P-ADIC STRING THEORY

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Abstract. This expository manuscript presents an introduction to p-adic string theory with an emphasis on p-adic string scattering amplitudes. After briefly reviewing the basic framework of bosonic string theory and the Veneziano, Koba-Nielsen, and Bardakci-Ruegg amplitudes, we discuss their generalizations to the p-adic field $\mathbb{Q}_p$. Particular attention is devoted to the role of ultrametricity, p-adic sign functions, and quadratic extensions in the structure of p-adic amplitudes, together with illustrative examples of four-point and five-point scattering amplitudes. The manuscript is intended primarily for readers familiar with p-adic analysis who may not necessarily have prior background in string theory.

1. Introduction

The discovery that many structures in theoretical physics admit meaningful formulations over non-Archimedean number fields has led to a rich interaction between modern physics and number theory. Among the most notable examples is p-adic string theory [1–3], where scattering amplitudes of ordinary string theory [4–6] are generalized from the real numbers to the field of p-adic numbers [7]. These theories provide remarkable examples of physical models whose fundamental analytic structure is governed by ultrametric geometry [8] rather than ordinary Archimedean geometry.

Historically, p-adic string theory emerged from attempts to understand the arithmetic structure underlying string scattering amplitudes. The Veneziano amplitude and its higher-point generalizations possess forms that naturally admit p-adic analogues obtained by replacing real integration domains and ordinary absolute values with p-adic ones. This observation led to the construction of p-adic open-string amplitudes by Volovich, Freund, Olson, Witten, and others, eventually revealing deep adelic relations between ordinary and p-adic string theories.

One of the striking features of p-adic string amplitudes is that many otherwise complicated integrals simplify dramatically due to the ultrametric property of the p-adic norm. As a result, amplitudes that are highly nontrivial in ordinary string theory often admit closed-form expressions in the p-adic setting. At the same time, the theory retains many of the characteristic features of ordinary string scattering, including pole structures, crossing symmetries, and nonlocal effective dynamics. These simplifications make p-adic string theory an important laboratory for studying nonlocal field theories, tachyon dynamics, and hidden algebraic structures in string amplitudes.

The central theme of this manuscript is that string scattering amplitudes, which are generally difficult objects to compute in ordinary Archimedean string theory, often become remarkably tractable when reformulated over the p-adic field. Our goal is not to construct

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a complete theory of p-adic strings, but rather to illustrate how familiar string-theoretic scattering amplitudes admit natural p-adic analogues and how the ultrametric structure of $\mathbb{Q}_p$ dramatically simplifies their evaluation.

We therefore begin by reviewing the minimal ingredients of bosonic string theory [4–6] required to understand tree-level scattering amplitudes. We then introduce the Veneziano amplitude [9] and its Koba–Nielsen [10] and Bardakci–Ruegg [11] representations, emphasizing their interpretation as scattering amplitudes for interacting open strings. Only after this physical motivation has been established do we pass to the p-adic setting, where the real number field and ordinary absolute value are replaced by $\mathbb{Q}_p$ and the p-adic norm.

The resulting amplitudes provide a striking illustration of how arithmetic structure can simplify physical computations. In particular, the ultrametric property of the p-adic norm causes many contributions that are complicated in the Archimedean theory to collapse into simple closed-form expressions. Through illustrative four-point and five-point examples, we demonstrate how these simplifications arise and how characteristic p-adic contact-interaction terms emerge naturally.

Our emphasis throughout is on exposing the analytic and algebraic structures underlying these amplitudes while requiring only a modest amount of prior familiarity with string-theoretic methods.

2. String Theory

In this section we briefly review the aspects of ordinary bosonic string theory that are needed later. Our primary objective is not to develop string theory in full detail, but rather to explain the origin of string scattering amplitudes. The amplitudes discussed in Section 3 arise from the interaction of strings and may be viewed as string-theoretic analogues of scattering amplitudes in quantum field theory. We therefore focus on the worldsheet description of strings and the integral representations of their tree-level scattering amplitudes, which will later admit p-adic generalizations.

