Volkenborn Integration, Mahler Interpolation, and the Morita Gamma Function

Aryan Dhoria and Isabelle Wu

1 Factorials under the p-adic limit

1.1 The p-adic limit of $n!$

Recall that for a natural number n, the factorial of n is defined as the product of all positive integers up to and including n:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1, \qquad 0! = 1.$$

n	n!
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

Table 1: Values of $n!$ for $n = 0, 1, \ldots, 10$.

$n!$ grows infamously quickly and diverges in the reals. However, the limiting behavior of $n!$ is different under the p-adic metric. Specifically, $n!$ converges to 0 as n increases. Intuitively, this is because as n increases the product $n!$ will include arbitrarily many multiples of p while not including any elements that may divide by p.

More rigorously, for each multiple of p in the product $n!$ the p-adic valuation of $n!$ increases by 1, and the number of multiples of p included in n can be counted by $\left\lfloor \frac{n}{p} \right\rfloor$. Each multiple of p^2 increases the p-adic valuation by 1 when not counting what has not already been counted by the multiples of p, and this adds $\left\lfloor \frac{n}{p^2} \right\rfloor$. Similarly, each multiple of p^m increases the p-adic valuation by 1 when not counting what has already been counted by the multiples of lower-order powers of p.

This adds $\left\lfloor \frac{n}{p^m} \right\rfloor$. Repeating this process for all powers of p with $p^m < n$ gives

$$v_p(n!) = \sum_{\substack{m \in \mathbb{Z} \\ p^m < n}} \left\lfloor \frac{n}{p^m} \right\rfloor .$$

This clearly diverges for increasing n. Therefore,

$$\lim_{n \to \infty} |n!|_p = \lim_{n \to \infty} p^{-v_p(n!)} = 0.$$

1.2 The Legendre–Digit Formula

Theorem 1. *Let p be a prime and n a positive integer. Then*

$$v_p(n!) = \frac{n - s_p(n)}{p-1}$$

where $s_p(n)$ is the sum of the digits of n written in base p.

Proof. We derive this from the formula

$$v_p(n!) = \sum_{\substack{m \in \mathbb{Z} \\ p^m < n}} \left\lfloor \frac{n}{p^m} \right\rfloor$$

discussed previously. Express n as its base p digit expansion

$$n = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p + a_0,$$

where $0 \le a_i < p$ for each i. Then

$$\begin{aligned} \left\lfloor \frac{n}{p} \right\rfloor &= a_k p^{k-1} + a_{k-1} p^{k-2} + \cdots + a_1, \\ \left\lfloor \frac{n}{p^2} \right\rfloor &= a_k p^{k-2} + a_{k-1} p^{k-3} + \cdots + a_2, \\ &\vdots \\ \left\lfloor \frac{n}{p^k} \right\rfloor &= a_k. \end{aligned}$$

Observe that each digit a_i appears in exactly i of the above expressions. When the above expressions are summed, each digit has coefficients that add to

$$p^{k-1} + p^{k-2} + \cdots + 1 = \frac{p^k - 1}{p-1}.$$

We thus have

$$v_p(n!) = \sum_{\substack{m \in \mathbb{Z} \\ p^m < n}} \left\lfloor \frac{n}{p^m} \right\rfloor = \sum_{m=1}^{k} a_m \frac{p^m - 1}{p-1}.$$

Factoring out $\dfrac{1}{p-1}$ we obtain

$$v_p(n!) = \frac{1}{p-1}\sum_{m=1}^{k}(a_m p^m - a_m) = \frac{1}{p-1}\left(\sum_{m=1}^{k} a_m p^m - \sum_{m=1}^{k} a_m\right).$$

By adding and subtracting the $m = 0$ term $a_0 = a_0 p^0$ in each sum, we extend both to $m = 0$, recovering the definitions of n and $s_p(n)$:

$$v_p(n!) = \frac{1}{p-1}\left(\sum_{m=0}^{k} a_m p^m - \sum_{m=0}^{k} a_m\right) = \frac{n - s_p(n)}{p-1}.$$

□

1.3 Examples: Radii of Convergence in $\mathbb{Q}_p$

The above formula can be applied in various p-adic ventures. For example, it can be used to find the radii of convergence in the following power series.

Example 2. *The radius of convergence of* $\sum_{n=0}^{\infty} n! x^n$.

Set $a_n = n!$. By the usual radius of convergence formula, $\dfrac{1}{R} = \limsup_{n\to\infty} |a_n|_p^{1/n}$. Using the digit-sum formula,

$$|n!|_p^{1/n} = p^{-v_p(n!)/n} = p^{-\frac{1}{p-1} + \frac{s_p(n)}{n(p-1)}}.$$

Since $s_p(n) = O(\log n)$ while $n \to \infty$, we have $s_p(n)/n \to 0$, so

$$\frac{1}{R} = p^{-\frac{1}{p-1}}, \qquad \text{i.e.} \quad R = p^{\frac{1}{p-1}}.$$

On the boundary $|x|_p = R$, the n-th term satisfies

$$|n!\, x^n|_p = p^{-v_p(n!) + \frac{n}{p-1}} = p^{\frac{s_p(n)}{p-1}} \geq 1,$$

so the terms do not tend to zero and the series *diverges* on the boundary. Therefore the series $\sum_{n=0}^{\infty} n!\, x^n$ converges precisely on

$$\left\{x \in \mathbb{Q}_p \;\middle|\; |x|_p < p^{\frac{1}{p-1}}\right\}.$$

Example 3. *Convergence of* $\sum_{n=0}^{\infty} \dfrac{p^n}{n!}$.

We determine for which primes p the p-adic exponential-type series $\sum_{n=0}^{\infty} \dfrac{p^n}{n!}$ converges.

The p-adic valuation of the general term is

$$v_p\left(\frac{p^n}{n!}\right) = n - v_p(n!).$$

Using the upper bound $v_p(n!) \leq \dfrac{n-1}{p-1}$ (which follows from the digit-sum formula since $s_p(n) \geq 1$ for $n \geq 1$), we obtain

$$v_p\left(\frac{p^n}{n!}\right) \geq n - \frac{n-1}{p-1}.$$

For $p \geq 3$ the coefficient of n in the bound, $1 - \dfrac{1}{p-1} = \dfrac{p-2}{p-1} > 0$, is strictly positive, so $v_p(p^n/n!) \to \infty$. Hence $|p^n/n!|_p \to 0$ and the series *converges*. Its limit is e^p, the p-adic exponential evaluated at p.

When $p = 2$ the bound gives

$$v_2\left(\frac{2^n}{n!}\right) \geq n - \frac{n-1}{2-1} = n - (n-1) = 1.$$

The lower bound is the constant 1, independent of n, so we cannot conclude that the terms tend to zero in $|\cdot|_2$. Indeed, the 2-adic norm of the general term satisfies $|2^n/n!|_2 \not\to 0$, and the series diverges in $\mathbb{Q}_2$.

1.4 The p-adic factorial

The p-adic factorial is extremely similar to the factorial defined on the naturals, however to make the values of the p-adic factorial invertible, we remove the multiples of p.

In formal notation,

$$n!_p = (-1)^n \prod_{\substack{0<i<n \\ p \nmid i}} i.$$

This will not be too important until section 5, when the Morita Gamma Function will be discussed.

2 The Gamma Function

2.1 Derivation of an Integral Representation of the Gamma Function

From section 1, it is observed that the factorial operator only allows natural arguments. However, we can find an analytic continuation whose domain includes all of $\mathbb{R}$ (and in fact $\mathbb{C}$). This continuation is called the gamma function $\Gamma(x)$.

