PROFINITE FIBONACCI NUMBERS

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Abstract. This paper presents an exposition of the profinite integers and their expansion, which would be the profinite fibonacci numbers. We introduce the factorial number system and use it to construct the ring $\hat{\mathbb{Z}}$ of profinite integers. We then explain how a periodicity of the Fibonacci sequence, called the Pisano period, which allows us to assign a well-defined Fibonacci number F_s to every $s \in \hat{\mathbb{Z}}$. Lastly, we derive a power series expansion for F_s and classify all eleven fixed points of the map $s \mapsto F_s$ in $\hat{\mathbb{Z}}$.

Contents

1. The Factorial Number System and Profinite Integers

To understand profinite Fibonacci numbers, we first need to understand the ring $\hat{\mathbb{Z}}$ of profinite integers. We will construct it through using the factorial number system. Consider the following definition.

Definition 1.1. The factorial number system (or factoriadic) is a positional numeral system with place values $1!, 2!, 3!, 4!, \ldots$ Every positive integer n has a unique representation

$$n = c_k \cdot k! + c_{k-1} \cdot (k-1)! + \cdots + c_2 \cdot 2! + c_1 \cdot 1!,$$

where the digits satisfy $0 \leq c_i \leq i$ and can be also written as $n = (c_k \, c_{k-1} \cdots c_2 \, c_1)_!$.

Example 1.1. We can convert 41 into a factoriadic like the following:

$$41 = 1 \cdot 24 + 2 \cdot 6 + 2 \cdot 2 + 1 \cdot 1 = 1 \cdot 4! + 2 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$$

As a result, $41 = (1221)_!$

Now, notice how divisibility can be inferred through the form of factoriadic. To demonstrate, every term $c_j \cdot j!$ with $j \geq k$ is divisible by k, since $k \mid j!$ for all $j \geq k$. This means that $n \pmod k$ by the digits after c_k, which would be $\ldots, c_{k-1}$.

Now, notice that an ordinary integer has a finite factoriadic representation, which is to say only finitely many digits are nonzero. A profinite integer is what we get by allowing infinitely many nonzero digits.

Definition 1.2. A profinite integer is an infinite sequence of digits $(\dots c_4\, c_3\, c_2\, c_1)_!$ with $0 \leq c_i \leq i$ for all $i \geq 1$. The set of all profinite integers, equipped with addition and multiplication defined componentwise, where carrying is allowed, forms a commutative ring called the ring of profinite integers, denoted $\hat{\mathbb{Z}}$. Every ordinary integer is also included in $\hat{\mathbb{Z}}$ as its factoriadic representation simply has infinitely many leading zeros.

Now let us consider addition with carrying. If we add two profinite integers $(\ \dots c_3\, c_2\, c_1)_!$ and $(\dots d_3\, d_2\, d_1)_!$, we add digit by digit starting from $c_1 + d_1$. If the sum exceeds the allowed range for that position, we carry to the next position, exactly as in ordinary addition. Since there are infinitely many digits to absorb any carry, this always terminates.

There is an equivalent way to think about $\hat{\mathbb{Z}}$ that is useful for what comes later. An element of $\hat{\mathbb{Z}}$ can be viewed as a system of residues where a sequence $(a_1, a_2, a_3, \dots)$ where $a_n \in \mathbb{Z}/n!\mathbb{Z}$ and $a_{n+1} \equiv a_n \pmod{n!}$ for all n. The compatibility condition says the residues are consistent across all moduli. Therefore,

$$\hat{\mathbb{Z}} = \varprojlim_{n \geq 1} \mathbb{Z}/n!\mathbb{Z}.$$

We can then notice that given the sequence (c_i), we set $a_n = \sum_{i=1}^{n-1} c_i \cdot i!$, the integer formed by the first $n-1$ digits.

In addition, $\hat{\mathbb{Z}}$ decomposes as a product over all primes. Using the Chinese Remainder Theorem, we can notice that for any n we have

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{a_r}\mathbb{Z},$$

Taking an inverse limit over all n, this gives

$$\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ is the ring of p-adic integers. Therefore a profinite integer is simply a choice of p-adic integer for each prime p that are collected and assembled..

2. Pisano Periods

The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and the recurrence $F_n = F_{n-1} + F_{n-2}$:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \dots$$

To explore Profinite Fibonacci numbers, or express F_s for profinite integer s, we have to use the concept of Pisano Periods.