We can introduce string theory starting from the Lagrangian describing a relativistic point mass particle. Let us consider a D dimensional Minkowski space $\mathbb{R}^{D-1}$, equipped with a metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \ldots, +1). \tag{2.1}$$

If we fix a frame with coordinates $X^\mu = (t, \vec{x})$ then the action is simple:

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}. \tag{2.2}$$

To see why this is correct we can compute the momentum $\vec{p}$ conjugate to $\vec{x}$, and the energy E which is equal to the Hamiltonian,

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}}, \quad E = \sqrt{m^2 + p^2}. \tag{2.3}$$

Although eqn.(2.2) is correct, it is not fully satisfactory because the time t and space $\vec{x}$ play very different roles in the Langrangian. The position $\vec{x}$ is a dynamical degree of freedom.

In contrast, time t is merely a parameter providing a label for the position. Yet Lorentz transformations are supposed to mix up t and $\vec{x}$ and such symmetries are not completely obvious in eqn.(2.2). A viable action that resolves this issue is given by

$$S = -m \int dt \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}, \tag{2.4}$$

where $\mu = 0, 1, \ldots D-1$ and $\dot{X}^\mu = dX^\mu/d\tau$. We have introduced a new parameter τ which labels the position along the worldline of the particle.

Naively it looks as if we now have D physical degrees of freedom rather than $D-1$ because the time direction $X^0 = t$ is among our dynamical variables. However, this is an illusion because the action given by eqn.(2.2) has an important property of reparameterization invariance. This means that we can pick a different parameter $\tilde{\tau}$ on the worldline, related to τ by a monotonic function

$$\tilde{\tau} = \tilde{\tau}(\tau). \tag{2.5}$$

The upshot of this is that not all D degrees of freedom X^μ are physical. One could find a solution of the system so that you know how X^0 changes with τ and how X^1 changes with τ and so on. Not all of that information is meaningful since τ itself is not meaningful. In particular, we can use our reparameterization invariance to simply set

$$\tau = X^0(\tau) = t. \tag{2.6}$$

If we plug this choice into the action (2.4) then we recover our initial action (2.2).

The fact that one of the degrees of freedom is a fake also shows up if we look at the momenta,

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{m\dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\dot{X}^\lambda \dot{X}^\rho \eta_{\lambda\rho}}}. \tag{2.7}$$

These momenta aren't all independent. They satisfy

$$p_\mu p^\mu + m^2 = 0. \tag{2.8}$$

This is a constraint on the system. It is, of course, the mass-shell constraint for a relativistic particle of mass m. From a worldline perspective, it tells us that the particle isn't allowed to sit still in Minkowski space, at the very least, it had better keep moving in a timelike direction $(p^0)^2 \geq m^2$.

Just like how a particle sweeps out a worldline in Minkowski space, a string sweeps out a world sheet. We will parametrize this worldsheet by one timelike coordinate τ and one spacelike coordinate σ. For closed strings, σ is taken to be periodic with range

$$\sigma \in [0, 2\pi). \tag{2.9}$$

We will sometimes package the two worldsheet coordinates together as $\sigma^\alpha = (\tau, \sigma)$, $\alpha = 0, 1$. Then, the string sweeps out a surface n spacetime which defines a map from the worldsheet to Minkowski space, $X^\mu(\sigma, \tau)$ with $\mu = 0, \ldots D-1$.

The string sweeps out a surface in spacetime which defines a map from the world sheet to the Minkowski space, $X^\mu(\sigma, \tau)$ with $\mu = 0, \ldots, D-1$. We need an action that describes the dynamics of this string. The key property that we will ask for is that nothing depends on the coordinates σ that we choose on the worldsheet. In other words, the string action should be reparameterization invariant. For the point particle, he action was proportional to the length of the worldline. The obvious generalization is that the action for the string should be proportional to the area, A od the worldsheet.

Now, the worldsheet is curved surface embedded in spacetime. The induced metric, $\gamma_{\alpha\beta}$ on this surface is the pull-back of the flat metric on Minkowski space,

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}. \tag{2.10}$$

Then the action which is proportional to the area of the worldsheet is given by,

$$S = -T\int d^2\sigma\sqrt{-\det\gamma}. \tag{2.11}$$

Here T is a constant of proportionality and can be interpreted as the tension of the string, meaning the mass per unit length.

We can write this action a little more explicitly. The pull-back of the metric is given by,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}, \tag{2.12}$$

where $\dot{X}^\mu = \partial X^\mu/\partial\tau$ and $X'^\mu = \partial X^\mu/\partial\sigma$. The action then takes the form,

$$S = -T\int d^2\sigma\sqrt{-(\dot{X})^2(X')^2 + (\dot{X}\cdot X')^2}. \tag{2.13}$$

This is the Nambu-Goto action of a relativistic string. The tension T is often written as

$$T = \frac{1}{2\pi\alpha'} \tag{2.14}$$

due to historical reasons.