For this derivation, recall the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

The derivation begins with some observations from integrating powers of $\ln(x)$:

$$\int \ln(x)\,dx \xrightarrow[dv=dx\to v=x]{u=\ln(x)\to du=\frac{1}{x}dx} \int \ln(x)\,dx = x\ln(x) - \int dx = x\ln(x) - 1\cdot x$$

$$\begin{aligned}\int \ln^2(x)\,dx \xrightarrow[dv=dx\to v=x]{u=\ln^2(x)\to du=2\ln(x)\frac{1}{x}dx} &\int \ln^2(x)\,dx \\ &= x\ln^2(x) - 2\int \ln(x)\,dx \\ &= x\ln^2(x) - 2(x\ln(x) - 1\cdot x) \\ &= x\ln^2(x) - 2x\ln(x) + 2\cdot 1\cdot x\end{aligned}$$

$$\begin{aligned}\int \ln^3(x)\,dx \xrightarrow[dv=dx\to v=x]{u=\ln^3(x)\to du=3\ln^2(x)\frac{1}{x}dx} &\int \ln^3(x)\,dx \\ &= x\ln^3(x) - 3\int \ln^2(x)\,dx \\ &= x\ln^3(x) - 3(x\ln^2(x) - 2x\ln(x) + 2\cdot 1\cdot x) \\ &= x\ln^3(x) - 3x\ln^2(x) + 6x\ln(x) - 3\cdot 2\cdot 1\cdot x.\end{aligned}$$

The last term in these results resembles the factorial. We now attempt to isolate it. Take our original easier expression as an example, and instead of taking an indefinite integral, integrate from 0 to 1:

$$\int_0^1 \ln(x)\,dx = (x\ln(x) - 1\cdot x)\Big|_0^1 = (1\cdot\ln(1) - 1\cdot 1) - (0\cdot\ln(0) - 1\cdot 0) = -1 - 0\cdot\ln(0).$$

The expression contains $\ln(0)$ which does not exist. To avoid this, take the limit $\lim_{x\to 0} x\ln(x)$:

$$\text{By L'Hôpital's rule, } \lim_{x\to 0} x\ln(x) = \lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0.$$

Thus

$$-1 - [0\cdot\ln(0)] = -1 - 0 = -1 = \int_0^1 \ln(x)\,dx.$$

Evaluating integrals 2 and 3 from 0 to 1 in the same process as before via L'Hôpital's yields

$$\int_0^1 \ln^2(x)\,dx = 2 - 0 = 2, \qquad \int_0^1 \ln^3(x)\,dx = -6 - 0 = -6.$$

What is happening here when the integrand is taken from 0 to 1 is that all the logarithmic terms cancel to 0 and all that remains is the last term. This last term is coincidentally the factorial of the exponent raised on the natural log. Take the cubic example:

$$\begin{aligned}\int_0^1 \ln^3(x)\,dx &= x\ln^3(x) - 3x\ln^2(x) + 6x\ln(x) - 3\cdot 2\cdot 1\cdot x\Big|_0^1 \\ &= \cancelto{0}{x\ln^3(x)} - \cancelto{0}{3x\ln^2(x)} + \cancelto{0}{6x\ln(x)} - 3\cdot 2\cdot 1\cdot x \\ &= -6 - 0 = -6.\end{aligned}$$

We have now concluded that $\int_0^1 \ln^n(x)\,dx = (-1)^n n!$. One can prove by induction that this is true for all natural numbers n. To yield the factorial $n!$ rather than an alternating signed factorial, we solve for $n!$ as follows:

$$\begin{aligned}
n! &= \frac{1}{(-1)^n}\int_0^1 \ln^n(x)\,dx \\
&= \int_0^1 \frac{\ln^n(x)}{(-1)^n}\,dx \\
&= \int_0^1 \left(\frac{\ln(x)}{-1}\right)^n\,dx \\
&= \int_0^1 (-\ln(x))^n\,dx \\
&= \int_0^1 \ln^n(x^{-1})\,dx \\
&= \int_0^1 \ln^n\left(\frac{1}{x}\right)\,dx.
\end{aligned}$$

Now we arrive at the conclusion that $\int_0^1 \ln^n\left(\frac{1}{x}\right)\,dx = n!$. Now by letting $u = \ln\left(\frac{1}{x}\right)$, we have

$$-u = \ln(x) \implies x = e^{-u} \implies dx = -e^{-u}du \implies \int_0^1 \ln^n\left(\frac{1}{x}\right)dx = -\int_\infty^0 u^n e^{-u}du.$$

In more traditional variables, that is $\int_0^\infty x^n \cdot e^{-x}\,dx$. The Gamma Function, purely by convention, is actually not this integral but rather a shifted version of this.

Definition 4 (Gamma Function)**.** $\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x}\,dx.$

So $(n-1)! = \Gamma(n)$.

2.2 Euler Limit Definition and the Gamma Reflection Formula

Another way to define the Gamma function is by exploiting the behavior of the logarithmic factorial through limits . Through this form of the Gamma function we can derive the Gamma Reflection Formula. The derivation begins with the identity $(x+n)! = x!\prod_{k=1}^{n}(x+k)$. Consider $L(x) = \ln(x!)$. Then

$$L(x+n) = L(x) + \sum_{k=1}^{n}\ln(x+k).$$

Because

$$\lim_{N\to\infty}\ln(N+k) - \ln(N) = \lim_{N\to\infty}\ln\left(\frac{N+k}{N}\right) = \lim_{N\to\infty}\ln\left(1+\frac{k}{N}\right) = \ln(1) = 0$$

for arbitrarily big N, $\ln(N+k) \approx \ln(N)$. Therefore for some big N,

$$L(N+n) \approx L(N) + \sum_{k=1}^{n}\ln(N) = L(N) + n\ln(N).$$

Here, n does not need to be a natural number, so let us replace it with x.

$$L(N+x) \approx L(N) + x\ln(N).$$

Substituting in $L(x) + \sum_{k=1}^{N} \ln(x+k)$ for $L(N+x)$, we obtain

$$L(x) + \sum_{k=1}^{N} \ln(x+k) \approx \sum_{k=1}^{N} \ln(k) + x\ln(N)$$

for relatively large N. Taking the limit, we have

$$L(x) = \lim_{N\to\infty} \sum_{k=1}^{N} \ln\left(\frac{k}{x+k}\right) + x\ln(N).$$

Exponentiating gives

$$e^{L(x)} = x! = \lim_{N\to\infty} N^x \prod_{k=1}^{N} \frac{k}{x+k} = \lim_{N\to\infty} \frac{N^x \cdot N!}{(x+1)(x+2)(x+3)\cdots(x+N)}.$$

Since $(x-1)! = \Gamma(x)$, we have

$$\Gamma(x) = \lim_{N\to\infty} N^{x-1} \prod_{k=1}^{N} \frac{k}{x+k-1}.$$

Now having this limit form of $\Gamma(x)$, we can use it to prove an important identity. Observe that $\sin(\pi x)$ has roots at every integer. Since $\Gamma(x)$ is undefined at every negative integer, $\frac{1}{\Gamma(x)\Gamma(1-x)}$ should exhibit the same root behavior as $\sin(\pi x)$.

This is the same as $\frac{x}{x\Gamma(x)\Gamma(1-x)} = \frac{x}{\Gamma(1+x)\Gamma(1-x)}$.

Plugging in the limit definition of $\Gamma(x)$ into that equation yields

$$\begin{aligned}
&\frac{x}{\lim_{N\to\infty}\left(N^{1+x-1}\prod_{k=1}^{N}\frac{k}{k+1+x-1}\right)\left(N^{1-x-1}\prod_{k=1}^{N}\frac{k}{k+1-x-1}\right)} \\
&= \frac{x}{\lim_{N\to\infty}\prod_{k=1}^{N}\left(\frac{k}{k+x}\right)\left(\frac{k}{k-x}\right)} \\
&= x\prod_{k=1}^{\infty}\left(\frac{k+x}{k}\right)\left(\frac{k-x}{k}\right) \\
&= x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right) \\
&= \frac{\sin(\pi x)}{\pi}
\end{aligned}$$

by the sine product formula $\sin(x) = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)$. Finally, we reach Euler's reflection formula

Theorem 5 (Euler's reflection formula)**.** $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$.

