Linear Programming Foundations

Simplex, Interior Point, and Duality Theory

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Outline

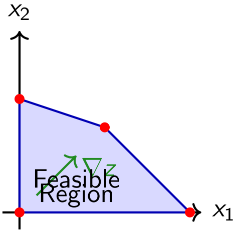
What is Linear Programming?

Linear Programming (LP) optimizes a linear objective subject to linear constraints.

Standard Form:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$
- Feasible region = convex polytope
- Optimum occurs at a vertex



Historical Context

1939 – Leonid Kantorovich

- First mathematical study of scheduling
- Laid theoretical foundations

1947 – George Dantzig

- Developed the Simplex Method
- US Air Force logistics
- With von Neumann: duality theory

1984 – Narendra Karmarkar

- First polynomial-time interior point algorithm
- Revolutionized large-scale optimization

Today

- Logistics, economics, CS, ML
- Integer and nonlinear programming

A Concrete Example: The Factory Problem

Manufacturing Optimization

A factory produces two components:

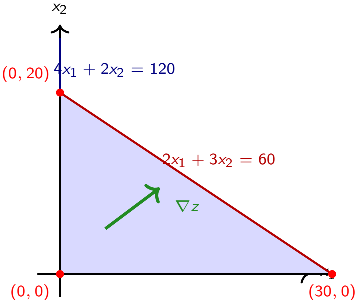
- Component 1 (x_1): profit \$40/unit
- Component 2 (x_2): profit \$30/unit

Resource Constraints:

- Assembly: $2x_1 + 3x_2 \leq 60$ hours
- Materials: $4x_1 + 2x_2 \leq 120$ units
- Non-negativity: $x_1, x_2 \geq 0$

The Linear Program

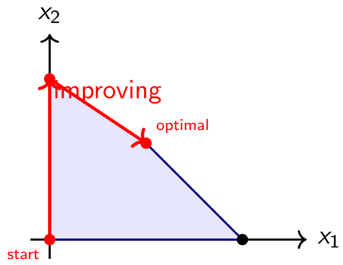
$$\begin{aligned} \max \quad & z = 40x_1 + 30x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 60 \\ & 4x_1 + 2x_2 \leq 120 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Optimal: $x_1 = 30$, $x_2 = 0$, $z = 1200$

Assembly fully used; materials slack.

Simplex: The Geometric Intuition



The Simplex Idea

1. Start at a **vertex**
2. Check each edge — does it improve?
3. Walk along best edge to new vertex
4. Repeat until no improvement

Key Insight

A pivot = walk from one vertex to an adjacent vertex along an edge.

Simplex: Algebraic Foundations

Definition (Basic Feasible Solution)

Given a basis B (m linearly independent columns of A), the **BFS** sets $x_i = 0$ for $i \notin B$ and solves $A_B x_B = b$.

Theorem (Fundamental Theorem of LP)

Basic feasible solutions correspond exactly to **vertices** *of the feasible polytope.*

Definition (Reduced Cost)

For nonbasic variable x_j: $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$
Measures rate of objective change if x_j enters the basis.

Optimality Criterion

A BFS is optimal $\Leftrightarrow$ $\bar{c}_j \geq 0$ for all nonbasic j.

The Pivot Operation

When $\bar{c}_j < 0$ for some nonbasic j

1. **Entering variable:** x_j enters the basis
2. Increase $x_j = t$ from 0:
$$x_B(t) = A_B^{-1}b - t \cdot A_B^{-1}A_j$$
3. **Ratio test:** $t^* = \min\left\{\frac{(A_B^{-1}b)_i}{(A_B^{-1}A_j)_i} : (A_B^{-1}A_j)_i > 0\right\}$
4. **Leaving variable:** basic variable that hits zero first
5. Update basis: $B' = B \setminus \{i\} \cup \{j\}$

Geometric Meaning

Each pivot moves from one vertex to an adjacent vertex along an improving edge.

Complexity: The Klee–Minty Result

Theorem (Klee–Minty, 1972)

There exist LPs where Simplex (under Dantzig's rule) visits all $2^d - 1$ vertices.

Property	**Simplex**
Worst-case iterations	Exponential
Typical iterations (practice)	$O(m)$ to $O(3m)$
Iteration cost	$O(m^2 n)$ (sparse)
Warm starting	Natural
Trajectory	Boundary (vertices)

Theoretical Gap

No polynomial worst-case bound for standard pivot rules. This motivated **interior point methods**.