Consider the following: fix a positive integer m and consider the Fibonacci sequence modulo m. At each step, the next term is determined by the previous two, so the pair

$$(F_n \bmod m, F_{n+1} \bmod m)$$

determines the sequence as it progresses. Since there are only

$$m^2$$

possible pairs, as there are m possible residues available, by the pigeonhole principle some pair must repeat. As a result, once a pair repeats, the sequence modulo m is periodic from that point on. Moreover, since the pair $(F_0, F_1) = (0, 1)$ is the first pair, we can also run the recurrence backwards, which shows the sequence is periodic from the very start.

Definition 2.1. The Pisano period $\pi(m)$ is the smallest positive integer T such that $F_{n+T} \equiv F_n \pmod{m}$ for all $n \geq 0$.

Example 2.1. Let us compute the Fibonacci sequence modulo 3 or (F_n mod 3) like the following:

$$0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ \ldots$$

We can notice that the subsequence $0, 1, 1, 2, 0, 2, 2, 1$ repeats, and as a result, $\pi(3) = 8$. Notice that $F_n \pmod 3$ depends only on $n \pmod 8$ where the Pisano period for this case would be 8.

Example 2.2. Using the similar method of bashing, we can find several Pisano periods for the following moduli:

$$\pi(1) = 1, \quad \pi(2) = 3, \quad \pi(3) = 8, \quad \pi(4) = 6, \quad \pi(5) = 20, \quad \pi(6) = 24.$$

Notice how $F_n \pmod m$ depends only on $n \pmod{\pi(m)}$. Therefore, to know what the Fibonacci sequence does modulo m, we only need to know the modlo $\pi(m)$.

Now, a profinite integer $s \in \hat{\mathbb{Z}}$ is a compatible system of residues modulo all integers simultaneously. In particular, for any m, the element s has a well-defined residue modulo $\pi(m)$, which determines $F_s \pmod m$.

3. Extension of the Fibonacci Sequence to $\hat{\mathbb{Z}}$

Definition 3.1. For $s \in \hat{\mathbb{Z}}$, define the profinite Fibonacci number F_s by

$$F_s = \lim_{i \to \infty} F_{n_i},$$

where (n_i) is any sequence of positive integers converging to s in $\hat{\mathbb{Z}}$ and the limit is taken in $\hat{\mathbb{Z}}$.

For this definition to make sense, we need to check two things: that the limit exists, and that independently of which sequence we choose that would converge to s, the limit is consistent.

Proposition 3.2. *The limit in Definition 3.1 exists and is independent of the choice of sequence* (n_i).

Proof. Let (n_i) be a sequence of positive integers converging to s in $\hat{\mathbb{Z}}$. Let us have any positive integer k. We want to show that the sequence (F_{n_i} mod $k!$) eventually converges and the limit exists.

Since $n_i \to s$, we know that $n_i \equiv s \pmod{k!}$ for all large enough i. In particular, for all large i, n_i all lie in the same residue class modulo $k!$. Now, $\pi(k!)$ divides some multiple of $k!$, so the residue class of n_i modulo $\pi(k!)$ also converges. Since $F_{n_i} \pmod{k!}$ depends only on $n_i \pmod{\pi(k!)}$, we conclude that $F_{n_i} \pmod{k!}$ eventually converges, and therefore the limit eventually does exist.

Next, let us show that this is true, independent to whatever sequence n_i which we choose. Since k was arbitrary, for every $k \geq 1$ the sequence

$$\bigl(F_{n_i} \bmod k!\bigr)$$

is eventually constant. Hence, for every k, there exists N_k such that

$$i, j \geq N_k \quad \Longrightarrow \quad F_{n_i} \equiv F_{n_j} \pmod{k!}.$$

Therefore (F_{n_i}) is Cauchy in $\widehat{\mathbb{Z}}$. Since $\widehat{\mathbb{Z}}$ is complete, (F_{n_i}) converges. If (n_i') is another sequence converging to s, then $n_i \equiv n_i' \pmod{k!}$ for all large i, so both sequences produce the same eventual residue modulo $k!$. The limit is thus independent of the choice of approximating sequence. □

The fact that the map $s \mapsto F_s$ is continuous and defined on all of $\hat{\mathbb{Z}}$ lets us extend this identity. If an identity holds for all ordinary integers, and both sides are continuous functions of the indices, then the identity holds for all profinite integers too, since $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$.

Theorem 3.3 (Fibonacci identities in $\hat{\mathbb{Z}}$). *For all* $m, n \in \hat{\mathbb{Z}}$*:*

(i) $F_n = F_{n-1} + F_{n-2}$.

(ii) $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$.

(iii) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$,
where $(-1)^n$ *is defined to be* 1 *if the first digit* c_1 *of* n *is even, and* -1 *if* c_1 *is odd.*

Proof. Each identity is classical and holds for all integers m, n. We show that the identities extend to $\widehat{\mathbb{Z}}$.