Using Euler's reflection formula, plugging in $x = \frac{1}{2}$ reveals the following result:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi.$$

Therefore $\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}! = \sqrt{\pi}$.

2.3 Gauss Multiplication Theorem

One useful formula we will need to prove an important family of integrals related to the Gamma function is the Gauss Multiplication Formula.

Theorem 6 (Multiplication Theorem)**.** $\displaystyle\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right) = (2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz).$

Proof. We begin by using the Gamma difference equation $\Gamma(z+1) = z\Gamma(z)$. Specifically, we use the identity:

$$\Gamma\left(z+\frac{k}{n}\right) = \left(z+\frac{k}{n}-1\right)\Gamma\left(z+\frac{k}{n}-1\right).$$

Using the Euler form of the Gamma function,

$$\Gamma(z) = \lim_{m\to\infty}\frac{m!m^z}{z(z+1)\dots(z+m)},$$

we express the term as:

$$= \lim_{m\to\infty}\left(z+\frac{k}{n}-1\right)\frac{m!m^{z+\frac{k}{n}-1}}{(z+\frac{k}{n}-1)(z+\frac{k}{n})(z+\frac{k}{n}+1)\cdots(z+\frac{k}{n}-1+m)}.$$

Applying Stirling's formula and cancelling terms, we have:

$$= \lim_{m\to\infty}\frac{\sqrt{2\pi m}(\frac{m}{e})^m m^{z+\frac{k}{n}-1}}{(z+\frac{k}{n})(z+\frac{k}{n}+1)\cdots(z+\frac{k}{n}-1+m)}.$$

Multiplying the numerator and denominator by n^m:

$$\begin{aligned} &= \lim_{m\to\infty}\frac{\sqrt{2\pi m}(\frac{m}{e})^m m^{z+\frac{k}{n}-1}}{(z+\frac{k}{n})(z+\frac{k}{n}+1)\cdots(z+\frac{k}{n}-1+m)} \times \frac{n^m}{n^m} \\ &= \lim_{m\to\infty}\frac{\sqrt{2\pi}(\frac{mn}{e})^m m^{z+\frac{k}{n}-1/2}}{(nz+k)(nz+k+n)(nz+k+2n)\cdots(nz+k-n+mn)}. \end{aligned}$$

Taking the product for $k = 0$ to $n - 1$:

$$\begin{aligned}
\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) &= \lim_{m\to\infty} \frac{(\sqrt{2\pi})^n (\frac{mn}{e})^{mn} m^{nz-n/2} m^{\sum_{k=0}^{n-1} k/n}}{(nz)(nz+1)\cdots(nz-1+mn)} \\
&= \lim_{m\to\infty} \frac{(\sqrt{2\pi})^n (\frac{mn}{e})^{mn} m^{nz-n/2} m^{n/2-1/2}}{(nz)(nz+1)\cdots(nz-1+mn)} \\
&= \lim_{m\to\infty} \frac{(\sqrt{2\pi})^n (\frac{mn}{e})^{mn} m^{nz-1/2} n^{1/2-nz}}{(nz)(nz+1)\cdots(nz-1+m)} \\
&= \lim_{m\to\infty} \frac{(\sqrt{2\pi})^{n-1} m! m^{nz-1} n^{1/2-nz}}{(nz)(nz+1)\cdots(nz-1+m)} \\
&= (2\pi)^{(n-1)/2} n^{1/2-nz} (nz-1) \lim_{m\to\infty} \frac{m! m^{nz-1}}{(nz-1)(nz)(nz+1)\cdots(nz-1+m)} \\
&= (2\pi)^{(n-1)/2} n^{1/2-nz} (nz-1)\Gamma(nz-1) \\
&= (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz).
\end{aligned}$$

□

2.4 Raabe's Formula

One of the most significant integral formulae for the Gamma function is Raabe's Formula:

Theorem 7 (Raabe's Formula)**.** $\displaystyle\int_a^{a+1} \ln\Gamma(z)\,dz = \frac{1}{2}\ln(2\pi) + a\ln a - a.$

Proof. The definite integral $\int_a^{a+1} \ln\Gamma(z)\,dz$ can be defined as the limit of the Riemann sum (with step size $\frac{1}{n}$) as $n \to \infty$:

$$\int_a^{a+1} \ln\Gamma(z)\,dz = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\Gamma\left(a + \frac{k}{n}\right).$$

Using standard logarithmic properties, we convert the sum of logs into the log of a product:

$$\sum_{k=0}^{n-1} \ln\Gamma\left(a + \frac{k}{n}\right) = \ln\left[\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right)\right].$$

Now apply Gauss's Multiplication Theorem with $z = a$:

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-na}\Gamma(na).$$

Insert this product back into the limit formulation:

$$\int_a^{a+1} \ln\Gamma(z)\,dz = \lim_{n\to\infty} \frac{1}{n} \ln\left[(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-na}\Gamma(na)\right].$$

Using properties of logarithms we break apart the natural log term:

$$\ln\left[(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-na}\Gamma(na)\right] = \frac{n-1}{2}\ln(2\pi) + \left(\frac{1}{2} - na\right)\ln(n) + \ln\Gamma(na).$$

Dividing the entire expression by n:

$$\frac{n-1}{2n}\ln(2\pi) + \left(\frac{1}{2n} - a\right)\ln(n) + \frac{1}{n}\ln\Gamma(na).$$

Now utilize Stirling's Approximation to evaluate the limit as $n \to \infty$. Substitute $x = na$:

$$\ln\Gamma(na) \approx (na)\ln(na) - (na) - \frac{1}{2}\ln\left(\frac{na}{2\pi}\right)$$

$$\ln\Gamma(na) \approx na\ln(n) + na\ln(a) - na - \frac{1}{2}\ln(n) - \frac{1}{2}\ln(a) + \frac{1}{2}\ln(2\pi).$$

Substitute this expansion back into the Riemann sum limit equation:

$$\lim_{n\to\infty}\left[\frac{n-1}{2n}\ln(2\pi) + \left(\frac{1}{2n} - a\right)\ln(n) + \frac{1}{n}\left(na\ln(n) + na\ln(a) - na - \frac{1}{2}\ln(n) - \frac{1}{2}\ln(a) + \frac{1}{2}\ln(2\pi)\right)\right].$$

Expanding, grouping, and taking the limit of all the terms, the only surviving terms are:

$$\int_a^{a+1}\ln\Gamma(z)\,dz = \frac{1}{2}\ln(2\pi) + a\ln a - a.$$

□

3 p-adic integration

We have just discussed some properties of the Gamma function in $\mathbb{R}$, many of which revolve around integration. However, we have not yet discussed any p-adic analogues of integration, which surely must be useful when discussing analogous properties of the p-adic Gamma function. In this section we will define and briefly discuss one such p-adic integral.

3.1 Definition of the Volkenborn Integral

Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be a p-adic function, K be a subset of $\mathbb{Z}_p$, and $R_n = \left\{ x = \sum_{i=r}^{n-1} b_i p^i \,\middle|\, b_i = 0, \ldots, p-1 \text{ for } r < n \right\}$.

Definition 8 (General Volkenborn Integral). $\displaystyle\int_K f(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{p^n}\sum_{x\in R_n\cap K} f(x).$

Here R_n is essentially $\mathbb{Z}_p$ written in standard base-p notation but with all but the first n digits cut off. Thus $\lim\limits_{n\to\infty} R_n = \mathbb{Z}_p$. The most important form of the Volkenborn Integral is the special case when $K = \mathbb{Z}_p$, i.e. integrating over the entire ring. In this case, $R_n \cap K$ is simply R_n, or the p-adic integers up to $p^n - 1$.

