = [0, 1]$. Then

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right)$$

We define the cantor set to be

$$C = \bigcap_{n=0}^{\infty} C_n$$

This definition is the same construction, but written in a more algebraic form. It says that C_n is made from two smaller copies of C_{n-1}. One copy is put in the left third of the interval, and the other copy is put in the right third.

This form is useful because it makes the self-similarity of the Cantor set more visible. It also makes it easier to generalize the construction. Instead of keeping two intervals out of three, we can keep p alternating intervals out of $2p - 1$ intervals.

Now let p be a prime. We divide the interval into $2p - 1$ equal pieces. Then we keep the pieces with even indices and remove the alternating open intervals. This gives a Cantor-type set where p intervals are kept at each stage.

The reason for using $2p - 1$ is that the even digits

$$0, 2, 4, \ldots, 2p - 2$$

give exactly p choices. These p choices will match the p possible digits in a p-adic expansion.

Definition 2.3. Let $C_0^{(p)} = [0, 1]$. Then

$$C_n^{(p)} = \bigcup_{j=0}^{p-1} \left(\frac{2j}{2p-1} + \frac{C_{n-1}^{(p)}}{2p-1} \right)$$

Then

$$C^{(p)} = \bigcap_{n=0}^{\infty} C_n^{(p)}$$

When $p = 2$, this gives the usual Cantor set. In that case, $2p - 1 = 3$, and the allowed digits are 0 and 2. So $C^{(2)}$ is exactly the usual middle-thirds Cantor set.

For larger primes p, the construction is similar, but more intervals are kept at each stage.

3 Digit Expansions and the Connection to $\mathbb{Z}_p$

We first want to get some idea as to what numbers are in the cantor set. We will begin with the special case of $C^{(2)}$. Considering that the Cantor set is constructed by taking the removing the middle third interval, it intuitively makes sense that the Cantor set would contain the numbers with no 1 in their base three expansion.

Lemma 3.1. *$C^{(2)}$ consists of the real numbers in $[0,1]$ such that their base 3 expansions only contain the digits 0 and 2.*

Proof. At the first step of the Cantor set construction, we keep the left third and the right third of $[0,1]$. These are exactly the points whose first base 3 digit can be chosen to be 0 or 2. At the second step, we again keep the left and right thirds inside each remaining interval. This means the second base 3 digit can also be chosen to be 0 or 2.

Continuing in this way, a point survives through the first n stages exactly when its first n base 3 digits can be chosen from the set $\{0,2\}$. Therefore a point survives all stages exactly when it has a base 3 expansion using only 0 and 2.

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Lemma 3.2. *$C^{(p)}$ consists of the real numbers in $[0,1]$ which have a base $2p-1$ expansion containing only the digits $0, 2, \ldots, 2p-2$.*

Proof. Let $b = 2p-1$. At the first stage, we divide $[0,1]$ into b equal pieces and keep the pieces with even indices. These correspond to the first base b digit being one of

$$0, 2, 4, \ldots, 2p-2.$$

At the next stage, the same process is repeated inside each remaining interval. Therefore the second base b digit must also be one of $0, 2, 4, \ldots, 2p-2$. Continuing inductively, a point survives through the first n steps exactly when its first n base b digits can be chosen from the set of even digits.

Thus a point belongs to $C^{(p)}$ if and only if it survives every stage. This happens exactly when the point admits a base $2p-1$ expansion using only the digits

$$0, 2, 4, \ldots, 2p-2.$$

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The previous lemma shows that a point of $C^{(p)}$ is described by an infinite sequence of digits $a_0, a_1, a_2, \ldots$, where each $a_i \in \{0, 1, \ldots, p-1\}$. This is also the same kind of digit sequence that describes an element of $\mathbb{Z}_p$.

The difference is that the same sequence is interpreted in two different ways. In $\mathbb{Z}_p$, it gives $a_0 + a_1 p + a_2 p^2 + \cdots$. In $C^{(p)}$, we use the digits $2a_i$ in base $2p-1$. So the same sequence gives the real number $\sum_{i=0}^{\infty} \frac{2a_i}{(2p-1)^{i+1}}$. This is the idea behind the map in the next theorem.

