Expanders and Ramanujan Graphs

Sparse Graphs with Strong Connectivity

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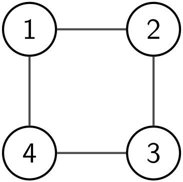
Euler Circle

Graphs

Definition

A *graph* is a pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges* connecting pairs of vertices.



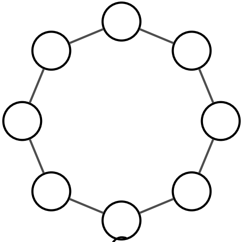
In this talk, all graphs are finite, simple, and undirected, so an edge is an unordered pair $\{u, v\}$.

For this graph,

$$V = \{1, 2, 3, 4\}, \qquad E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}.$$

Cycle Graphs

The graph above is the cycle graph C_4. More generally, C_n has n vertices arranged in a closed loop.



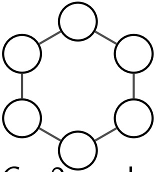
C_8

Every vertex in C_n has exactly two neighbors.

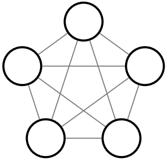
Degree and Regularity

Definition

The *degree* of a vertex is the number of neighbors it has. A graph is *k-regular* if every vertex has degree k.



C_6: 2-regular



K_5: 4-regular

Regular graphs are uniform, which makes their eigenvalues easier to interpret.

The Handshaking Lemma

Lemma

For any finite graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof.

The left side counts endpoint-edge incidences. Each edge has exactly two endpoints, so each edge is counted twice. ∎

Fixed Degree Means Sparse

If G is k-regular with n vertices, then

$$kn = 2|E|, \qquad \text{so} \qquad |E| = \frac{kn}{2}.$$

For fixed k, this is linear in n. By contrast, the complete graph has

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2},$$

which is quadratic in n.

Connectedness Is Not Enough

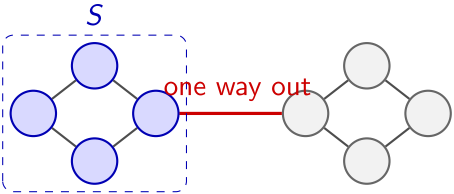
Definition

A graph is *connected* if every two vertices can be joined by a path.

Connectedness means the graph is in one piece. But a connected graph can still have a bottleneck: a large region attached to the rest by only a few edges.

Expansion is a quantitative way to rule out bottlenecks.

Bottleneck Picture



The set S is large, but only one edge leaves it.

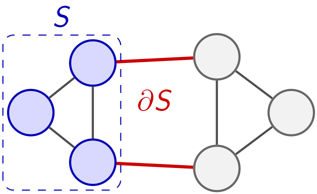
Edge Boundary

Definition

Let $G = (V, E)$ and let $S \subseteq V$. The *edge boundary* of S is

$$\partial S = \{\{u, v\} \in E : u \in S,\ v \notin S\}.$$

Expansion compares $|\partial S|$ with $|S|$.



Edge Expansion

Definition

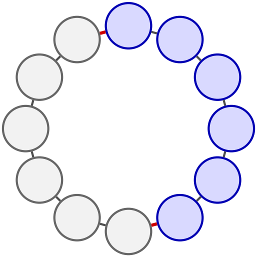
The *edge expansion* of G is

$$h(G) = \min_{0<|S|\leq|V|/2} \frac{|\partial S|}{|S|}.$$

The restriction $|S| \leq |V|/2$ avoids checking both S and its complement. Large $h(G)$ means that every small set has many edges leaving it.

Computing Expansion: Cycles

In C_n, choose S to be $n/2$ consecutive vertices, assuming n is even.



