Profinite Fibonacci Numbers

Abdul Rafay

June 8, 2026

Abstract

The Fibonacci sequence is one of the most familiar integer sequences in mathematics. While usually studied as a sequence indexed by ordinary integers, it can also be extended to profinite integers. This extension leads to the notion of profinite Fibonacci numbers introduced by Hendrik Lenstra. In this paper we discuss Fibonacci numbers modulo integers, introduce profinite integers through compatible congruence classes and factorial notation, explain the meaning of the profinite Fibonacci function F_s, and present Lenstra's theorem that the equation $F_s = s$ has exactly eleven solutions. The paper is intended as an elementary exposition for beginning graduate students.

1 Introduction

The Fibonacci sequence is defined by

$$F_0 = 0, \qquad F_1 = 1,$$

and

$$F_{n+2} = F_{n+1} + F_n$$

for all integers $n \geq 0$.

The first few Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The sequence appears throughout mathematics, from combinatorics and number theory to dynamical systems and algebra. One particularly interesting aspect of the Fibonacci sequence is its behavior modulo integers.

The purpose of this paper is to explain how congruence properties of Fibonacci numbers naturally lead to profinite integers and to Lenstra's theory of profinite Fibonacci numbers.

2 Fibonacci Numbers Modulo Integers

For a positive integer m, consider the sequence

$$F_0, F_1, F_2, \dots \pmod{m}.$$

Since the Fibonacci recurrence is determined by consecutive pairs

$$(F_n, F_{n+1}),$$

and only finitely many such pairs exist modulo m, the sequence must eventually repeat.

The period of the Fibonacci sequence modulo m is called the *Pisano period* and is denoted by $\pi(m)$.

Example 1. *Reducing Fibonacci numbers modulo* 5 *gives*

$$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, \dots$$

The pair $(0, 1)$ *reappears after* 20 *terms, so*

$$\pi(5) = 20.$$

Hence

$$F_{n+20} \equiv F_n \pmod{5}.$$

Example 2. *Reducing modulo* 8 *gives*

$$0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, \dots$$

The period is 12*, so*

$$F_{n+12} \equiv F_n \pmod{8}.$$

These periodicity properties suggest that Fibonacci numbers are naturally compatible with congruences, which motivates the introduction of profinite integers.

3 Profinite Integers and Factorial Notation

A profinite integer may be viewed as a compatible collection of residue classes modulo every positive integer.

Formally,

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

Thus a profinite integer consists of residue classes

$$(a_n)_{n\geq 1}$$

such that whenever m divides n, the residue class a_n reduces to a_m modulo m.

Every ordinary integer determines such a compatible family and therefore may be viewed as a profinite integer.

Example 3. *The integer* 7 *corresponds to*

$$7 \bmod 1, \quad 7 \bmod 2, \quad 7 \bmod 3, \quad 7 \bmod 4, \ldots$$

and hence defines a profinite integer.

Lenstra uses factorial notation to make profinite integers more concrete.

Every positive integer can be written uniquely as

$$n = c_k k! + c_{k-1}(k-1)! + \cdots + c_1 1!,$$

where

$$0 \leq c_i \leq i.$$

The integer is then written as

$$(c_k c_{k-1} \cdots c_1)!.$$

Example 4. *Since*

$$5 = 2 \cdot 2! + 1 \cdot 1!,$$

we write

$$5 = (21)!.$$

Similarly,

$$25 = 1 \cdot 4! + 0 \cdot 3! + 0 \cdot 2! + 1 \cdot 1!,$$

so

$$25 = (1001)!.$$

A profinite integer is obtained by allowing infinitely many factorial digits:

$$(\dots c_5c_4c_3c_2c_1)!.$$

This notation makes profinite integers resemble infinite decimal expansions.

4 The Profinite Fibonacci Function

The periodicity properties discussed earlier imply that the Fibonacci sequence behaves well with respect to congruences.

As a result, the Fibonacci sequence extends naturally from ordinary integers to profinite integers.

If

$$s \in \widehat{\mathbb{Z}},$$

the corresponding value of the extended Fibonacci function is denoted by

$$F_s.$$

One should think of s as a generalized index.

When s is an ordinary integer, the notation agrees with the classical sequence:

$$F_5 = 5, \qquad F_{10} = 55.$$

The extension therefore enlarges the domain of the Fibonacci function without changing its values on ordinary integers.