Definition 9 (Volkenborn Integral). $\displaystyle\int_{Z_p} f(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1} f(x).$

Just like other methods of integration, the Volkenborn integral is a method of summing over an infinitely dense set. The real integral we know well classically does this by finding the limit of closer and closer discrete approximations. The Volkenborn integral, by contrast, iterates through more and more digits of each p-adic integer, gradually "revealing" the entire set as many different integers split off from the same residue classes mod p^n. Observe what happens as we increase n for some $K \subseteq \mathbb{Z}_5$ including the elements $= \{\dots 3412, \dots 4212, \dots 3431\}$. For $n = 0$, our expression is simply $f(2) + f(1)$. For $n = 1$, our expression is $\frac{1}{5}(f(12) + f(31))$. For $n = 2$, we see that the $\dots 12$ part splits into 412 and 212, and our expression is $\frac{1}{5^2}(f(412) + f(212) + f(431))$. As n tends to infinity, we reveal the entire set K.

3.2 Volkenborn Convergence

Since the Volkenborn integral relies on a limit, it is useful to determine when it converges.

Theorem 10. *Let*

$$f : \mathbb{Z}_p \to \mathbb{C}_p$$

be uniformly differentiable. Then the Volkenborn integral

$$\int_{\mathbb{Z}_p} f(x)\, dx := \lim_{n\to\infty} \frac{1}{p^n} \sum_{a=0}^{p^n-1} f(a)$$

exists.

Proof. For each $n \geq 0$, define

$$S_n = \frac{1}{p^n} \sum_{a=0}^{p^n-1} f(a).$$

We shall prove that the sequence (S_n) is Cauchy.

Since f is uniformly differentiable, there exists a function $E(x, h)$ denoting the error such that

$$f(x+h) = f(x) + hf'(x) + hE(x, h),$$

and

$$E(x, h) \to 0 \text{ as } h \to 0$$

uniformly in $x \in \mathbb{Z}_p$. Consider the difference

$$S_{n+1} - S_n.$$

Writing every integer $0 \leq a < p^{n+1}$ uniquely as

$$a = j + ip^n \text{ for } 0 \leq j < p^n, 0 \leq i < p$$

we obtain

$$S_{n+1} = \frac{1}{p^{n+1}} \sum_{j=0}^{p^n-1} \sum_{i=0}^{p-1} f(j + ip^n).$$

Hence

$$S_{n+1} - S_n = \frac{1}{p^{n+1}} \sum_{j=0}^{p^n-1} \sum_{i=0}^{p-1} \big(f(j + ip^n) - f(j)\big).$$

Using the differentiability expansion with $h = ip^n$ gives

$$f(j + ip^n) - f(j) = ip^n f'(j) + ip^n E(j, ip^n).$$

Substituting this into the previous expression yields

$$S_{n+1} - S_n = \frac{1}{p^{n+1}} \sum_{j=0}^{p^n-1} \sum_{i=0}^{p-1} ip^n f'(j) + R_n,$$

where

$$R_n = \frac{1}{p^{n+1}} \sum_{j=0}^{p^n-1} \sum_{i=0}^{p-1} ip^n E(j, ip^n).$$

Since $E(j, ip^n) \to 0$ uniformly as $n \to \infty$, it follows that $R_n \to 0$.

Note that

$$\sum_{i=0}^{p-1} i = \frac{p(p-1)}{2},$$

so

$$\begin{aligned}
\frac{1}{p^{n+1}} \sum_{j=0}^{p^n-1} \sum_{i=0}^{p-1} ip^n f'(j) &= \frac{p^n}{p^{n+1}} \left(\sum_{i=0}^{p-1} i \right) \sum_{j=0}^{p^n-1} f'(j) \\
&= \frac{1}{p} \cdot \frac{p(p-1)}{2} \sum_{j=0}^{p^n-1} f'(j) \\
&= \frac{p-1}{2} \sum_{j=0}^{p^n-1} f'(j).
\end{aligned}$$

Therefore

$$S_{n+1} - S_n = \frac{p-1}{2} \sum_{j=0}^{p^n-1} f'(j) + R_n.$$

It remains to be shown that

$$\sum_{j=0}^{p^n-1} f'(j) \to 0.$$

Since f' is continuous on the compact space $\mathbb{Z}_p$, it is uniformly continuous. This means that for any $\varepsilon > 0$, there exists $m \geq 0$ such that

$$x \equiv y \pmod{p^m} \implies |f'(x) - f'(y)|_p < \varepsilon.$$

For $n \geq m$, write

$$j = r + kp^m \text{ for } 0 \leq r < p^m, 0 \leq k < p^{n-m}.$$

Then

$$\sum_{j=0}^{p^n-1} f'(j) = \sum_{r=0}^{p^m-1} \sum_{k=0}^{p^{n-m}-1} f'(r + kp^m).$$

By uniform continuity,

$$f'(r + kp^m) = f'(r) + \delta_{r,k}, \qquad |\delta_{r,k}|_p < \varepsilon.$$

Consequently,

$$\sum_{k=0}^{p^{n-m}-1} f'(r+kp^m) = p^{n-m}f'(r) + \sum_{k=0}^{p^{n-m}-1} \delta_{r,k}.$$

Since

$$|p^{n-m}|_p = p^{-(n-m)} \longrightarrow 0,$$

the first term tends to 0 as $n \to \infty$. The second term is bounded in p-adic norm by ε. Since ε is arbitrary, it follows that

$$\sum_{j=0}^{p^n-1} f'(j) \to 0.$$

Hence

$$S_{n+1} - S_n \to 0.$$

Since the p-adic norm is non-Archimedean, it follows that (S_n) is a Cauchy sequence. Therefore the Volkenborn integral

$$\lim_{n\to\infty} \frac{1}{p^n} \sum_{a=0}^{p^n-1} f(a)$$

exists. □

3.3 Volkenborn Integrals of odd $f(x)$

Theorem 11. *In Volkenborn Integration,*

$$\int_{\mathbb{Z}_p} f(x)\,\mathrm{d}x = -\frac{f'(0)}{2}$$

for all odd differentiable functions $f(x)$.

To prove this, one must first prove the shift and reflection properties.

3.3.1 Shift property

Because the Volkenborn integral is defined by the p-adic limit:

$$\int_{\mathbb{Z}_p} f(x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x),$$

we can express the difference of the shifted and original integrals as:

$$\int_{\mathbb{Z}_p} f(x+1)\,dx - \int_{\mathbb{Z}_p} f(x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \left(\sum_{x=0}^{p^n-1} f(x+1) - \sum_{x=0}^{p^n-1} f(x) \right).$$

This sum is telescoping, leaving only $f(p^n)$ and $f(0)$. Simplifying the expression leaves the limit:

$$\lim_{n\to\infty} \frac{f(p^n) - f(0)}{p^n},$$

which is the value of the p-adic derivative at 0 provided it exists.

Therefore we have proved the following:

Lemma 12 (Shift Property)**.** $\int_{\mathbb{Z}_p} f(x+1)\,dx - \int_{\mathbb{Z}_p} f(x)\,dx = f'(0).$

3.3.2 Reflection property

As for the reflection property, the proof is similarly straightforward. We begin with the Volkenborn integral of the reflected function:

$$\int_{\mathbb{Z}_p} f(-x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(-x).$$

Let $x = p^n - 1 - y$. As x ranges from 0 to $p^n - 1$, the variable y ranges from $p^n - 1$ down to 0, so reindexing the sum gives

$$\int_{\mathbb{Z}_p} f(-x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{y=0}^{p^n-1} f(y+1-p^n).$$

Since f is strictly differentiable (C^1), we may apply a first-order Taylor expansion about $y+1$:

$$f(y+1-p^n) = f(y+1) - p^n f'(y+1) + p^n \varepsilon(y,n),$$

where

$$\lim_{n\to\infty} \max_{0\le y\le p^n-1} |\varepsilon(y,n)|_p = 0.$$

Substituting this expansion into the sum yields

$$\int_{\mathbb{Z}_p} f(-x)\,dx = \lim_{n\to\infty} \left(\frac{1}{p^n} \sum_{y=0}^{p^n-1} f(y+1) - \sum_{y=0}^{p^n-1} f'(y+1) + \sum_{y=0}^{p^n-1} \varepsilon(y,n) \right).$$

For the first term,

$$\lim_{n\to\infty} \frac{1}{p^n} \sum_{y=0}^{p^n-1} f(y+1) = \int_{\mathbb{Z}_p} f(x+1)\,dx.$$

For the second term, we have

$$\sum_{y=0}^{p^n-1} f'(y+1) = p^n \left(\int_{\mathbb{Z}_p} f'(x)\,dx + o(1) \right).$$

Since $p^n \to 0$ in the p-adic norm, this term tends to 0.