(i) Let $n \in \widehat{\mathbb{Z}}$, and choose a sequence of integers (n_i) converging to n. By continuity of the Fibonacci function,

$$F_{n_i} \to F_n, \qquad F_{n_i-1} \to F_{n-1}, \qquad F_{n_i-2} \to F_{n-2}.$$

Since

$$F_{n_i} = F_{n_i-1} + F_{n_i-2}$$

for every i, c

$$F_n = F_{n-1} + F_{n-2}.$$

(ii) Let $m, n \in \widehat{\mathbb{Z}}$, and choose sequences of integers (m_i) and (n_i) converging to m and n, respectively. Since

$$F_{m_i+n_i} = F_{m_i}F_{n_i+1} + F_{m_i-1}F_{n_i}$$

for every i,

$$F_{m+n} = F_m F_{n+1} + F_{m-1}F_n.$$

(iii) Let $n \in \widehat{\mathbb{Z}}$, and choose a sequence of integers (n_i) converging to n. The classical identity

$$F_{n_i+1}F_{n_i-1} - F_{n_i}^2 = (-1)^{n_i}$$

holds for every i. Since the function $n \mapsto (-1)^n$ depends only on the parity of n, it is continuous on $\widehat{\mathbb{Z}}$. Passing to the limit gives

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

□

4. Power Series Expansion of F_s

In the classical setting, Binet's formula expresses F_n as

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. We cannot directly use this formula in $\hat{\mathbb{Z}}$ since φ is irrational. However, we can relate the Fibonacci sequence to a power series expansion through using p-adic logarithms.

For each prime $p \neq 5$, the number φ lies in $\mathbb{Z}_p$ (the p-adic integers), since its polynomial representation $x^2 - x - 1$ has a root that is p-adically close to an integer for $p \neq 5$. We can therefore define $\log_p \varphi$ using the p-adic logarithm series. Define

$$l = \frac{\log \varphi}{\varphi - \psi} \in \hat{\mathbb{Z}},$$

where the numerator is assembled from the p-adic logarithm $\log_p \varphi$ at each prime $p \neq 5$, and $\varphi - \psi = \sqrt{5}$.

Proposition 4.1. *The element* $l \in \hat{\mathbb{Z}}$ *is divisible by every prime* $p \neq 5$.

This divisibility is what makes the power series below converge in $\hat{\mathbb{Z}}$. For a fixed prime p, the p-adic valuation of l is positive, and the factorial denominators $(2i+1)!$ in the series grow fast enough to ensure that each term eventually contributes zero modulo any fixed power of p.

4.1. **The Lucas numbers.** The Lucas numbers are the sequence defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$:

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ \ldots$$

They satisfy the identity $L_n = F_{n-1} + F_{n+1}$ for all n, which we use to define profinite Lucas numbers. For $s \in \hat{\mathbb{Z}}$, set

$$L_s = F_{s-1} + F_{s+1}.$$

In particular, $L_0 = F_{-1} + F_1 = 1 + 1 = 2$ and $L_1 = F_0 + F_2 = 0 + 1 = 1$.

Theorem 4.2 (Power series expansion of F_s)**.** *For any $s_0 \in \hat{\mathbb{Z}}$, we have*

$$F_s = \sum_{i=0}^{\infty} \left[5^i\, l^{2i}\, F_{s_0}\, \frac{(s-s_0)^{2i}}{(2i)!} + 5^i\, l^{2i+1}\, L_{s_0}\, \frac{(s-s_0)^{2i+1}}{(2i+1)!} \right].$$

Setting $s_0 = 0$ and using $F_0 = 0$, $L_0 = 2$ gives the simplified form

$$F_s = \sum_{i=0}^{\infty} \frac{2 \cdot 5^i\, l^{2i+1}\, s^{2i+1}}{(2i+1)!}.$$

The series converges in $\hat{\mathbb{Z}}$: for each modulus m and each $k \geq 1$, all but finitely many terms are divisible by m^k.

Now we can notice why this sequence would converge. Fix a prime p and look at the p-adic valuation v_p of the i-th term, which would be the following:

$$\frac{2 \cdot 5^i l^{2i+1} s^{2i+1}}{(2i+1)!}$$

.

Since $v_p(l) \geq 1$ for $p \neq 5$, we have

$$v_p(l^{2i+1}) \geq 2i + 1$$

. Meanwhile, $v_p((2i+1)!) \leq \frac{2i+1}{p-1}$ by Legendre's formula. So the valuation of the i-th term grows approximately

$$(2i+1)\left(1 - \frac{1}{p-1}\right)$$

, which goes to infinity as $i \to \infty$. This shows the terms go to zero p-adically and the series converges.

