On Ordinals and Order Types

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Euler Circle

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Introduction of Ordinals

- Ordinal numbers offer a clear way to describe order and position in well-ordered sets.
- Developed by Georg Cantor, a German mathematician.
- John von Neumann introduced the standard set-theoretic construction of ordinals we use today.

Background Information

Given S, it is said to have a **partial ordering** for $\forall s \in S$:

- s is transitive, meaning that $s \leq s$.
- s is asymmetrical, meaning if $t \leq s$ and $s \leq t$, then $s = t$.
- s is reflective, meaning if $s \leq t$, $t \leq w$ then $s \leq w$.

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Finally, the set is said to have a **well ordering** if:

- S has total ordering.
- If S is nonempty, S has a least element.

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Example

Given a set S,

$$\begin{aligned} S &= \{1, 2, 3, 4\} \\ 1 &= \{2, 3, 4\} \\ 2 &= \{4\} \\ 3 &= \{2\} \\ 4 &= \{2, 3\} \end{aligned}$$

S is transitive.

Property of Ordinals

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Example

For example, this is the first 3 Ordinal numbers:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} \\ 2 &= \{0, 1\}. \end{aligned}$$

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Note that ordinals are worked in the standard Zermelo-Fraenkel set theory with the axiom of choice (ZFC), so all elements in sets are themselves sets.

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Axiom

For all non-empty sets A, $\exists x \in A$ such that $x \cap A = \emptyset$.

This axiom prevents any containment in the form of

$$x_1 \ni x_2 \ni x_3 \ldots$$

Property of Ordinals

Theorem

$x \notin x$ for all set x.

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Theorem

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Proof.

Assume $x \in x$, then this would result in a infinite chain of x shown as $x \ni x \ni x \ldots$. This would violate foundation, making it a contradiction. ■

Von Neumann's Construction

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Proposition

$\emptyset$ is an ordinal.

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Proof.

For $\emptyset$ to be transitive, $x \in y \in \emptyset$ and $x \in \emptyset$. $\emptyset$ does not contain any elements so it would not contain x or y, making it vacuously true. $\emptyset$ is also well ordered since it is not non-empty (empty), the condition is vacuously true. Therefore $\emptyset$ is a ordinal. ∎

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For all ordinals, every member of a ordinal is a ordinal

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Example

Recall the first three ordinals:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} \\ 2 &= \{0, 1\}. \end{aligned}$$

Von Neumann's Construction

Proposition

If α is a ordinal, then $\alpha \cup \{\alpha\}$ is also a ordinal.

Proof.

For a set α to be transitive, if $x \in y \in \alpha$, then $x \in \alpha$. Assume that $x \in y \in \alpha \cup \{\alpha\}$, then either $x \in y \in \alpha$ or $x \in y = \alpha$. $x \in \alpha$ either way, so the set of $\alpha \cup \{\alpha\}$ is transitive. Every element of $\alpha \cup \{\alpha\}$ is transitive, making this a ordinal ■

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$$4 = \{0, 1, 2\} \cup \{3\} = \{0, 1, 2, 3\}$$

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1. The ordinal 0: The $\emptyset$ itself.
2. Successor Ordinal: any ordinal that can be expressed as $\{\alpha\} \cup \alpha = \alpha' + 1$.
3. Limit Ordinal: any ordinal that is non-zero and not a successor ordinal.

Limit Ordinals

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There exists a set containing all the natural numbers,

$$\exists x(\emptyset \in x \text{ and } \forall y \in x(y' + 1 \in x)).$$

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ω is non-zero. Assume that $\omega = \alpha' + 1$ for some $\alpha \in \omega$, then α is a finite natural number and the largest element in ω. $\alpha' + 1$ is a successor ordinal and also a finite ordinal. This means $\alpha' + 1 \in \omega$ since ω contains all the finite natural numbers. This creates a contradiction. ∎

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We can now construct ordinals in the form of $\omega + 1, \omega + 2 \ldots$ Notice that ω is a limit ordinal while $\omega + 1$ is however, a successor ordinal.