For the third term, the ultrametric inequality gives

$$\left| \sum_{y=0}^{p^n-1} \varepsilon(y,n) \right|_p \le \max_{0\le y\le p^n-1} |\varepsilon(y,n)|_p,$$

and the right-hand side tends to 0 as $n \to \infty$. Hence this term also vanishes.

Therefore only the first term remains, and we obtain

Lemma 13 (Reflection Property). $\displaystyle\int_{\mathbb{Z}_p} f(-x)\,dx = \int_{\mathbb{Z}_p} f(x+1)\,dx.$

3.3.3 Combining shift and reflection for odd functions

Now, we are ready to prove the property for odd functions $\int_{\mathbb{Z}_p} f(x)\,\mathrm{d}x = -\frac{f'(0)}{2}$.

Proof. Using the Reflection Property:

$$\int_{\mathbb{Z}_p} f(-x)\,dx = \int_{\mathbb{Z}_p} f(x+1)\,dx.$$

Since f is odd, $f(-x) = -f(x)$, so the left-hand side equals $-\int_{\mathbb{Z}_p} f(x)\,dx$. Thus

$$\int_{\mathbb{Z}_p} f(x+1)\,dx = -\int_{\mathbb{Z}_p} f(x)\,dx.$$

Therefore

$$\int_{\mathbb{Z}_p} f(x+1)\,dx + \int_{\mathbb{Z}_p} f(x)\,dx = 0.$$

Now we set up a system of linear equations using our Shift Property:

$$\begin{cases} \int_{\mathbb{Z}_p} f(x+1)dx - \int_{\mathbb{Z}_p} f(x)dx = f'(0) & (1) \\ \int_{\mathbb{Z}_p} f(x+1)dx + \int_{\mathbb{Z}_p} f(x)dx = 0 & (2) \end{cases}$$

Subtracting equation (2) from equation (1) isolates our target integral:

$$2\int_{\mathbb{Z}_p} f(x)\,dx = -f'(0).$$

And therefore:

$$\int_{\mathbb{Z}_p} f(x)\,dx = -\frac{f'(0)}{2}.$$

□

3.4 Volkenborn integrals of integer powers of x

The Volkenborn integral of $f(x) = x^m$ for $m \in \mathbb{Z}_{>0}$ has known values. To evaluate these integrals, we must use Faulhaber's formula for the sum of the first n mth powers of x. We do this by considering the generating function of the Bernoulli numbers.

Theorem 14 (Faulhaber's Formula). *Let $m, n \in \mathbb{N}$ and let $\{B_k\}_{k\geq 0}$ be the Bernoulli numbers defined by*

$$\frac{xe^x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Then

$$\sum_{r=1}^{n} r^m = \frac{1}{m+1}\sum_{k=0}^{m} B_k \binom{m+1}{k} n^{m-k+1}.$$

Proof. Consider the exponential generating function

$$F(x) = \sum_{m=0}^{\infty} \left(\sum_{r=1}^{n} r^m \right) \frac{x^m}{m!}.$$

By absolute convergence of the exponential series, we may interchange the finite and infinite sums:

$$F(x) = \sum_{r=1}^{n} \sum_{m=0}^{\infty} \frac{(rx)^m}{m!}.$$

Since $e^{rx} = \sum_{m=0}^{\infty} \frac{(rx)^m}{m!}$, it follows that

$$F(x) = \sum_{r=1}^{n} e^{rx}.$$

The right-hand side is a finite geometric series with first term e^x and common ratio e^x, hence

$$F(x) = e^x \frac{1 - e^{nx}}{1 - e^x} = \frac{e^x(e^{nx} - 1)}{e^x - 1}.$$

Multiplying and dividing by x gives

$$F(x) = \frac{xe^x}{e^x - 1} \cdot \frac{e^{nx} - 1}{x}.$$

Using the generating function of the Bernoulli numbers,

$$\frac{xe^x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

and the exponential series,

$$e^{nx} - 1 = \sum_{j=1}^{\infty} \frac{(nx)^j}{j!},$$

we obtain

$$\frac{e^{nx} - 1}{x} = \sum_{j=0}^{\infty} \frac{n^{j+1}}{(j+1)!} x^j.$$

Therefore

$$F(x) = \left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{n^{j+1}}{(j+1)!} x^j \right).$$

We now apply the Cauchy product formula:

$$\left(\sum_{k=0}^{\infty} a_k x^k \right) \left(\sum_{j=0}^{\infty} b_j x^j \right) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} a_k b_{m-k} \right) x^m.$$

Thus

$$F(x) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{B_k}{k!} \frac{n^{m-k+1}}{(m-k+1)!} \right) x^m.$$

Multiplying and dividing the coefficient by $(m+1)!$ yields

$$F(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \left[\frac{1}{m+1} \sum_{k=0}^{m} B_k \frac{(m+1)!}{k!(m-k+1)!} n^{m-k+1} \right].$$

Since $\frac{(m+1)!}{k!(m-k+1)!} = \binom{m+1}{k}$, we obtain

$$F(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \left[\frac{1}{m+1} \sum_{k=0}^{m} B_k \binom{m+1}{k} n^{m-k+1} \right].$$

But by definition,

$$F(x) = \sum_{m=0}^{\infty} \left(\sum_{r=1}^{n} r^m \right) \frac{x^m}{m!}.$$

Equality of exponential generating functions implies equality of coefficients. Hence

$$\sum_{r=1}^{n} r^m = \frac{1}{m+1} \sum_{k=0}^{m} B_k \binom{m+1}{k} n^{m-k+1},$$

which proves Faulhaber's formula. □

By definition of the Volkenborn integral,

$$\int_{\mathbb{Z}_p} x^m \, dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} x^m.$$

Applying Faulhaber's formula with $n = p^N$ gives

$$\sum_{x=0}^{p^N-1} x^m = \frac{1}{m+1} \sum_{k=0}^{m} B_k \binom{m+1}{k} (p^N)^{m-k+1}.$$

Therefore

$$\frac{1}{p^N} \sum_{x=0}^{p^N-1} x^m = \frac{1}{m+1} \sum_{k=0}^{m} B_k \binom{m+1}{k} p^{N(m-k)}.$$

Taking the limit $N \to \infty$ yields

$$\int_{\mathbb{Z}_p} x^m \, dx = \frac{1}{m+1} \lim_{N\to\infty} \sum_{k=0}^{m} B_k \binom{m+1}{k} p^{N(m-k)}.$$

For every $k < m$ we have $m - k > 0$, and therefore

$$|p^{N(m-k)}|_p = p^{-N(m-k)} \longrightarrow 0.$$

Hence $p^{N(m-k)} \to 0$ for $k < m$. Consequently every term with $k < m$ vanishes in the limit. The only surviving term is $k = m$:

$$\int_{\mathbb{Z}_p} x^m \, dx = \frac{1}{m+1} B_m \binom{m+1}{m}.$$

Since $\binom{m+1}{m} = m+1$, we obtain

$$\int_{\mathbb{Z}_p} x^m \, dx = \frac{1}{m+1} B_m(m+1) = B_m.$$

Thus

$$\int_{\mathbb{Z}_p} x^m \, dx = B_m.$$

4 Mahler Expansions

4.1 Mahler's Theorem

Throughout real analysis, the applications of Taylor's Theorem and the existence of uniformly convergent Taylor series for many of our common functions is endless. Likewise, Mahler expansion in the p-adic analysis world is equally fundamental.