To derive the equations of motion for the Nambu-Goto string, we first introduce the momenta which we call Π,

$$\Pi^\tau_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}} \tag{2.15}$$

$$\Pi^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu} = -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (X')^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}} \tag{2.16}$$

The equations of motion are then given by,

$$\frac{\partial\Pi^\tau_\mu}{\partial\tau} + \frac{\partial\Pi^\sigma_\mu}{\partial\sigma} = 0. \tag{2.17}$$

A simpler way to write this is

$$\partial_\alpha(\sqrt{-\det\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0. \tag{2.18}$$

The square-root in the Nambu-Goto action means that it's rather difficult to quantize using path integral techniques. However, there is another form of the string action which is classically equivalent to the Nambu-Goto action. It eliminates the square root at the expense of introducing another field,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.19}$$

where $g = \det g$. This is the Polyakov action.

The new dynamical filed introduced here is $g_{\alpha\beta}$. From the perspective of the worldsheet, the Polyakov action is a bunch of scalar fields X coupled to 2d gravity.

The equation of motion of X^μ is

$$\partial_\alpha(\sqrt{-g} g^{\alpha\beta} \partial_\beta X) = 0, \tag{2.20}$$

which coincides with the equation of motion (2.18) rom the Nambu-Goto action, except that $g_{\alpha\beta}$ is now an independent varaible fixed by its own equation of motion,

$$g_{\alpha\beta} = 2f(\sigma) \partial_\alpha X \partial_\beta X, \tag{2.21}$$

where the function $f(\sigma)$ is given by,

$$f^{-1} = g^{\rho\sigma} \partial_\rho X \partial_\sigma X. \tag{2.22}$$

Let us now dwell into string interactions and in particular, the scattering amplitudes of two interacting strings. Remarkably, the scattering amplitude of interacting strings doesn't explicitly include any extra non-linear terms, unlike the case in Quantum Field Theory. To see that this is at least feasible, note that in Feynman diagrams of Quantum Field Theory, the information about interactions is contained in vertices, where different lines meet. In the case of an interacting string, there are no such points and locally every part of the diagram looks like a freely propagating string. That being said, scattering amplitudes of two interacting strings only depend on their Polyakov action and their topology. Moreover, computing scatting amplitude involves summing over worldsheets equipped with different topologies. This is what gives rise to the perturbative expansion in String Theory. Different topologies weigh differently. Moreover,

$$S_{\text{string}} = S_{\text{Polyakov}} + \lambda\chi \tag{2.23}$$

where λ is a Real number and for a closed string, χ is the Euler number (which may be familiar from Differential Geometry) defined by

$$\chi = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} R, \tag{2.24}$$

where R is the Ricci scalar of the worldsheet metric. For a worldsheet without a boundary (i.e for a closed string), χ counts the number of handles h of the surface. The Gauss-Bonnet theorem states χ is a topological invariant and can be expressed as

$$\chi = 2 - 2h = 2(1 - g) \tag{2.25}$$

for worldsheets without a boundary (i.e for a closed string). Here h is the number of handles of the surface and g is the genus of the surface.

However, for a open string, χ is defined as

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\sqrt{g}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} dsk, \tag{2.26}$$

where k is the geodesic curvature of the boundary.

Now, the Gauss-Bonnet theorem has an extension to surfaces with boundaries. For surfaces with h handles and b boundaries, the Euler character is defined by

$$\chi = 2 - 2h - b. \tag{2.27}$$

Now suppose that we want to compute the S-matrix of m states, we label them as Λ_i and assign them a spacetime momenta p_i. Each has a corresponding vertex operator $V_{\Lambda_i}(p_i)$.

The Vertex amplitude for a 4-points tachyon scattering scenario is given by

$$V(p_i) = \sqrt{g_s}\int dx e^{ip_i\cdot X}, \tag{2.28}$$

where the integral $\int dx$ is over the boundary and $p^2 = 1/\alpha'$ is the on-shell condition for an open-string tachyon.