Relations of Ordinals

Proposition

$\alpha \leq \beta \iff \alpha \subseteq \beta$

Proposition

$\alpha < \beta \iff \alpha \subset \beta$

Proposition

If α and β are ordinals, then $\alpha = \beta$, $\alpha \in \beta$ or $\beta \in \alpha$

Bureli-Forti's Paradox

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Construct a set ON such that:

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$\alpha \in ON$ for any arbitrary ordinal α. This also implies that $\alpha < ON$. ON is greater than every ordinal.

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We claim that $x \in y \in ON$, this means that y is a ordinal. this implies x is a ordinal. And thus, by the definition of ON, $x \in ON$, proving that ON is transitive.
ON is also well ordered and non-empty, making it seemingly a ordinal.

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This contradiction shows that there is no largest ordinal. It caused mathematicians to distinguish between two types of collections: sets, and proper classes.

Addition in the ordinals

We define addition in the ordinals recursively as:

1. $$\alpha + 0 = \alpha$$
2. $$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$
3. $$\alpha + \beta = \mathit{sup}(\alpha + \gamma : \gamma < \beta)(\beta \text{ is a limit ordinal}).$$

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Let us consider the addition $1 + \omega$. We can write this as $\sup(1 + \gamma : \gamma < \omega)$ This is, however, just the ordinal that is greater than the sum of the ordinals less than ω. However, the sum of the ordinals less than a limit ordinal is always going to be below the limit ordinal so $1 + \omega$ would just be equal to ω.

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However, if we are doing addition on $\omega + 1$. This would mean that $\omega + 1 = \omega \cup \{\omega\}$. This would mean that $\omega \in \omega + 1$. This implies that $\omega + 1 > \omega$.

Addition in the ordinals

Figure: $1 + \omega$

Addition in the ordinals

Figure: $1 + \omega$

Figure: $\omega + 1$

Multiplication on the Ordinals

Multiplication follows a similar format:

1. $$\alpha * 0 = 0$$
2. $$\alpha * (\beta + 1) = \alpha * \beta + \alpha$$
3. $$\alpha * \beta = \text{sup}(\alpha * \gamma : \gamma < \beta)(\beta \text{ is a limit ordinal}).$$

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If A is a ordinal, then $\bigcup A$ *or* $\sup(A)$ *is also a ordinal.*

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If A is a ordinal, then $\bigcup A$ or $\sup(A)$ is also a ordinal.

Figure: $\omega * 2$

Exponentiation on the Ordinals

1.
$$\alpha^0 = 1$$

2.
$$\alpha^{\beta+1} = \alpha^\beta + \alpha$$

3.
$$\alpha^\beta = \sup(\alpha^n : n < \beta)(\beta \text{ is a limit ordinal}).$$

Exponentiation on the Ordinals

Proposition

$a^{\omega} = \omega$ *for* $1 < a < \omega$

Exponentiation on the Ordinals

Figure: ω^2

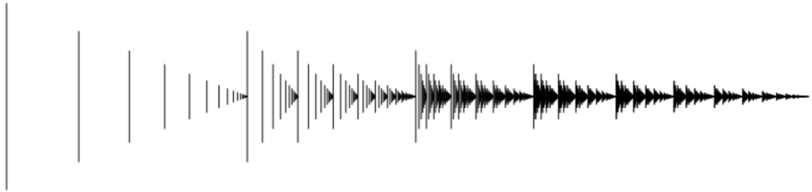

Figure: ω^ω

ε Numbers

Definition

A ordinal ε is called a epsilon number if ε is the fixed point of an exponential map.

In other words, epsilon numbers are a collection of transfinite numbers such that

$$\omega^{\varepsilon} = \varepsilon.$$

The ordinal ε_0 may be constructed by repeatedly exponentiating ω by itself. Define a recursive sequence of ordinals (α_n) by:

$$a_0 = \omega$$

$$a_1 = \omega^{a_0} = \omega^{\omega}$$

$$a_2 = \omega^{a_1} = \omega^{\omega^{\omega}}.$$

ε Numbers

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$$\varepsilon_0 = \sup(a_0, a_1, a_2 \ldots) = \sup_{n<\omega}(a_n).$$

$$\varepsilon_0 = \sup(\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}} \ldots).$$