Theorem 15 (Mahler)**.** *Every continuous function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ can be written in the form*

$$f(x) = \sum_{n \geq 0} a_n \binom{x}{n} = a_0 + a_1 x + a_2 \binom{x}{2} + a_3 \binom{x}{3} + \cdots \tag{1}$$

for all $x \in \mathbb{Z}_p$, where $a_n \in \mathbb{Q}_p$ and $a_n \to 0$ as $n \to \infty$.

The expansion (1) is called the *Mahler expansion* of f and the numbers a_n are called the *Mahler coefficients* of f. Though this paper will not feature a proof, the properties of Mahler expansions are more relevant to this paper as we will use them to prove further results, so therefore we will prove things about Mahler expansions, and not the existence of them as described in Mahler's Theorem.

4.2 Mahler Expansion Continuity and Convergence

Let us see why an infinite series $\sum_{n\geq 0} a_n \binom{x}{n}$ with $a_n \to 0$ in $\mathbb{Q}_p$ is a continuous function on $\mathbb{Z}_p$.

Step 1. *When $a_n \to 0$ in $\mathbb{Q}_p$, the infinite series $\sum_{n\geq 0} a_n \binom{x}{n}$ converges for all $x \in \mathbb{Z}_p$.*

The key point is that even though $\binom{x}{n} = x(x-1)\cdots(x-(n-1))/n!$ has $n!$ in the denominator, it is not bad when $x \in \mathbb{Z}_p$ because $\binom{x}{n} \in \mathbb{Z}_p$ by p-adic continuity of polynomials and its values on the dense subset $\mathbb{N}$. Thus $\left|\binom{x}{n}\right|_p \leq 1$ for $x \in \mathbb{Z}_p$, so $\left|a_n\binom{x}{n}\right|_p \leq |a_n|_p$, which proves $\left|a_n\binom{x}{n}\right|_p \to 0$ from $|a_n|_p \to 0$. Therefore $\sum_{n\geq 0} a_n \binom{x}{n}$ converges in $\mathbb{Q}_p$ for each $x \in \mathbb{Z}_p$.

Step 2. *When $a_n \to 0$ in $\mathbb{Q}_p$, the function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ defined by $f(x) = \sum_{n\geq 0} a_n \binom{x}{n}$ is continuous.*

For $x_0 \in \mathbb{Z}_p$ we want to prove f is continuous at x_0. Pick $\varepsilon > 0$. Since $a_n \to 0$, there is an N such that $|a_n|_p < \varepsilon$ for $n \geq N$. Each of the finitely many functions $\binom{x}{n}$ for $0 \leq n \leq N-1$ is continuous at x_0, so by taking the minimal δ used for each of them in the ε-δ definition of continuity at x_0, there is a single $\delta > 0$ such that

$$|x - x_0|_p < \delta \implies \left|\binom{x}{n} - \binom{x_0}{n}\right|_p < \varepsilon \quad \text{for } n \in \{0, 1, \ldots, N-1\}. \tag{2}$$

If $|x - x_0|_p < \delta$,

$$|f(x) - f(x_0)|_p = \left|\sum_{n\geq 0} a_n \left(\binom{x}{n} - \binom{x_0}{n}\right)\right|_p \leq \max_{n\geq 0} |a_n|_p \left|\binom{x}{n} - \binom{x_0}{n}\right|_p.$$

Since $\binom{x}{n} - \binom{x_0}{n} \in \mathbb{Z}_p$ we have $|a_n|_p \left|\binom{x}{n} - \binom{x_0}{n}\right|_p \le |a_n|_p < \varepsilon$ for $n \ge N$. For the terms preceding the N-th term, we instead say $|a_n|_p \left|\binom{x}{n} - \binom{x_0}{n}\right|_p \le |a_n|_p \cdot \varepsilon$ by (2). Let $A = \max_{n\ge 0} |a_n|_p$ (this exists since the a_n's tend to 0), so $|a_n|_p \le A$ for all n. Thus

$$|x - x_0|_p < \delta \implies |f(x) - f(x_0)|_p \le \max(\varepsilon, A\varepsilon) = \max(1, A)\varepsilon,$$

so f is continuous at x_0. As x_0 was arbitrary, f is continuous on $\mathbb{Z}_p$.

4.3 Formula for Mahler Coefficients a_n

Next we derive a formula for the coefficients in $f(x) = \sum_{n\ge 0} a_n \binom{x}{n}$ when $a_n \to 0$ in $\mathbb{Q}_p$. It is easy to get the value of the constant term: setting $x = 0$ in (1) gives $a_0 = f(0)$, since all terms vanish in the Mahler expansion except the one at $n = 0$. Similarly, if $x = 1$ then all terms in (1) vanish except for the first two: $f(1) = a_0 + a_1$, so $a_1 = f(1) - a_0 = f(1) - f(0)$. Setting $x = 2$, all terms vanish except the first three: $f(2) = a_0 + 2a_1 + a_2$, so

$$a_2 = f(2) - a_0 - 2a_1 = f(2) - f(0) - 2(f(1) - f(0)) = f(2) - 2f(1) + f(0).$$

These formulas suggest we can write a_n in terms of $f(0), f(1), \ldots, f(n)$, and we will see this is true.

The basic mechanism behind a formula for a_n is the *discrete difference operator* Δ: for any function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$, define $\Delta f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ by

$$(\Delta f)(x) = f(x+1) - f(x).$$

This is also a function $\mathbb{Z}_p \to \mathbb{Q}_p$, and it can be iterated to give functions $\Delta^n f$ for $n \ge 2$: $\Delta^2 f = \Delta(\Delta f)$, and more generally $\Delta^n f = \Delta(\Delta^{n-1} f)$. Set $\Delta^0 f = f$, which is analogous to the zeroth derivative $f^{(0)}$ of a function f being the function itself.

The discrete difference operator Δ behaves nicely on the binomial coefficient polynomials because it shifts them down by one: $\Delta\binom{x}{n} = \binom{x}{n-1}$ for $n \ge 1$, and $\Delta\binom{x}{0} = \Delta(1)$ is the zero function. Indeed, by the Pascal's triangle recursion for binomial coefficients, if $n \ge 1$ then

$$\Delta\binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1} + \binom{x}{n} - \binom{x}{n} = \binom{x}{n-1}.$$

Applying Δ to a function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ having a Mahler expansion $\sum_{n\ge 0} a_n \binom{x}{n}$ where $a_n \to 0$,

$$(\Delta f)(x) = \sum_{n\ge 0} a_n \binom{x+1}{n} - \sum_{n\ge 0} a_n \binom{x}{n} = \sum_{n\ge 0} a_n \left(\binom{x+1}{n} - \binom{x}{n}\right) = \sum_{n\ge 1} a_n \binom{x}{n-1},$$

where a_0 drops out and the other coefficients shift down one position. Reindexing the series to start at $n = 0$,

$$\Delta \sum_{n\ge 0} a_n \binom{x}{n} = \sum_{n\ge 0} a_{n+1} \binom{x}{n} = a_1 + a_2 x + a_3 \binom{x}{2} + a_4 \binom{x}{3} + \cdots.$$

The effect of applying Δ to a Mahler expansion m times is to shift coefficients m positions:

$$\Delta^m \sum_{n\ge 0} a_n \binom{x}{n} = \sum_{n\ge 0} a_{n+m} \binom{x}{n} = a_m + a_{m+1} x + a_{m+2} \binom{x}{2} + a_{m+3} \binom{x}{3} + \cdots.$$

Setting $x = 0$ leaves only the constant term a_m, so we have proved the following theorem.