From a physical perspective, scattering amplitudes encode the probabilities for incoming particles or strings to interact and emerge in specified outgoing states. In ordinary quantum field theory these quantities are computed from Feynman diagrams. In string theory the corresponding amplitudes arise from summing over worldsheets connecting the incoming and outgoing strings. The Veneziano amplitude is historically significant because it provides the tree-level four-point scattering amplitude for open-string tachyons and was one of the earliest indications that string-like objects could furnish a consistent framework for particle interactions.

We now find that the amplitude is given by

$$\mathcal{A}^{(4)} \sim \frac{g_s}{\mathrm{Vol}(SL(2;\mathbf{R})}\delta^{26}(\sum_i p_i)\int\prod_{i=1}^{4} dx_i\Pi_{j<l}|x_j - x_l|^{2\alpha' p_j p_l}. \tag{2.29}$$

After summing over the different orderings of vertex operators, the end result for the amplitude for open string tachyon scattering is,

$$\mathcal{A}^{(4)} \sim g_s[B(-\alpha' s - 1, \alpha' t - 1) + B(-\alpha' s - 1, -\alpha' u - 1) + B(-\alpha' t - 1, -\alpha' u - 1)], \tag{2.30}$$

where $B(\cdot,\cdot)$ is the Beta function. This is the famous Venezio's Amplitude.

One may generalize the Venzio Amplitude to N-point tachyon interactions, to obtain the Koba-Nielson formula

$$A_N = \int\left(\prod_{i=1}^{N} dz_\circ\right)\frac{1}{dV_{abc}}\prod_{1\leq i<j\leq N}|z_i - z_j|^{k_i k_j} \tag{2.31}$$

where the volume element dV_{abc} is given by

$$dV_{abc} = \frac{dz_a dz_b dz_c}{|z_a - z_b||z_b - z_c||z_c - z_a|} \tag{2.32}$$

and the points z_a, z_b and z_c are three arbitrary points which can be fixed at will due to the symmetries underlying the theory. Let us now fix $z_1 = 0$,$z_2 = 1$ and $z_3 = \infty$. We can now write

$$A_N^{(\mathrm{KN})} = \int \prod_{i=4}^{N} dz_i \prod_{i=4}^{N} |z_i|^{a_{1i}-1} |1 - z_i|^{a_{2i}-1} \prod_{4 \le i < j \le N} |z_i - z_j|^{a_{ij}-1}, \tag{2.33}$$

where $a_{ij} = k_i k_j + 1$ and the integration variables are restricted to obey $1 > z_4 > z_5 > \cdots > z_N$. There is another form, equivalent to eqn.(2.33), which is obtained from the Koba Neilson form by changing variables, $z_4 = x_2, z_5 = z_4 z_3, \ldots$. The new variables x_i all range in unit interval on the real line. This is the Baradecki Ruegg form of the scattering amplitude, it reads

$$B_N^{BR} = \int_0^1 \prod_{i=2}^{N-2} dx_i x_i^{a(i)-1} \prod_{2 \le i < j \le N-1} |1 - x_i x_{i+1} \ldots x_{j-1}|_{a_{ij}-1}, \tag{2.34}$$

where $a(i) = (k_1 + k_2 + \ldots k_i)^2 - 1$. There is a third formula which gives the full scattering amplitude (summing over non-cylic permutations of the momemnta), and is best written in the Koba-Nielson form

$$A_N = \int_{\mathbb{R}} \prod_{i=4}^{N} dz_i |z_i|^{a_{1i}-1} |1 - z_i|^{a_{2i}-1} \prod_{4 \le i < j \le N} |z_i - z_j|^{a_{ij}-1}. \tag{2.35}$$

3. P-adic Strings

We now arrive at the genuinely p-adic part of the discussion. Up to this point, all amplitudes have been the standard Archimedean amplitudes of ordinary open-string theory. The basic idea of p-adic string theory is surprisingly simple: one replaces the real number field by the p-adic field Qp and replaces ordinary absolute values by p-adic norms. The resulting integrals define new scattering amplitudes whose analytic structure is governed by ultrametric geometry rather than ordinary Euclidean geometry.

The motivation for doing so is twofold. First, many string amplitudes possess arithmetic structures that naturally admit p-adic analogues. Second, the ultrametric property of the p-adic norm often simplifies the resulting integrals dramatically, allowing explicit computations that are difficult or impossible in the Archimedean theory. The remainder of this section illustrates these ideas through several examples.