Remark 1. *The notation F_s does not define a new sequence. It is simply the classical Fibonacci sequence viewed as a function on a larger arithmetic space.*

5 Lenstra's Fixed Point Theorem

Once the Fibonacci sequence has been extended to profinite integers, it becomes natural to ask whether there exist profinite integers satisfying

$$F_s = s.$$

Such elements are called fixed points of the profinite Fibonacci function.

Among ordinary integers, the equation

$$F_n = n$$

has exactly three solutions:

$$0, \qquad 1, \qquad 5.$$

A natural question is whether additional solutions appear in the profinite setting.

Lenstra answered this question completely.

Theorem 1 (Lenstra)**.** *The equation*

$$F_s = s$$

has exactly eleven solutions in the ring of profinite integers.

At first sight one might expect infinitely many new fixed points to appear after enlarging the domain from the integers to the profinite integers. Lenstra's theorem shows that this does not happen. The fixed-point set remains finite and can be described explicitly.

Three of the fixed points are the ordinary integers

$$0, \qquad 1, \qquad 5.$$

The remaining eight are genuinely profinite integers.

Lenstra denotes them by

$$z_{1,-5}, z_{1,-1}, z_{1,0}, z_{1,5}, z_{5,-5}, z_{5,-1}, z_{5,0}, z_{5,1}.$$

These fixed points cannot be represented by ordinary integers and exist only in the profinite completion.Each point is determined by two pieces of congruence data. Roughly speaking, the first index records how the fixed point behaves modulo powers of 6, while the second index records how it behaves modulo powers of 5. For example, the fixed point $z_{1,0}$ behaves like 1 when viewed modulo increasingly large powers of 6, but simultaneously behaves like 0 modulo increasingly large powers of 5. No ordinary integer can have this behaviour. If an integer is divisible by arbitrarily large powers of 5, then it must be equal to 0. However, 0 is not congruent to 1 modulo 6. Thus such an object can exist only in the profinite completion. Similarly, the fixed point $z_{5,-1}$ behaves like 5 modulo powers of 6 and like -1 modulo powers of 5. Again, no ordinary integer can satisfy all of these conditions simultaneously. The eight non-classical fixed points should therefore be viewed as infinite compatible systems of congruences. They are genuine profinite integers rather than ordinary integers, and their existence demonstrates that the profinite completion contains new arithmetic information that is invisible in the classical setting.

Example 5. *Consider the fixed point* $z_{1,0}$*.*

Modulo 5 *it behaves like*

$$0.$$

Modulo 25 *it behaves like*

$$0.$$

Modulo 125 *it behaves like*

$$0.$$

In fact, it is divisible by every power of 5*.*

At the same time,

$$z_{1,0} \equiv 1 \pmod{6},$$

$$z_{1,0} \equiv 1 \pmod{36},$$

$$z_{1,0} \equiv 1 \pmod{216},$$

and similarly for all higher powers of 6*.*

This illustrates how a profinite integer can simultaneously encode infinitely many congruence conditions that cannot be realized by any ordinary integer.

‘‘‘

6 The Significance of the Fixed Points

The fixed points illustrate how enlarging a familiar mathematical object can reveal new structure.

The ordinary Fibonacci sequence already contains rich arithmetic information. By extending the indexing set from integers to profinite integers, one discovers eight additional fixed points that are invisible in the classical setting.

The result is particularly striking because the number of fixed points remains finite. Instead of producing infinitely many new solutions, the profinite extension produces exactly eight new ones.

This phenomenon highlights the subtle relationship between congruence information and global arithmetic structure.

7 Conclusion

The Fibonacci sequence is one of the simplest recurrence sequences in mathematics, yet it exhibits surprisingly rich arithmetic behavior.

By studying Fibonacci numbers modulo integers, one is naturally led to the ring of profinite integers. In that setting the Fibonacci sequence extends to a function defined on profinite indices.

Lenstra's theorem shows that the resulting fixed-point equation

$$F_s = s$$

has exactly eleven solutions. Three are ordinary integers, while eight are genuinely profinite objects.

The theory of profinite Fibonacci numbers therefore provides an elegant example of how classical number-theoretic ideas can lead to unexpected structures when viewed through a broader arithmetic perspective.

References

[1] H. W. Lenstra Jr., *Profinite Fibonacci Numbers*, Nieuw Archief voor Wiskunde, Series 5, Volume 6 (2005), 297–300.

[2] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, American Mathematical Society, 2003.

[3] P. Ribenboim, *The Little Book of Bigger Primes*, 2nd ed., Springer, 2004.