Theorem 3.3. *$C^{(p)}$ is homeomorphic to $\mathbb{Z}_p$.*

Proof. Every element $x \in \mathbb{Z}_p$ has a unique p-adic expansion

$$x = a_0 + a_1 p + a_2 p^2 + \cdots,$$

where each $a_i \in \{0, 1, \ldots, p-1\}$. Define a map $\Phi : \mathbb{Z}_p \to C^{(p)}$ by

$$\Phi(x) = \sum_{i=0}^{\infty} \frac{2a_i}{(2p-1)^{i+1}}.$$

This is a real number in $[0, 1]$. Its base $2p-1$ expansion has digits

$$2a_0, 2a_1, 2a_2, \ldots.$$

Since each $a_i \in \{0, 1, \ldots, p-1\}$, each digit $2a_i$ belongs to

$$\{0, 2, 4, \ldots, 2p-2\}.$$

By the previous lemma, $\Phi(x) \in C^{(p)}$. So Φ is well-defined.

We next show that Φ is onto. Let $y \in C^{(p)}$. By the previous lemma, y has a base $2p-1$ expansion using only the digits $0, 2, \ldots, 2p-2$. Thus we can write

$$y = \sum_{i=0}^{\infty} \frac{2a_i}{(2p-1)^{i+1}},$$

where each $a_i \in \{0, 1, \ldots, p-1\}$. Then

$$x = \sum_{i=0}^{\infty} a_i p^i$$

defines an element of $\mathbb{Z}_p$, and by construction $\Phi(x) = y$. Therefore Φ is surjective.

We now show that Φ is injective. Suppose $x, x' \in \mathbb{Z}_p$, and write

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad x' = \sum_{i=0}^{\infty} a'_i p^i.$$

If $x \neq x'$, then there is a first index m such that $a_m \neq a'_m$. Then the base $2p-1$ expansions of $\Phi(x)$ and $\Phi(x')$ first differ in the digit $2a_m$ versus $2a'_m$. These digits differ by at least 2. The remaining tail of the expansion is too small to cancel this first difference, because the largest possible tail after the m-th digit is

$$\sum_{i=m+1}^{\infty} \frac{2p-2}{(2p-1)^{i+1}} = \frac{1}{(2p-1)^{m+1}}.$$

But the first differing digit contributes at least

$$\frac{2}{(2p-1)^{m+1}}.$$

So the two real numbers $\Phi(x)$ and $\Phi(x')$ cannot be equal. Hence Φ is injective.

Finally, we show that Φ is continuous. If two p-adic integers are congruent modulo p^N, then their first N p-adic digits agree. Their images under Φ therefore have the same first N base $2p-1$ digits. Hence their images lie in the same interval of length $(2p-1)^{-N}$. So as two points get close in $\mathbb{Z}_p$, their images get close in $C^{(p)}$. Thus Φ is continuous.

Since $\mathbb{Z}_p$ is compact and $C^{(p)} \subseteq [0,1]$ is Hausdorff, a continuous bijection $\Phi : \mathbb{Z}_p \to C^{(p)}$ is a homeomorphism. Therefore $C^{(p)}$ is homeomorphic to $\mathbb{Z}_p$. ■

This theorem shows that $\mathbb{Z}_p$ has the same topology as the generalized Cantor set $C^{(p)}$. The map is not supposed to preserve addition or multiplication. It only preserves the topology.

The reason this works is that closeness is based on the same kind of digit agreement in both spaces. In $\mathbb{Z}_p$, two numbers are close if many of their first p-adic digits agree. In $C^{(p)}$, two points are close if many of their first base $2p-1$ digits agree. So the digit structure gives the homeomorphism.

4 Cantor Spaces

This motivates looking at Cantor sets more generally. A space does not have to literally be the middle-thirds Cantor set inside $[0,1]$ in order to have the same topology. If it has the same topological structure, we call it a Cantor space.

Definition 4.1. A space is a Cantor space if it is homeomorphic to a Cantor Set.