Each of the above scattering amplitude formulas can be generalized to the p-adic number field. The generalization of eqn.(2.35) is the most straight-forward because all one has to do is replace the region of integration with the p-adic field $\mathbb{Q}_p$ and replace the standard absolute value with the p-adic one $|\cdot|_p$. To generalize the remaining two formulas to the p-adic number field we have to rewrite them in such a way that the integration region for each variable becomes the real line, rather than a segment thereof. One can do this by introducing a sign function. Suppose we have a function $F(x)$ which is real for $x \in [0,1]$ and is such that $|F(x)| = F(x)$. Then, consider the integral

$$\int_0^1 dx F(x) = \int_0^1 dx |F(x)|. \tag{3.1}$$

Then we can use the sign function to write the integral as

$$\int_{\mathbb{R}} dx |F(x)| \frac{1}{2}(1 \pm \text{sign}(\pm 1)\text{sign}(x)\text{sign}(1-x)) \tag{3.2}$$

The purpose of this particular combination of sign functions we have used is to make contributions from regions $x \in (-\infty, 0)$ and $x \in (1, \infty)$ vanish. The only problem with the rewriting of this integral is that it is not unique in the sense that we can just as well use the combination

$$\frac{1}{2}(1 \pm \text{sign}(\pm 1)\text{sign}(x)\text{sign}(1-x))G(x) \tag{3.3}$$

for any $G(x)$ such that $G(x) = 1$ for $x \in [0,1]$.

The p-adic analog of the amplitude (2.35) is given by

$$A_N^{(p)} = \int_{\mathbb{Q}_p} \prod_{i=4}^{N} dz_i \prod_{j=4}^{N} \frac{|z_{1j}|_p^{a_{ij}} |z_{2j}|^{a_{2j}}}{|z_{1j}|_p ||z_{2j}|_p} \prod_{4 \leq i < j \leq N} \frac{|z_{ij}|_p^{a_{ij}}}{|z_{ij}|_p}, \tag{3.4}$$

where $z_{ij} = z_i - z_j$. As examples, the four-point and five-point p-adic scattering amplitudes are now given by

$$A_4^{(p)} = \int_{\mathbb{Q}_p} dx |x|_p^{a_{14}-1} |1-x|_p^{a_{24}-1}, \tag{3.5}$$

$$A_5^{(p)} = \int_{\mathbb{Q}_p} dx dy |x|_p^{a_{14}-1} |1-x|_p^{a_{24}-1} |y|_p^{a_{15}-1} |1-y|_p^{a_{25}-1} |x-y|_p^{a_{45}-1}. \tag{3.6}$$

After some manupilations, the foru-point amplitude can be found to be

$$A_4^{(p)} = \int_{|x|_p<1} dx |x|_p^{a-1} |1-x|_p^{b-1} + \int_{|x|_p<1} dx |x|_p^{b-1} |1-x|_p^{c-1} + \int_{|x|_p<1} dx |x|_p^{c} |1-x|_p^{a-1} + \left(\int_{|x|_p=1} - \int_{|x|_p<1} \right) dx, \tag{3.7}$$

where $a = a_{34}, b = a_{14}$ and $c = a_{24}$.

Before proceeding further, it is useful to to note down the following identities for p-adic integrals

$$\int_{|x|_p \leq q^k} dx = q^k \quad \text{and} \quad \int_{|x|_p = q^k} dx = q^k - q^{k-1}. \tag{3.8}$$

These formulas can be used as a definition of p-adic integrals. In particular, if $|x|_p^a$ is the charecter then, we have

$$\int_{|x|_p \leq q^k} |x|_p^a dx = \sum_{m=-k+1}^{\infty} \int_{|x|_p = q^{-m}} |x|_p^a dx = \left(\frac{q-1}{q} \right) \frac{q^{(k-1)(a-1)}}{1 - q^{-(a+1)}}. \tag{3.9}$$

Lets now turn our attention back to eqn.(3.7). The first three terms simplify considerably. This is because of the ultrametric property of p-adics. In particular, one has $|1-x|_p = 1$ if $|x|_p < 1$. With this observations, the integrals are easy to compute using eqn.(3.9). The last term of the integral is known as the "contact-interaction term". Eqn(3.9) then explicitly reads

$$A_4^{(p)} = \frac{1-p}{p}\frac{1}{1-p^a} + \frac{1-p}{p}\frac{1}{1-p^b} + \frac{1-p}{p}\frac{1}{1-p^c} + \frac{p-2}{p}. \tag{3.10}$$