Example 4.1. At $s_0 = 0$, the first few terms of the expansion are

$$F_s = 2ls + \frac{2 \cdot 5 \cdot l^3 s^3}{3!} + \frac{2 \cdot 5^2 \cdot l^5 s^5}{5!} + \cdots$$

At $s = 1$, in normal Fibonacci sequence, $F_1 = 1$. The series gives $F_1 = 2l + \frac{10l^3}{6} + \cdots$, which converges to 1 in $\hat{\mathbb{Z}}$ and we can definitely see that this works.

5. Fixed Points of the Fibonacci Map

A fixed point of the Fibonacci map is an element $s \in \hat{\mathbb{Z}}$ satisfying $F_s = s$. We want to find all such elements.

5.1. **Ordinary fixed points.** Among ordinary integers, we can find fixed points by direct computation. We have

$$F_0 = 0, \quad F_1 = 1, \quad F_5 = 5,$$

so 0, 1, and 5 are fixed points. Let us check that these are the only ones. The Fibonacci sequence grows roughly like $\varphi^n/\sqrt{5}$, so for large n the sequence grows much faster than n itself. A direct check of F_n for small n shows:

$$F_2 = 1 \neq 2, \quad F_3 = 2 \neq 3, \quad F_4 = 3 \neq 4, \quad F_6 = 8 \neq 6.$$

For $n \geq 7$ we have $F_n > n$, and for $n < 0$ one can similarly check there are no solutions. So the only integer fixed points are 0, 1, and 5.

5.2. **Profinite fixed points.** The Fibonacci map has additional fixed points in $\hat{\mathbb{Z}}$ that do not correspond to any ordinary integer. To find them, we use the decomposition $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ and solve $F_s = s$ one prime at a time.

The idea is to solve the fixed-point equation separately modulo powers of 2 and 3 (bundled as the condition modulo 6^k) and modulo powers of 5. The solutions to $F_s \equiv s \pmod{6^k}$ converge to elements $a \in \{1, 5\}$ as $k \to \infty$, and the solutions to $F_s \equiv s \pmod{5^k}$ converge to elements in $\{-5, -1, 0, 1, 5\}$.

Theorem 5.1. *There are exactly eleven solutions to* $F_s = s$ *in* $\widehat{\mathbb{Z}}$.

Three are the ordinary integers 0, 1, *and* 5. *The remaining eight are profinite fixed points* $z_{a,b}$, *where*

$$a \in \{1, 5\}, \qquad b \in \{-5, -1, 0, 1, 5\} \setminus \{a\},$$

uniquely determined by

$$z_{a,b} \equiv a \pmod{6^k}, \qquad z_{a,b} \equiv b \pmod{5^k}$$

for every $k \geq 1$.

Since there are $2 \cdot 5 = 10$ possible pairs (a, b), excluding $(1, 1)$ and $(5, 5)$ leaves eight nontrivial profinite fixed points. Together with 0, 1, and 5, this gives eleven fixed points in total.

Example 5.1. Let $z_{1,-1}$ be the fixed point satisfying

$$z_{1,-1} \equiv 1 \pmod{6^k}, \qquad z_{1,-1} \equiv -1 \pmod{5^k}$$

for all k.

Modulo 30, these conditions become

$$n \equiv 1 \pmod{6}, \qquad n \equiv 4 \pmod{5},$$

whose solution is $n \equiv 19 \pmod{30}$. Thus 19 is the first integer approximation to $z_{1,-1}$.

Example 5.2. Similarly, $z_{5,-1}$ satisfies

$$z_{5,-1} \equiv 5 \pmod{6^k}, \qquad z_{5,-1} \equiv -1 \pmod{5^k}$$

for all k.

Reducing modulo 30 gives

$$n \equiv 5 \pmod{6}, \qquad n \equiv 4 \pmod{5},$$

hence $n \equiv 29 \pmod{30}$. In this sense, $z_{5,-1}$ behaves like 5 modulo powers of 2 and 3, but like -1 modulo powers of 5.

As a result, for each prime p, we find the fixed points of the Fibonacci map modulo p and then use Hensel lifting.

6. Conclusion

We have seen that the Fibonacci sequence extends to a well-defined continuous map $F : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$. This was expanded through the Pisano period phenomenon, which guarantees that the map $n \mapsto F_n$ is uniformly continuous on $\mathbb{Z}$ with respect to the profinite topology. The extended map preserves all classical identities and also shows a power series expansion analogous to the hyperbolic sine formula.

Potential areas for more research could be about the periodic points of the Fibonacci map in $\hat{\mathbb{Z}}$, that is, elements s for which $F_s^{(k)} = s$ for some $k > 1$.

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