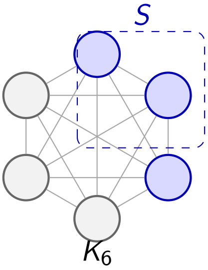
Then $|\partial S| = 2$ and $|S| = n/2$, so

$$h(C_n) \leq \frac{2}{n/2} = \frac{4}{n} \to 0.$$

Computing Expansion: Complete Graphs

In K_n, if $|S| = r$, every vertex in S connects to every vertex outside S.

$$|\partial S| = r(n-r), \qquad \frac{|\partial S|}{|S|} = n - r.$$



Since $r \leq n/2$,

$$h(K_n) \geq \frac{n}{2}.$$

Complete graphs expand extremely well, but their degree is $n-1$, so they are not sparse.

Expander Families

Definition

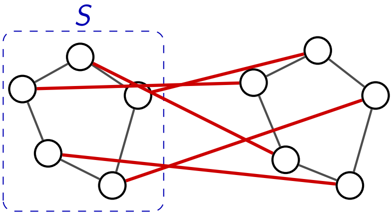
A family $\{G_m\}$ of finite k-regular graphs is an *expander family* if

$$|V(G_m)| \to \infty \qquad \text{and} \qquad h(G_m) \geq \varepsilon > 0$$

for every m.

The degree stays fixed, but the expansion stays positive. This is the sparse but robust behavior we want.

An Expander-Like Picture



Even if S contains many vertices, many edges cross from S to the rest of the graph.

Why Bring in Linear Algebra?

The number $h(G)$ is combinatorial, but it can be hard to compute directly.

Spectral graph theory gives another measurement: the eigenvalues of the adjacency matrix.

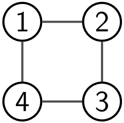
For regular graphs, these eigenvalues control expansion and random walks.

Adjacency Matrix

Definition

If G has vertices $v_1, \ldots, v_n$, its *adjacency matrix* is

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$



$$A_{C_4} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Why the Matrix Is Symmetric

Lemma

If G is a finite simple undirected graph, then A_G is symmetric and has zeros on the diagonal.

Proof.

Because the graph is undirected,

$$\{v_i, v_j\} \in E \iff \{v_j, v_i\} \in E,$$

so $(A_G)_{ij} = (A_G)_{ji}$. Since the graph is simple, no vertex is adjacent to itself, so $(A_G)_{ii} = 0$. ■

Walks from Matrix Powers

Proposition

The entry $(A_G^m)_{ij}$ is the number of walks of length m from v_i to v_j.

Example

For C_4, the entry $(A_{C_4}^2)_{11}$ is 2, corresponding to

$$1 \to 2 \to 1 \qquad \text{and} \qquad 1 \to 4 \to 1.$$

Eigenvalues and Spectrum

A number λ is an eigenvalue of a matrix A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some nonzero vector $\mathbf{x}$.

The eigenvalues of A_G are called the *spectrum* of G. Since A_G is symmetric, they are real, and we write

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

The Trivial Eigenvalue

Proposition

If G is k-regular, then k is an eigenvalue of A_G with eigenvector $\mathbf{1} = (1, 1, \ldots, 1)$.

Proof.

Every row of A_G has exactly k entries equal to 1, so every row sum is k. Therefore

$$A_G \mathbf{1} = k\mathbf{1}.$$

∎

The Largest Eigenvalue

If G is connected and k-regular, then Perron–Frobenius implies

$$\lambda_1 = k.$$

So the eigenvalue k is unavoidable and does not measure expansion. It only records that every vertex has degree k.

This is why we focus on the eigenvalues other than k.

Bipartite Symmetry

Proposition

If G is bipartite and λ is an eigenvalue of A_G, then $-\lambda$ is also an eigenvalue.

Proof.

With respect to the bipartition,

$$A_G = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Changing the signs of the coordinates on one side of the bipartition turns an eigenvector for λ into one for $-\lambda$. ∎

Trivial and Nontrivial Eigenvalues

For a connected k-regular graph, the eigenvalue k is called *trivial*.

If the graph is bipartite, then $-k$ is also treated as trivial.

All other eigenvalues are called *nontrivial*; these are the eigenvalues that measure expansion.

The Number $\lambda(G)$

For a connected k-regular graph, define

$$\lambda(G) = \max\{|\lambda| : \lambda \text{ is a nontrivial eigenvalue of } A_G\}.$$

Small $\lambda(G)$ means all nontrivial eigenvalues are small in absolute value.