Similarly, the 5-point amplitude is explictly given by

$$A_5^{(p)} = \sum_{[(ij)(kl)]} \frac{(1-p)/p}{1-p^{a_{ij}}}\frac{(1-p)p}{1-p^{a_{kl}}} + \left(\frac{p-2}{p}\right)\sum_{(ij)}\frac{(1-p)/p}{1-p^{a_{ij}}} + \left(\frac{p-2}{p}\right)\left(\frac{p-3}{p}\right). \tag{3.11}$$

The first to examples are highly instructive on what the general N-point amplitude A_N looks like. Indeed, it turns out that each term of the N point amplitude contains a factor

$$D^{(p)}(a) = \int_{|x|_p<1} dx|x|_p^{a-1} = \frac{(1-p)/p}{1-p^a}. \tag{3.12}$$

Furthermore, the contact term for the a N-point interaction is given by

$$V^p(N) = \prod_{j=2}^{N-2}\left(\frac{p-j}{p}\right). \tag{3.13}$$

Let us now move on to the next p-adic string theory where the Baradecki Ruegg formula is generalized to the p-adic number field. In order to do so, we shall first have to generalize the sign function to the p-adic sign function.

Now, the real sign function can be constructed to a quadratic extension of $\mathbb{R}$. Moreover, the statement $x > 0$ and $x \in \mathbb{R}$ means that the number x can be written as the modulus of a complex number i.e $x = (a-ib)(a+ib) = a^2 - i^2b^2$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. If a real number x cannot be expressed as the modulus of a complex number, then we call it a negative real number.

The above definition of the sign can be extended to the p-adic field. However, the difference is that in general (for $p \neq 2$) there are three independent quadratic extensions for the p-adic field. They are denoted $\mathbb{Q}_p(\sqrt{D})$ with $D = \epsilon$ or $D = p$ or $D = \epsilon p$, where ϵ is the $(p-1)$st root of 1. In general, there are three possible choices for the sign function that one can use for the p-adic number. The choice depends on the quadratic extension we consider. Therefore, in general there is the following definition of the sign of the p-adic number

$$\mathrm{sign}_D(x) = \begin{cases} +1 & \text{if } x = z\bar{z} \text{ with } z \in \mathbb{Q}_p(\sqrt{D}) \\ -1 & \text{otherwise} \end{cases} \tag{3.14}$$

where in the above definition we have used the notation $z = a + \sqrt{D}b \in \mathbb{Q}_p(\sqrt{D})$ and $a, b \in \mathbb{Q}_p$, and $\bar{z} = a - \sqrt{D}b$.

In p-adic string theory, it is more suitable to write the Koba-Nielson amplitude (2.33) in the form of (3.2) with $-\mathrm{sign}(-1)$ instead of $+1$, so as to have a more complete analysis of the hidden algebraic structures. The N point Koba Neilson scattering amplitude for p-adic theory is then given by

(3.15)
$$B_N^{KN}(p) = \int_{\mathbb{Q}_p} \prod_{i=4}^{N} dz_i |z_i|_p^{a_{1i}-1} |1-z_i|_p^{a_{2i}-1} \prod_{4\le j\le N} |z_i - z_j|_p^{a_{ij}-1} \prod_{i=4}^{N} \frac{1}{2}\left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(z_i)\mathrm{sign}_D(1-z_i)\right) G(x).$$

To examine the theory it is useful to start with the four- and five- point functions,

$$B_N^{KN}(p) = \frac{1}{2}\int_{\mathbb{Q}_p} dx |x|_p^{a-1} |1-x|_p^{b-1} \left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(x)\mathrm{sign}_D(1-x)\right) G(x). \tag{3.16}$$

$$\begin{aligned} B_N^{KN}(p) = \frac{1}{8}\int_{\mathbb{Q}_p} dxdy |x|_p^{a_{14}-1} |1-x|_p^{a_{24}-1} |y|_p^{a_{15}-1} |1-y|_p^{a_{25}-1} |x-y|_p^{a_{45}-1} \\ \left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(x)\mathrm{sign}_D(1-x)\right)\left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(y)\mathrm{sign}_D(1-y)\right) \\ \left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(x-y)\right) G(x,y). \end{aligned} \tag{3.17}$$