Theorem 16. *If $a_n \to 0$ in $\mathbb{Q}_p$ and $f(x) = \sum_{n\ge 0} a_n \binom{x}{n}$ for $x \in \mathbb{Z}_p$, then in terms of the function f we have $a_n = (\Delta^n f)(0)$.*

4.4 Formula for nth degree forward differences of $f(x)$

Let us first work out formulas for $(\Delta^2 f)(x)$ and $(\Delta^3 f)(x)$:

$$\begin{aligned}(\Delta^2 f)(x) &= (\Delta(\Delta f))(x) = (\Delta f)(x+1) - (\Delta f)(x)\\ &= (f(x+2) - f(x+1)) - (f(x+1) - f(x))\\ &= f(x+2) - 2f(x+1) + f(x),\end{aligned}$$

$$\begin{aligned}(\Delta^3 f)(x) &= (\Delta(\Delta^2 f))(x) = (\Delta^2 f)(x+1) - (\Delta^2 f)(x)\\ &= (f(x+3) - 2f(x+2) + f(x+1)) - (f(x+2) - 2f(x+1) + f(x))\\ &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x).\end{aligned}$$

These hold for all functions $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$. Notice the binomial coefficients and alternating signs.

Theorem 17. *Let $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ be any function. For $n \geq 0$ and $x \in \mathbb{Z}_p$,*

$$(\Delta^n f)(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k).$$

Proof. When $n = 0$ the formula says $(\Delta^0 f)(x) = f(x)$, and when $n = 1$ the formula says $(\Delta^1 f)(x) = f(x+1) - f(x)$. These are true by definition (Δ^1 means Δ).

If the formula is true for some n and all $x \in \mathbb{Z}_p$, then

$$\begin{aligned}(\Delta^{n+1} f)(x) &= (\Delta(\Delta^n f))(x) = (\Delta^n f)(x+1) - (\Delta^n f)(x)\\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f((x+1)+k) - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k)\\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+(k+1)) + \sum_{k=0}^{n} (-1)^{n-k+1} \binom{n}{k} f(x+k)\\ &= \sum_{k=1}^{n+1} (-1)^{n-(k-1)} \binom{n}{k-1} f(x+k) + \sum_{k=0}^{n} (-1)^{n-(k-1)} \binom{n}{k} f(x+k).\end{aligned}$$

The term in the first sum at $k = n+1$ is $f(x+n+1)$. The term in the second sum at $k = 0$ is $(-1)^{n+1} f(x)$. The remaining terms in both sums run from $k = 1$ to $k = n$, and together equal

$$\sum_{k=1}^{n} (-1)^{n-(k-1)} \left[\binom{n}{k-1} + \binom{n}{k}\right] f(x+k) = \sum_{k=1}^{n} (-1)^{n-(k-1)} \binom{n+1}{k} f(x+k).$$

The terms $f(x+n+1)$ and $(-1)^{n+1} f(x)$ fit into this sum at $k = n+1$ and $k = 0$, so

$$(\Delta^{n+1} f)(x) = \sum_{k=0}^{n+1} (-1)^{n-(k-1)} \binom{n+1}{k} f(x+k) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} f(x+k).$$

Thus the formula holds for $n+1$, completing the induction. □

Plugging in $x = 0$ we get

$$a_n = (\Delta^n f)(0) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

5 The Morita Gamma Function

5.1 Definition of Morita's Gamma Function

Recall from section 1 that we introduced the p-adic factorial $n!_p$ as follows:

$$n!_p = (-1)^n \prod_{\substack{0<i<n \\ p\nmid i}} i.$$

We simply define $\Gamma_p(x)$ to be the continuous extension of the p-adic factorial from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$. As the positive integers are dense with respect to the p-adic topology in $\mathbb{Z}_p$, $\Gamma_p(x)$ can be extended uniquely to the whole of $\mathbb{Z}_p$. Note that the codomain is the p-adic integers' group of units specifically because of the removal of multiples of p in the p-adic factorial.

5.2 Closed form of $\Gamma_p(n+1)$

Though $\Gamma_p(x)$ is defined for all $x \in \mathbb{Z}_p$, it is still worth finding a closed formula for natural inputs.

Theorem 18. *Let Γ_p denote the Morita p-adic gamma function, defined for $n \in \mathbb{N}$ by*

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{1\le k\le n \\ p\nmid k}} k.$$

Then for every integer $n \geq 2$,

$$\Gamma_p(n+1) = \frac{(-1)^{n+1} n!}{\left\lfloor \frac{n}{p} \right\rfloor ! \, p^{\left\lfloor \frac{n}{p} \right\rfloor}}.$$

Proof. Let $m = \left\lfloor \frac{n}{p} \right\rfloor$.

By definition,

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{1\le k\le n \\ p\nmid k}} k.$$

We now express the ordinary factorial $n!$ as the product of the integers divisible by p and those not divisible by p.

Since $n! = \prod_{k=1}^{n} k$, we may write

$$n! = \left(\prod_{\substack{1\le k\le n \\ p\nmid k}} k \right) \left(\prod_{\substack{1\le k\le n \\ p\mid k}} k \right).$$

The integers between 1 and n divisible by p are precisely p, $2p$, $3p, \ldots, mp$, where $m = \left\lfloor \frac{n}{p} \right\rfloor$. Therefore

$$\prod_{\substack{1\le k\le n \\ p\mid k}} k = \prod_{j=1}^{m} (jp).$$

Factoring out p from each term gives

$$\prod_{j=1}^{m}(jp) = p^m \prod_{j=1}^{m} j = p^m m!.$$

Substituting this into the factorization of $n!$ yields

$$n! = \left(\prod_{\substack{1 \le k \le n \\ p \nmid k}} k \right) p^m m!.$$

Solving for the product over the integers not divisible by p gives

$$\prod_{\substack{1 \le k \le n \\ p \nmid k}} k = \frac{n!}{p^m m!}.$$

Substituting this expression into the definition of $\Gamma_p(n+1)$, we obtain

$$\Gamma_p(n+1) = (-1)^{n+1} \frac{n!}{p^m m!}.$$

Finally, replacing m by $m = \left\lfloor \frac{n}{p} \right\rfloor$ gives

$$\Gamma_p(n+1) = \frac{(-1)^{n+1} n!}{\left\lfloor \frac{n}{p} \right\rfloor ! \, p^{\left\lfloor \frac{n}{p} \right\rfloor}}.$$

□

5.3 The Morita reflection formula

Like the regular gamma function, there is a p-adic analog of the reflection formula. However, in the p-adic world, the factor $\frac{\pi}{\sin(\pi x)}$ has been replaced with a scaled down discrete version.

Theorem 19 (Morita reflection formula)**.** *For any prime p and any p-adic integer $x \in \mathbb{Z}_p$,:*

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)},$$

where $R(x) \in \{1, 2, \dots, p\}$ is the first digit in the canonical p-adic expansion of x.

Proof. Let $\Phi(x) = \Gamma_p(x)\Gamma_p(1-x)$ be our target product function. We find the Mahler expansion of $\Phi(x)$ (if it exists). By definition, Morita's Gamma function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous extension of the sequence defined on $\mathbb{Z}^+$ by

$$\Gamma_p(n) = (-1)^n \prod_{i<n,\, p \nmid i} i.$$

It satisfies the fundamental continuous functional equation

$$\Gamma_p(x+1) = h_p(x)\Gamma_p(x) \quad \text{for all } x \in \mathbb{Z}_p,$$

where $h_p(x) = -x$ if $|x|_p = 1$ and $h_p(x) = -1$ if $|x|_p < 1$. Because the product of continuous functions is continuous, $\Phi(x)$ is continuous on $\mathbb{Z}_p$.

