# Orthogonal Functions

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# Inner Products

**Definition (inner product)**

Let $V$ be a vector space (over $\mathbb{C}$). An inner product on $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that the following holds true:

1. $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all $\mathbf{v} \in V \setminus \{\mathbf{0}\}$.
2. $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$ for all $\mathbf{v}, \mathbf{u} \in V$.
3. $\langle \mathbf{v} + \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$.
4. $\langle \alpha \mathbf{v}, \mathbf{u} \rangle = \alpha \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\alpha \in \mathbb{C}$ and $\mathbf{v}, \mathbf{u} \in V$.

# Norm

## Definition (norm)

Let $V$ be a vector space (over $\mathbb{C}$). A norm on $V$ is a map $||\cdot|| : V \to \mathbb{R}$ such that the following holds true:

1. $||\mathbf{v}|| \geq 0$ for all $\mathbf{v} \in V$ with $||\mathbf{v}|| = 0 \iff \mathbf{v} = \mathbf{0}$.
2. $||\alpha \mathbf{v}|| = |\alpha| \cdot ||\mathbf{v}||$ for all $\alpha \in \mathbb{C}$ and all $\mathbf{v} \in V$.
3. $||\mathbf{v} + \mathbf{u}|| \leq ||\mathbf{v}|| + ||\mathbf{u}||$ for all $\mathbf{v}, \mathbf{u} \in V$.

$\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is a norm on $V$.

# Orthogonality

**Definition (orthogonal, orthonormal)**

Let $V$ be an inner product space. We say that vectors $\mathbf{v}, \mathbf{u} \in V$ are *orthogonal* if $\langle \mathbf{v}, \mathbf{u} \rangle = 0$. We say that a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$ is orthogonal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $1 \leq i, j \leq n, i \neq j$. We say that a set is *orthonormal* if it is orthogonal and $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ for all $\mathbf{v}_i \in S$.

# $L^2$ Space

**Definition ($L^2$ space)**

We say that $L^2$ space is the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(x)|^2 \, dx$ is finite. We say that $g : [a, b] \to \mathbb{C}$ is in $L^2([a, b])$ if $\int_a^b |f(x)|^2 \, dx$ is finite.

The inner product on $L^2$ is $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)} \, dx$, and the induced norm is $||f|| = \sqrt{\int_a^b |f(x)|^2 \, dx}$.

# Orthogonal Functions

## First Example

The set $\{e^{inx} : n \in \mathbb{Z}\}$ is orthogonal over $L^2[(-\pi, \pi)]$.

## Second Example

The Legendre polynomials $\{P_n\}$ are orthogonal over $L^2[(-1, 1)]$.
$1,\ x,\ \frac{3}{2}x^2 - \frac{1}{2},\ \frac{5}{2}x^3 - \frac{3}{2}x, \ldots$ are the first few.

# The Fourier Series

- Fourier Coefficent: $c_n = \langle f, e^{inx} \rangle$
- Fourier Series: $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$

## Square Wave Series

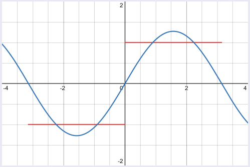


Figure: The square wave Fourier series: $\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$

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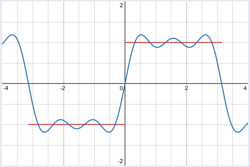


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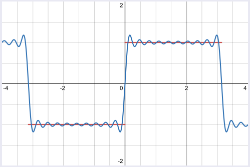


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# More Fourier Series

- Let $\{\phi_n\} \subseteq L^2$ be orthonormal.
- Fourier Coefficient: $c_n = \langle f, \phi_n \rangle$
- Fourier Series: $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$

## Legendre Square Wave Series

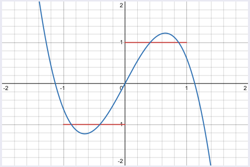


Figure: Square wave Fourier Series with respect to the Legendre Polynomials.

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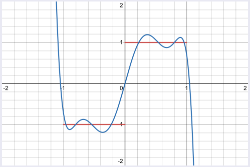

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# Bessel's Inequality

**Theorem (Bessel's inequality)**

*Let* $\{\phi_n\}$ *be orthonormal and* $f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x)$ *for some* $f \in L^2$. *Then*

$$\sum_{n=1}^{\infty} |c_n|^2 \leq ||f||^2$$

.

- $\langle f, t_n \rangle = \sum_{m=1}^{n} c_m \overline{d_m}$
- $\langle t_n, t_n \rangle = \sum_{m=1}^{n} d_m \overline{d_m}$
- $||f - t_n||^2 = ||f||^2 - \sum_{m=1}^{n} |c_m|^2 + \sum_{m=1}^{n} |d_m - c_m|^2$
- $d_m = c_m \implies ||f - t_n||^2 = ||f||^2 - \sum_{m=1}^{n} |c_m|^2$

# Orthonormal Bases

**Theorem**

*Let $\{\phi_n\}$ be orthonormal and linear combinations of $\{\phi_n\}$ be dense in $L^2$. Let $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$ for some $f \in L^2$. Then,*

$$\lim_{n\to\infty} ||f - s_n(f)||^2 = 0,$$

*where $s_n(f)$ denotes the nth partial sum of the Fourier series of $f$.*

- $||f - h|| < \frac{\varepsilon}{3}$ by completeness of $L^2$
- $||h - s_n(h)|| \leq ||h - P|| < \frac{\varepsilon}{3}$ by density of $\{\phi_n\}$
- $||s_n(h) - s_n(f)|| = ||s_n(h - f)|| \leq ||h - f|| < \frac{\varepsilon}{3}$
- $||f - s_n(f)|| < \varepsilon$ by triangle inequality

## Parseval's Identity

**Theorem (Parseval's identity)**

*Let $\{\phi_n\}$ be orthonormal and linear combinations of $\{\phi_n\}$ be dense in $L^2$. Let $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$. Then,*

$$||f||^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

- $|||f||^2 - \langle s_n(f), f\rangle| = |\langle f - s_n(f), f\rangle| \leq ||f - s_n(f)|| \cdot ||f||$
- $\langle s_n(f), f\rangle = \sum_{m=1}^{n} |c_m|^2$
- $\lim_{n\to\infty} |||f||^2 - \sum_{m=1}^{n} |c_m|^2| = 0$

This is a generalized Pythagorean theorem!