If G is connected and not bipartite, then

$$\lambda(G) = \max\{|\lambda_2|, |\lambda_n|\}.$$

The quantity $k - \lambda(G)$ is the *spectral gap*.

Example: Spectrum of C_4

The eigenvalues of C_4 are

$$2, \quad 0, \quad 0, \quad -2.$$

Here $k = 2$. Since C_4 is bipartite, both 2 and -2 are trivial eigenvalues.

The remaining nontrivial eigenvalues are 0 and 0.

Bottleneck Intuition

If a graph has a bottleneck, then a vector can be almost constant on one side and almost constant on the other.

That vector behaves almost like an eigenvector with a large nontrivial eigenvalue.

So small nontrivial eigenvalues prevent bottlenecks.

Rayleigh Quotient Viewpoint

For a symmetric matrix A, eigenvalues can be detected by quotients of the form

$$\frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

For a k-regular graph, vectors perpendicular to $\mathbf{1}$ ignore the trivial eigenvalue k.

A bottleneck produces a vector perpendicular to $\mathbf{1}$ with a large Rayleigh quotient.

Cheeger's Inequality

Theorem (Cheeger's inequality)

Let G be a connected k-regular graph, and let λ_2 be the second largest eigenvalue of A_G. Then

$$\frac{k - \lambda_2}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_2)}.$$

What Cheeger Tells Us

Cheeger's inequality turns eigenvalue information into expansion information.

If $\lambda_2 \leq k - \delta$, then

$$h(G) \geq \frac{k - \lambda_2}{2} \geq \frac{\delta}{2}.$$

A large spectral gap forces strong edge expansion.

The Optimality Question

We now know that small nontrivial eigenvalues imply expansion.

The next question is: for fixed degree k, how small can the nontrivial eigenvalues be in an infinite family of larger and larger graphs?

The answer is controlled by the Alon–Boppana theorem.

Theorem (Alon–Boppana)

If $\{G_m\}$ is an infinite family of connected k-regular graphs with $|V(G_m)| \to \infty$, then

$$\liminf_{m\to\infty} \lambda_2(G_m) \geq 2\sqrt{k-1}.$$

So $2\sqrt{k-1}$ is an unavoidable spectral barrier.

The Ramanujan Bound

The number

$$2\sqrt{k-1}$$

is called the *Ramanujan bound* for k-regular graphs.

k	$2\sqrt{k-1}$
3	$2\sqrt{2} \approx 2.828$
4	$2\sqrt{3} \approx 3.464$
5	4
10	6

Ramanujan Graphs

Definition

A finite connected k-regular graph G is a *Ramanujan graph* if

$$\lambda(G) \leq 2\sqrt{k-1}.$$

Equivalently, every nontrivial eigenvalue λ satisfies

$$|\lambda| \leq 2\sqrt{k-1}.$$

Unpacking the Definition

To check that a graph is Ramanujan, we need:

- the graph is connected and k-regular,
- the adjacency eigenvalues of the graph,
- the nontrivial eigenvalues only,
- the comparison bound $2\sqrt{k-1}$.

The bound depends only on the degree k.

Why Ramanujan Graphs Are Optimal

Alon–Boppana says infinite fixed-degree families cannot do better than

$$2\sqrt{k-1}$$

asymptotically.

Ramanujan graphs meet this threshold, so they are optimal from the spectral point of view.

Using Cheeger's inequality, a k-regular Ramanujan graph satisfies

$$h(G) \geq \frac{k-2\sqrt{k-1}}{2}.$$

Example: Complete Graphs

The complete graph K_n is $(n-1)$-regular and has spectrum

$$n-1, \quad -1, \quad -1, \quad \ldots, \quad -1.$$

The nontrivial eigenvalue is -1, and

$$|-1| = 1 \leq 2\sqrt{n-2}.$$

Thus every K_n is Ramanujan, but this is not a sparse fixed-degree family.