To find its exact values, we examine how the forward difference operator acts on $\Phi(x)$ by evaluating $\Phi(x+1)$:

$$\Phi(x+1) = \Gamma_p(x+1)\Gamma_p(1-(x+1)) = \Gamma_p(x+1)\Gamma_p(-x).$$

Substituting $\Gamma_p(x+1) = h_p(x)\Gamma_p(x)$ and isolating $\Gamma_p(-x) = \frac{\Gamma_p(1-x)}{h_p(-x)}$ from the functional equation yields

$$\Phi(x+1) = (h_p(x)\Gamma_p(x)) \cdot \left(\frac{\Gamma_p(1-x)}{h_p(-x)}\right) = \frac{h_p(x)}{h_p(-x)}\Phi(x).$$

We evaluate the multiplier $q(x) = \frac{h_p(x)}{h_p(-x)}$ based on the p-adic valuation of x:

- If $|x|_p = 1$ ($x \not\equiv 0 \pmod p$), then $|-x|_p = 1$, giving $h_p(x) = -x$ and $h_p(-x) = x$, which results in $q(x) = \frac{-x}{x} = -1$.
- If $|x|_p < 1$ ($x \equiv 0 \pmod p$), then $|-x|_p < 1$, giving $h_p(x) = -1$ and $h_p(-x) = -1$, which results in $q(x) = \frac{-1}{-1} = 1$.

This establishes that the step-property for the product function undergoes a sign change gated purely by multiples of p:

$$\Phi(x+1) = \begin{cases} -\Phi(x) & \text{if } x \not\equiv 0 \pmod p, \\ +\Phi(x) & \text{if } x \equiv 0 \pmod p. \end{cases}$$

Evaluating this systematically on the non-negative integers $k \in \mathbb{Z}_{\geq 0}$ via induction reveals a crucial boundary constraint. For the first block where $k \in \{1, 2, \ldots, p-1\}$, the condition $k \not\equiv 0 \pmod p$ holds, and the sign flips consecutively such that $\Phi(k) = (-1)^k$. However, this global alternation breaks at the boundary multiples of p. For instance, at $k = p-1$, we have $\Phi(p-1) = (-1)^{p-1}$. Moving to $k = p$, since $p - 1 \not\equiv 0 \pmod p$, the sign flips to $\Phi(p) = -\Phi(p-1) = (-1)^p$. But for the next step, since $p \equiv 0 \pmod p$, the multiplier goes back to $+1$, meaning $\Phi(p+1) = +\Phi(p) = (-1)^p$.

This tracking shows that $\Phi(k)$ does not equal $(-1)^k$ globally across $\mathbb{Z}$. If it did, the Mahler coefficients would evaluate via the binomial theorem to

$$a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (-1)^j = (-1)^n 2^n.$$

For an odd prime, $|2|_p = 1$, which means $\lim_{n\to\infty} |a_n|_p = 1 \neq 0$, a direct violation of Mahler's theorem confirming that a purely alternating global sequence is discontinuous in $\mathbb{Z}_p$. Instead, the boundary conditions constrain $\Phi(k)$ to be strictly periodic with period p on the integers, meaning $\Phi(k+p) = \Phi(k)$ for all $k \in \mathbb{Z}_{\geq 0}$.

Because $\Phi(k)$ is strictly periodic, its forward differences $\Delta^n\Phi(0)$ do not terminate to 0 for $n \geq p$. Rather, the differences of a periodic function on $\mathbb{Z}$ accumulate factors of p under binomial combinations, ensuring that $\lim_{n\to\infty} |\Delta^n\Phi(0)|_p = 0$. Thus, by Mahler's Theorem, $\Phi(x)$ possesses a well-defined, uniformly convergent infinite series expansion valid for all $x \in \mathbb{Z}_p$:

$$\Phi(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \quad \text{where } a_n = \Delta^n\Phi(0).$$

We now examine the function $f(x) = (-1)^{R(x)}$ where $R(x)$ returns the first digit in the p-adic expansion of x. Because $R(x)$ depends solely on the first digit of the p-adic expansion of x, $f(x)$ is a locally constant function on $\mathbb{Z}_p$. Like $\Phi(x)$, it is continuous and strictly periodic with period p when restricted to the non-negative integers.

By comparing the periodic outputs of both functions on the integers $k \in \mathbb{Z}_{\geq 0}$, our previous inductive evaluation shows an exact match: $\Phi(k) = (-1)^{R(k)}$ for every single non-negative integer. Because the subset $\mathbb{Z}_{\geq 0}$ is topologically dense in the ring of p-adic integers $\mathbb{Z}_p$, the Uniqueness Principle of Continuous Extensions dictates that any two continuous functions matching entirely on a dense subset must be identical across the entire domain.

Consequently, the unique Mahler coefficients of $\Phi(x)$ must match the Mahler coefficients of $(-1)^{R(x)}$ term-for-term, forcing the full identity to hold universally:

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)} \quad \forall x \in \mathbb{Z}_p.$$

□

5.4 $\Gamma_p(x)$ in relation to Number Theory

As seen with the traditional gamma function and its reflection formula, plugging in $\frac{1}{2}$ can result in magical identities. Here, a similar result of plugging in $\frac{1}{2}$ can be derived with the help of the Morita Reflection Formula.

Corollary 20. *Let p be an odd prime, and let $\Gamma_p(x)$ denote Morita's p-adic Gamma function. Then the following identity holds:*

$$\Gamma_p\left(\frac{1}{2}\right)^2 = -\left(\frac{-1}{p}\right),$$

where $\left(\frac{-1}{p}\right)$ is the Legendre symbol.

Proof. As we saw with the previous result, Morita's Gamma function has the following reflection formula:

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle x\rangle_p}, \tag{3}$$

where $\langle x\rangle_p$ is the unique integer chosen such that $1 \leq \langle x\rangle_p \leq p$ and $\langle x\rangle_p \equiv x \pmod{p\mathbb{Z}_p}$.

We substitute $x = \frac{1}{2}$ into equation (3). Since $1 - \frac{1}{2} = \frac{1}{2}$, the left-hand side simplifies to a square:

$$\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\langle 1/2\rangle_p}. \tag{4}$$

To determine the exponent $\langle 1/2\rangle_p$, we solve the congruence relation $2k \equiv 1 \pmod{p}$ for $1 \leq k \leq p$. Since p is an odd prime, $p+1$ is an even integer, allowing us to write:

$$2\left(\frac{p+1}{2}\right) = p+1 \equiv 1 \pmod{p}.$$

Because $1 \leq \frac{p+1}{2} \leq p$, it follows by definition that

$$\langle 1/2\rangle_p = \frac{p+1}{2}.$$

Substituting this value back into equation (4) yields

$$\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\frac{p+1}{2}}. \tag{5}$$

By factoring the exponent, we can express the right-hand side of equation (5) as:

$$(-1)^{\frac{p+1}{2}} = (-1)^{\frac{p-1}{2}+1} = (-1)^{\frac{p-1}{2}} \cdot (-1)^1 = -(-1)^{\frac{p-1}{2}}.$$

Finally, Euler's criterion states that for any odd prime p, the Legendre symbol $\left(\frac{-1}{p}\right)$ satisfies

$$\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \pmod{p}.$$

Since both sides belong to $\{-1, 1\}$, this is a strict equality of integers: $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$. Substituting this into our expression gives

$$\Gamma_p\left(\frac{1}{2}\right)^2 = -\left(\frac{-1}{p}\right).$$

This completes the proof. □

5.5 *p*-adic Raabe Formula

As discussed previously, the Raabe Formula is one of the most important theorems regarding the standard gamma function in real and complex analysis. There is an analogue of that theorem for the p-adic gamma function. Aside from Mahler's Theorem, the p-adic Raabe Formula is the only other topic in this paper that will not feature a proof. To prove it, one must dive deep into measure theory and Diamond's regularized log-gamma function, which is out of the scope of this paper. However, here is the formula stated.

Theorem 21 (p-adic Raabe)**.** *Let p be an odd prime, and let $\Gamma_p(x)$ denote Morita's p-adic Gamma function. Then*

$$\int_{\mathbb{Z}_p} \log \Gamma_p(x+t)\, dt = (x-1)(\log \Gamma_p)'(x) - x + \left\lceil \frac{x}{p} \right\rceil \qquad (x \in \mathbb{Z}_p),$$

where the ceiling function is to be understood as the p-adic limit $\lim_{n\to\infty} \left\lceil \frac{x_n}{p} \right\rceil$ such that $x_n \to x$ through rational integers.