Monsky's Theorem

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Abstract

In this paper, I study Monsky's theorem, which states that a square cannot be divided into an odd number of triangles of equal area. I focus on the 2-adic coloring argument behind the theorem. I first define dissections and equidissections, then introduce the 2-adic valuation as a way to measure powers of two. I also prove the version of Sperner's lemma needed for the argument. Putting these pieces together gives a contradiction when the number of equal-area triangles is odd.

1 Introduction

Drawing one diagonal cuts a square into two triangles of equal area. From there, it is not hard to make more examples with an even number of triangles. For example, one can divide the square into smaller rectangles and cut each rectangle along a diagonal. The harder question is whether an odd number of equal-area triangles can ever work.

Monsky's theorem says that it cannot. Paul Monsky proved this result in 1970 in *The American Mathematical Monthly* [1]. The statement is simple, but the method uses more than elementary geometry.

Theorem 1 (Monsky's theorem)**.** *A square cannot be dissected into an odd number of triangles of equal area.*

Equivalently, if a square is divided into n triangles of equal area, then n must be even. In this paper, I will prove this theorem using a 2-adic coloring of the plane and Sperner's lemma. The main point is that the 2-adic valuation records divisibility by two rather than ordinary size. This lets us separate points in the plane into three colors based on the 2-adic sizes of their coordinates.

Once the vertices of a triangulated square are colored, Sperner's lemma produces a triangle with one vertex of each color. The area formula for this triangle then gives a 2-adic absolute value greater than 1. If the square were cut into n equal-area triangles with n odd, each triangle would have area $1/n$, whose 2-adic absolute value is 1. The contradiction rules out the odd case.

2 Dissections and equidissections

We begin with the geometric setup.

Definition 2. *A **dissection** of a square into triangles is a finite collection of triangles whose interiors do not overlap and whose union is the entire square.*

Definition 3. *An **equidissection** is a dissection in which all the triangles have the same area.*

It is enough to work with the unit square $[0,1] \times [0,1]$. Scaling a square changes every area by the same factor, so it does not affect whether the areas are equal or whether the number of triangles is odd. If the unit square is dissected into n equal-area triangles, then every triangle has area

$$\frac{1}{n}.$$

Thus the theorem becomes the claim that this is impossible when n is odd.

For simplicity, I will use the word dissection to mean an edge-to-edge triangulation of the square. In other words, when two triangles meet, they meet along a full edge, at a vertex, or not at all.

3 The 2-adic valuation

The number theory in this paper comes from the 2-adic valuation. More generally, a p-adic valuation measures divisibility by a fixed prime p [4]. Since Monsky's theorem is about ruling out odd numbers of triangles, the relevant prime is 2.

Definition 4. *For a nonzero rational number x, write*

$$x = 2^k \frac{a}{b},$$

*where a and b are odd integers. The integer k is called the **2-adic valuation** of x, and it is written $v_2(x)$.*

The associated absolute value is

$$|x|_2 = 2^{-v_2(x)}$$

for $x \neq 0$, and we define $|0|_2 = 0$.

Example 5. *For some basic examples,*

$$v_2(8) = 3, \quad v_2(6) = 1, \quad v_2(3) = 0, \quad v_2\left(\frac{1}{4}\right) = -2.$$

So

$$|8|_2 = \frac{1}{8}, \quad |6|_2 = \frac{1}{2}, \quad |3|_2 = 1, \quad \left|\frac{1}{4}\right|_2 = 4.$$

These examples show that 2-adic size is different from usual size. The number 8 is large in the usual sense, but it is small 2-adically because it has many factors of 2. The number 1/4 is small in the usual sense, but it is large 2-adically because the power of 2 appears in the denominator.

Two facts about $|\cdot|_2$ will be used later. First,

$$|xy|_2 = |x|_2|y|_2.$$

Second,

$$|x + y|_2 \le \max(|x|_2, |y|_2).$$

The second inequality is the ultrametric inequality. It is stronger than the usual triangle inequality and is one of the basic features of p-adic absolute values [4].

The parity information follows directly from the definition. If m is odd, then $v_2(m) = 0$, so

$$|m|_2 = 1 \quad \text{and} \quad \left|\frac{1}{m}\right|_2 = 1.$$

If m is even, then $v_2(m) > 0$, so $|m|_2 < 1$.