Example: Cycle Graphs

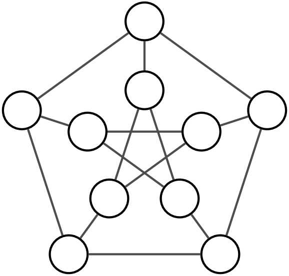
The cycle graph C_n is 2-regular and has eigenvalues

$$2\cos\left(\frac{2\pi j}{n}\right), \qquad j = 0, 1, \ldots, n-1.$$

Since $|2\cos(2\pi j/n)| \leq 2$ and the Ramanujan bound for $k = 2$ is 2, cycles satisfy the Ramanujan inequality.

But $\{C_n\}$ is not an expander family, because $h(C_n) \to 0$.

The Petersen Graph



The Petersen graph is 3-regular on 10 vertices.

The Petersen Graph Is Ramanujan

The spectrum of the Petersen graph is

$$\{3, 1^{(5)}, (-2)^{(4)}\}.$$

The Ramanujan bound for 3-regular graphs is

$$2\sqrt{3-1} = 2\sqrt{2} \approx 2.828.$$

The nontrivial eigenvalues are 1 and -2, and both have absolute value at most $2\sqrt{2}$.

Bipartite Ramanujan Graphs

If G is bipartite and k-regular, then $-k$ is also a trivial eigenvalue.

So a bipartite k-regular graph is Ramanujan when every remaining eigenvalue satisfies

$$|\lambda| \leq 2\sqrt{k-1}.$$

This distinction matters for the Marcus–Spielman–Srivastava theorem.

Individual Graphs vs. Families

A single graph can have unusually good eigenvalues.

The deeper problem is to construct infinite families of fixed-degree Ramanujan graphs with

$$|V(G_m)| \to \infty.$$

That is why Ramanujan graph constructions are a major part of the subject.

Cayley Graphs

Many explicit expander constructions use groups.

Definition

Given a group Γ and a generating set S, the *Cayley graph* has vertex set Γ, with edges connecting g to gs for $s \in S$.

Group structure can force strong expansion in a graph that is still regular and sparse.

Lubotzky–Phillips–Sarnak

Lubotzky, Phillips, and Sarnak constructed explicit Ramanujan graphs using groups such as

$$\mathrm{PSL}_2(\mathbb{F}_q) \qquad \text{and} \qquad \mathrm{PGL}_2(\mathbb{F}_q).$$

For suitable primes p and q, their graphs are $(p+1)$-regular, so the Ramanujan bound is

$$2\sqrt{(p+1)-1} = 2\sqrt{p}.$$

Their construction achieves this bound for every nontrivial eigenvalue.

Marcus–Spielman–Srivastava

Marcus, Spielman, and Srivastava proved that for every degree $k \geq 2$, there exist infinite families of bipartite k-regular Ramanujan graphs.

Their proof uses interlacing polynomials and graph lifts rather than the number-theoretic machinery of LPS.

This answered a major existence question for Ramanujan graph families.

Networks

Ramanujan graphs are good models for efficient networks.

- Bounded degree means each node has few connections.
- Expansion means there are no easy bottlenecks.
- Small eigenvalues imply rapid spreading of information.

This is why expanders appear in network design and communication problems.

Error Correction and Algorithms

Expanders also appear in theoretical computer science.

- In coding theory, expansion helps spread local information globally.
- In algorithms, expanders help reduce randomness.
- In complexity theory, expander walks simulate independent random choices efficiently.

The same spectral condition supports many different applications.

Takeaway

The story is:

sparse graphs $\longrightarrow$ expansion $\longrightarrow$ eigenvalues $\rightarrow$ Ramanujan graphs

Ramanujan graphs are sparse graphs whose nontrivial eigenvalues are as small as possible in infinite fixed-degree families.

Their definition is spectral, but their meaning is combinatorial: small eigenvalues force strong global connectivity.

Thank You

Thank you!

Thank you to Euler Circle for this program.

Special thanks to Simon and to Shahzar Rizvi, my teaching assistant.