For simplicity, we will work with the choice $G(x) = 1$everywhere. For the case $D = \epsilon$, we have

$$B_{4\epsilon}^{KN}(p) = \frac{1}{2}\frac{1-p}{p}\left\{\frac{1}{1-p^a} - \frac{1}{1+p^a} + \frac{1}{1-p^b} - \frac{1}{1+p^b}\right\}, \tag{3.18}$$

which can be further rewritten using the "hyperbolic sine" function

$$B_{4\epsilon}^{KN(p)} = \frac{1}{2}\frac{p-1}{p}\left\{\frac{1}{\sinh(a\ln p)} + \frac{1}{\sinh(b\ln p)}\right\}. \tag{3.19}$$

By adding all non-cyclic permutations of momenta, one finds

$$A_{4\epsilon}^{KN(p)} = \frac{p-1}{p}\left\{\frac{1}{\sinh(a\ln p)} + \frac{1}{\sinh(b\ln p)} + \frac{1}{\sinh(c\ln p)}\right\}. \tag{3.20}$$

Now we return to the cases $D = p, p\epsilon$. We find that these two cases yield the same scattering amplitude. In particular,

$$B_{4\epsilon p,p}^{KN(p)} = \frac{1}{2}\frac{p-1}{p}\left\{-1 + \frac{1}{1-p^a} + \frac{1}{1-p^b}\right\}. \tag{3.21}$$

By adding all remaining channels, one finds

$$A_{4\epsilon p,p}^{KN(p)} = \frac{1+p}{2p} + \frac{1-p^{a-1}}{1-p^{-a}}\frac{1-p^{b-1}}{1-p^{-b}}\frac{1-p^{c-1}}{1-p^{-c}}. \tag{3.22}$$

Although the four point amplitude has a relatively simple and nice explicit form, the five-point amplitude is much more complicated and is beyond the scope of the present manuscript.

The next p-adic string theory is obtained by generalizing the Bardakci-Ruegg formula for real string amplitudes. The N-point scattering amplitude in this theory is given by

(3.23)
$$B_N^{BR(p)} = \frac{1}{2}\int_{\mathbb{Q}_p} \prod_{i=2}^{N-2} dz_i |z_i|_p^{a(i)-1} \prod_{2\le i<j\le N-1} |1 - \prod_{k=1}^{j-1} z_k|_p^{a_{ij}-1} \prod_{i=2}^{N-2} \left(1 - \mathrm{sign}_D(-1)\mathrm{sign}_D(z_i)\mathrm{sign}_D(1-z_i)\right) G(z_i)$$

Computing the explicit four point and five point amplitudes of the above amplitude is difficult and is beyond the scope of the present manuscript.

4. Conclusions

In this manuscript, we have presented an introductory overview of p-adic string scattering amplitudes with an emphasis on their analytic and algebraic structure. Beginning from the basic framework of bosonic string theory, we reviewed the Nambu-Goto and Polyakov formulations of relativistic strings and discussed the origin of perturbative string scattering amplitudes through worldsheet methods. We then introduced the Veneziano amplitude together with its Koba-Nielsen and Bardakci-Ruegg representations, which serve as the starting point for p-adic generalizations.

The central idea underlying p-adic string theory is that many of the integral formulas appearing in ordinary string amplitudes admit natural non-Archimedean analogues obtained by replacing the real number field with $\mathbb{Q}_p$ and the ordinary absolute value with the p-adic norm. Owing to the ultrametric structure of the p-adic field, these amplitudes often simplify considerably and can frequently be expressed in explicit closed form. In particular, we examined the structure of four-point and five-point amplitudes and illustrated the emergence of contact interaction terms characteristic of the p-adic formulation.

We also discussed generalized p-adic Koba-Nielsen amplitudes involving p-adic sign functions associated with quadratic extensions of $\mathbb{Q}_p$. These constructions reveal additional hidden algebraic structures and demonstrate that multiple inequivalent p-adic analogues of ordinary string amplitudes may arise depending on the arithmetic data chosen. Such features highlight the deep interplay between number theory, ultrametric analysis, and string-theoretic scattering amplitudes.

Although the present manuscript has focused primarily on tree-level amplitudes and their explicit evaluation, p-adic string theory extends far beyond these examples. Connections with adelic physics, nonlocal effective field theories, tachyon condensation, and arithmetic geometry continue to motivate ongoing research in the subject. Moreover, p-adic models often provide analytically tractable settings in which phenomena that are difficult to study in ordinary string theory can be explored in greater detail.

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