There is one technical issue to mention. The vertices of a dissection may have irrational coordinates, while the 2-adic absolute value was just defined on rational numbers. One uses an extension of the 2-adic absolute value to a field containing all the vertex coordinates. I will take this extension fact as given in the main proof, since developing that part of valuation theory would take us away from the main argument [2]. A brief note on this issue appears in the appendix.

4 Sperner's lemma

Next is Sperner's lemma. I will use a parity version for a triangulated polygon.

Definition 6. *Suppose the vertices of a triangulated polygon are colored with colors* 1, 2, *and* 3. *A small triangle is called* ***complete*** *if its three vertices have three different colors. An edge is called a* ***12-edge*** *if one endpoint has color* 1 *and the other endpoint has color* 2.

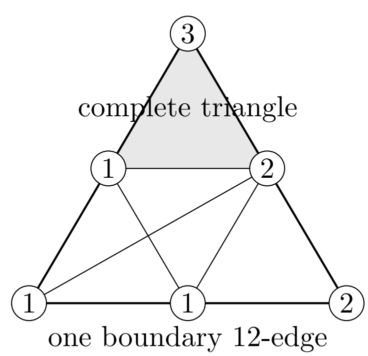

Figure 1: A small example of Sperner's lemma. The shaded triangle is complete because its vertices have colors 1, 2, and 3. The boundary also has one 12-edge, so both numbers are odd in this example.

This picture shows the kind of parity statement that Sperner's lemma is making. It does not predict exactly where the complete triangle has to be. It says that if the boundary has an odd number of 12-edges, then at least one complete triangle must appear somewhere in the triangulation.

Lemma 7 (Sperner's lemma, parity form)**.** *Let a polygon be triangulated, and color every vertex with one of the colors* 1, 2, *and* 3*. Then the number of complete triangles has the same parity as the number of 12-edges on the boundary of the polygon.*

Proof. We count certain triangle sides. Mark every side of every small triangle that is a 12-edge.

Counting by edges, an interior 12-edge is counted twice because it borders two small triangles. A boundary 12-edge is counted once because it borders only one small triangle. Therefore, modulo 2, the number of marked sides is the same as the number of boundary 12-edges.

Counting by triangles gives the same total in another way. A complete triangle has exactly one 12-edge. A triangle that is not complete has either zero 12-edges or two 12-edges. So, modulo 2, the number of marked sides is the same as the number of complete triangles.

The same marked sides were counted both ways. Therefore, the number of complete triangles has the same parity as the number of boundary 12-edges. □

In particular, if a triangulated polygon has an odd number of boundary 12-edges, then it contains at least one complete triangle. This is the form of Sperner's lemma used below.

5 The Monsky coloring

We now color the plane using the 2-adic absolute value. The three colors are called C_1, C_2, and C_3. A point (x, y) is assigned its color as follows:

$$C_1 = \{(x, y) : |x|_2 < 1 \text{ and } |y|_2 < 1\},$$

$$C_2 = \{(x, y) : |x|_2 \geq 1 \text{ and } |x|_2 \geq |y|_2\},$$

$$C_3 = \{(x, y) : |y|_2 \geq 1 \text{ and } |y|_2 > |x|_2\}.$$

These cases give every point exactly one color. If both coordinates are 2-adically small, the point is in C_1. Otherwise, at least one coordinate has 2-adic size at least 1. We then compare the two coordinates, putting ties into C_2.

For the four corners of the unit square, the colors are

$$(0, 0) \in C_1, \quad (1, 0) \in C_2, \quad (0, 1) \in C_3, \quad (1, 1) \in C_2.$$

The sides of the square also have restricted colors. On the bottom side, where $y = 0$, only C_1 and C_2 can appear. On the left side, where $x = 0$, only C_1 and C_3 can appear. On the top and right sides, only C_2 and C_3 can appear.

Now consider the bottom side. It begins at $(0, 0)$, which has color C_1, and ends at $(1, 0)$, which has color C_2. Since only colors C_1 and C_2 occur on this side, the color must switch from C_1 to C_2 an odd number of times. Equivalently, the boundary has an odd number of 12-edges. By Sperner's lemma, at least one triangle in the triangulation is complete.