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The next epsilon numbers can be constructed by similar logic:
Define a recursive function such $\beta_0 = \varepsilon_0 + 1$, $\beta_{n+1} = \omega^{\beta_n}$. Finally define ε_1 as:

$$\varepsilon_1 = \sup_{n<\omega}(\beta_n) = \{\omega^{\varepsilon_0+1}, \omega^{\omega^{\varepsilon_0+1}} ...\}.$$

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Now define another function γ such that $\gamma_0 = \epsilon_1 + 1, \gamma_{n+1} = \omega^{\gamma_n}$. Define ε_2 as

$$\varepsilon_2 = \sup_{n<\omega}(\gamma_n) = \{\omega^{\varepsilon_1+1}, \omega^{\omega^{\varepsilon_1+1}} ...\}.$$

ε Numbers

Theorem

The ordinals ε are a closed point in the function $f(\alpha) = \omega^\alpha$. That is:

$$\omega^\varepsilon = \varepsilon.$$

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Proposition

Every Epsilon number is a limit ordinal.

ε Numbers

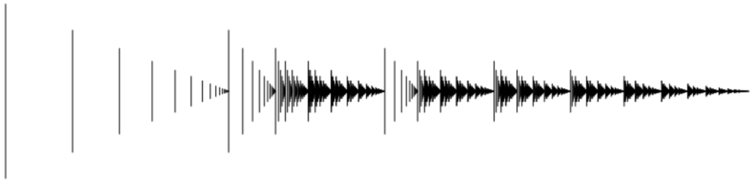

Figure: ε_0

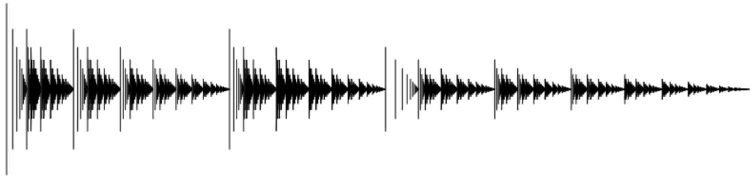

Figure: ε_1

Structure of all Ordinals

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots$$
$$\omega * 2, \omega * 2 + 1, \ldots, \omega * 3, \omega * 3 + 1, \ldots$$
$$\omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \ldots$$
$$\omega^3, \omega^3 + 1, \ldots, \omega^3 + \omega, \omega^3 + \omega + 1, \ldots$$
$$\omega^3 + \omega^2, \omega^3 + \omega^2 + 1, \ldots, \omega^3 + \omega^2 + \omega, \ldots$$
$$\omega^\omega, \ldots, \omega^\omega + \omega, \omega^\omega + \omega + 1, \ldots$$
$$\omega^\omega * 2, \omega^\omega * 2 + 1, \ldots, \omega^\omega * \omega, \omega^\omega * \omega + 1, \ldots$$
$$\omega^\omega * 2, \omega^\omega * 2 + 1, \ldots, \omega^\omega * \omega, \omega^\omega * \omega + 1$$
$$\omega^{\omega^2}, \ldots, \omega^{\omega^3}, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots$$
$$\varepsilon_0, \varepsilon_1, \ldots$$

Axiom of Choice

The ordinals can be applied in the axiom of choice.

Axiom (Axiom of Choice)

Every set has a choice function

Axiom (Well Ordering Principle)

All sets can be well ordered

Axiom (Zorn's Lemma)

if a partially ordered set has the property that every totally ordered subset has an upper bound within the set, then the set contains at least one maximal element

Axiom (Turkey's Lemma)

If a collection of sets U is of finite character then U contains maximal sets under the ordering $\subseteq$

Hartog's Number

Definition

Given a set S, Hartog's number for the set S, $h(S)$, is defined to be the least ordinal such that there is no one to one function such that $f : h(S) \to S$

Lemma

For a set S:

$$h(S) = \{\alpha \in ON : \exists \text{ Bijection } f : \alpha \to T\} \text{ for some } T \subseteq S$$

Well ordering theorem

Theorem

Given the set A is a collection of non-empty sets. If for any such set A, there is a choice function, then there exists a relation "$\preceq$" such that it is a well-ordering on S.

Questions?

Thank you for listening!
Any questions?