The next theorem gives a way to recognize Cantor spaces. It says that the Cantor set is determined, up to homeomorphism, by a few topological properties. These properties are compactness, metrizability, total disconnectedness, and having no isolated points.

The converse direction is the main part. It says that any space with these properties can be split again and again into smaller clopen pieces, just like the Cantor set.

Theorem 4.2. *A topological space is a Cantor space if and only if it is nonempty, perfect, compact, totally disconnected, and metrizable.*

Proof. First suppose X is a Cantor space. Then X is homeomorphic to the Cantor set. So it is enough to check these properties for the Cantor set itself.

The Cantor set is nonempty because it contains points such as 0 and 1. It is compact because it is a closed subset of the compact interval $[0,1]$. It is metrizable because it inherits the usual metric from $\mathbb{R}$. It is perfect because every point of the Cantor set is approached by other points of the Cantor set. Indeed, every basic interval in the construction contains smaller intervals from later stages, so no point is isolated. Finally, it is totally disconnected because any two distinct points eventually lie in different intervals at some finite stage of the construction, and those intervals are separated by an open gap. Therefore the only connected subsets are single points.

Since all of these properties are preserved under homeomorphism, every Cantor space is nonempty, perfect, compact, totally disconnected, and metrizable.

Conversely, suppose X is nonempty, perfect, compact, totally disconnected, and metrizable. We want to show that X is homeomorphic to the Cantor set.

Because X is compact, metrizable, and totally disconnected, it has many clopen sets. In particular, one can separate points using clopen neighborhoods. Since X is also perfect, no nonempty open set consists of just one point. This lets us repeatedly split X into smaller nonempty clopen pieces.

We construct a binary tree of nonempty clopen sets. First split X into two disjoint nonempty clopen sets, called U_0 and U_1. Then split each U_i into two disjoint nonempty clopen sets U_{i0} and U_{i1}. Continuing this way, for each finite binary string s, we get a nonempty clopen set U_s, and

$$U_s = U_{s0} \cup U_{s1}, \qquad U_{s0} \cap U_{s1} = \varnothing.$$

We can also choose these sets so that their diameters go to 0 as the length of s goes to infinity.

Now let $(\epsilon_1, \epsilon_2, \epsilon_3, \dots)$ be an infinite binary sequence. This sequence determines a nested chain

$$U_{\epsilon_1} \supseteq U_{\epsilon_1\epsilon_2} \supseteq U_{\epsilon_1\epsilon_2\epsilon_3} \supseteq \cdots.$$

Since X is compact, the intersection of these nested nonempty closed sets is nonempty. Since their diameters go to 0, the intersection contains exactly one point. Thus every infinite binary sequence gives one point of X.

This defines a map from the space of infinite binary sequences $\{0,1\}^{\mathbb{N}}$ to X. The construction also works backward: every point of X lies in exactly one set at each level, so it determines a unique infinite binary sequence. Therefore this map is a bijection. It is continuous because agreeing on the first n binary digits means lying in the same clopen set at level n, and these sets shrink to points. Its inverse is continuous for the same reason.

Thus X is homeomorphic to $\{0,1\}^{\mathbb{N}}$. But $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the usual Cantor set by sending a binary sequence (ϵ_i) to the ternary number

$$\sum_{i=1}^{\infty} \frac{2\epsilon_i}{3^i}.$$

Therefore X is homeomorphic to the Cantor set. Hence X is a Cantor space. ■

So the Cantor set can be understood in two ways. First, it is a subset of $[0,1]$ made by repeatedly removing middle intervals. Second, it is a space of infinite digit sequences. The second view is what connects it to p-adic numbers.

For $C^{(p)}$, the allowed real digits are $0, 2, 4, \dots, 2p-2$. These correspond to the p possible digits of a p-adic integer. This gives a natural homeomorphism between $C^{(p)}$ and $\mathbb{Z}_p$.

Thus the p-adic integers can be viewed topologically as a Cantor-type space. This is the main connection between Cantor sets and p-adic analysis in this paper.

References

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