At this stage, we know that some triangle has one vertex of each color. We now compute the 2-adic size of its area.

6 The area computation

The following lemma connects the coloring to the usual area formula for a triangle.

Lemma 8. *If a triangle is complete under the Monsky coloring, then its area A satisfies*

$$|A|_2 > 1.$$

Proof. Let the complete triangle have one vertex of each color. Suppose the C_1 vertex is (a, b). Since it has color C_1, we have

$$|a|_2 < 1 \quad \text{and} \quad |b|_2 < 1.$$

Translate the triangle by $(-a, -b)$. Translation does not change area. Also, the other two vertices keep their colors, because adding a number of 2-adic size less than 1 cannot change a coordinate whose 2-adic size is at least 1. This uses the ultrametric inequality.

After translating, the vertices may be written as

$$(0, 0), \quad (x_2, y_2), \quad (x_3, y_3),$$

where (x_2, y_2) has color C_2 and (x_3, y_3) has color C_3. The area is

$$A = \frac{x_2 y_3 - x_3 y_2}{2}.$$

From the definition of C_2,

$$|x_2|_2 \geq 1 \quad \text{and} \quad |x_2|_2 \geq |y_2|_2.$$

From the definition of C_3,

$$|y_3|_2 \geq 1 \quad \text{and} \quad |y_3|_2 > |x_3|_2.$$

Multiplying the corresponding inequalities gives

$$|x_2 y_3|_2 > |x_3 y_2|_2.$$

When two terms have different 2-adic absolute values, the ultrametric inequality implies that the larger term determines the size of the difference. Hence

$$|x_2 y_3 - x_3 y_2|_2 = |x_2 y_3|_2.$$

Therefore,

$$|A|_2 = \left| \frac{1}{2} \right|_2 |x_2 y_3 - x_3 y_2|_2 = 2|x_2 y_3|_2.$$

Since $|x_2|_2 \geq 1$ and $|y_3|_2 \geq 1$, we get

$$|A|_2 \geq 2 > 1.$$

This proves the lemma. $\square$

So the area of a complete triangle cannot have 2-adic absolute value 1.

7 Monsky's theorem

We now prove the theorem.

Proof. Assume, for contradiction, that the unit square has an equidissection into n triangles, where n is odd. Since the unit square has area 1, every triangle has area

$$\frac{1}{n}.$$

Apply the Monsky coloring to the vertices of the triangulation. From the boundary coloring and Sperner's lemma, there is a complete triangle. By the area lemma, the area A of this triangle satisfies

$$|A|_2 > 1.$$

On the other hand, the triangle is part of the equidissection, so its area is

$$A = \frac{1}{n}.$$

Since n is odd,

$$\left|\frac{1}{n}\right|_2 = 1.$$

This contradicts $|A|_2 > 1$. Therefore, the unit square cannot be dissected into an odd number of equal-area triangles. By scaling, the same is true for any square. $\square$

8 Conclusion

Monsky's theorem begins with a question about cutting up a square, but the main obstruction is arithmetic. The 2-adic valuation separates numbers according to how divisible they are by 2, and that information is built into a coloring of the plane. Sperner's lemma then gives a triangle with all three colors, and the area computation shows that this triangle has 2-adic area greater than 1.

If the square is cut into n equal pieces and n is odd, then each triangle has area $1/n$, which has 2-adic absolute value 1. This contradiction rules out the odd case. The result is an example of how a geometric question can depend on number theory in a fairly direct way.

A Note on extending the 2-adic valuation

The proof uses $|\cdot|_2$ on the vertex coordinates of the dissection. This is slightly technical because some of those coordinates may be irrational, while $|\cdot|_2$ was first defined only on $\mathbb{Q}$. To handle this, we work in a field containing all the vertex coordinates and use an extension of the 2-adic absolute value to that field [5]. The important point is that the extension still satisfies the same rules used in the proof:

$$|xy|_2 = |x|_2|y|_2 \quad \text{and} \quad |x+y|_2 \le \max(|x|_2, |y|_2).$$

References

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