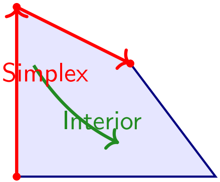
Interior Point: A New Philosophy

Simplex: Walks on the **boundary**

- Vertex $\rightarrow$ edge $\rightarrow$ vertex
- Combinatorial, discrete
- Stuck at corners (Klee–Minty)

Interior Point: Through the **interior**

- Smooth, continuous path
- No corners, no edges
- Calculus-friendly



The Logarithmic Barrier

Definition (Logarithmic Barrier)

For $x_j > 0$:

$$B(x) = -\sum_{j=1}^{n} \log(x_j)$$

Note: $B(x) \to +\infty$ as any $x_j \to 0^+$.

Definition (Barrier Problem)

For parameter $\mu > 0$:

$$\min \quad c^T x + \mu B(x) \quad \text{s.t.} \quad Ax = b$$

What This Achieves

- Inequalities $x \geq 0$ absorbed into objective
- Only equality $Ax = b$ remains
- Smooth problem — use Lagrangian methods

The Central Path

Definition (Central Path)

Let $x^*(\mu)$ be the unique minimizer of the barrier problem for $\mu > 0$. The set $\{x^*(\mu) : \mu > 0\}$ is the **central path**.

Optimality Conditions: With dual slack $s = c - A^T y$:

$$Ax = b, \quad A^T y + s = c, \quad XSe = \mu e, \quad x, s > 0$$

where $X = \text{diag}(x)$, $S = \text{diag}(s)$.

Key Observation

As $\mu \to 0$:

- $x^*(\mu) \to$ optimal primal solution
- $y^*(\mu) \to$ optimal dual solution
- Duality gap $x^T s = n\mu \to 0$

Newton's Method on the Central Path

Linearized System

Given (x, y, s) and target μ, solve for $(\Delta x, \Delta y, \Delta s)$:

$$A\Delta x = b - Ax$$
$$A^T\Delta y + \Delta s = c - A^T y - s$$
$$S\Delta x + X\Delta s = \mu e - XSe$$

Algorithm:

1. Initialize (x^0, y^0, s^0) with $x^0, s^0 > 0$
2. Choose μ and reduction factor $\sigma \in (0, 1)$
3. Solve Newton system for search direction
4. Step with length α preserving strict positivity
5. Reduce $\mu \leftarrow \sigma\mu$
6. Stop when duality gap $x^T s < \varepsilon$

Polynomial-Time Convergence

Theorem (Karmarkar, 1984)

Under short-step strategy, interior point achieves gap $< \varepsilon$ in

$$O\left(\sqrt{n} \cdot \log \frac{1}{\varepsilon}\right)$$

iterations, each costing $O(n^3)$ operations.

Property	**Simplex**	**Interior Point**
Worst-case iterations	Exponential	$O(\sqrt{n}\log(1/\varepsilon))$
Typical iterations	$O(m)$–$O(3m)$	$O(30)$–$O(100)$
Iteration cost	$O(m^2 n)$ sparse	$O(n^3)$ dense
Warm starting	Natural	Difficult
Trajectory	Boundary	Interior

The Dual Problem

Primal (minimization):

$$\min c^T x \quad \text{s.t.} \quad Ax = b,\ x \geq 0$$

Dual (maximization):

$$\max y^T b \quad \text{s.t.} \quad A^T y \leq c$$

Economic Interpretation: Shadow Prices

Dual variables y are **marginal values** of resources:

- $y_i =$ value of one additional unit of resource b_i
- The dual asks: what resource prices rationalize production?

The Three Duality Theorems

Theorem (Weak Duality)

For primal-feasible x and dual-feasible y:

$$y^T b \leq c^T x$$

Theorem (Strong Duality)

If primal has optimal x^*, then dual has y^* with:

$$c^T x^* = (y^*)^T b$$

Theorem (Complementary Slackness)

x^* and y^* are simultaneously optimal $\Leftrightarrow$ for all j:

$$x_j^* \cdot (c_j - (A^T y^*)_j) = 0$$

Duality and the Algorithms

Simplex Method

- Never explicitly solves dual
- At optimal basis B: $y^* = (c_B^T A_B^{-1})^T$
- Stopping $\bar{c}_j \geq 0 =$ dual feasibility
- Dual certificate "for free"

Interior Point Method

- Maintains dual (y, s) at every step
- Central path: $A^T y + s = c$ (dual feasibility)
- Duality gap $x^T s = n\mu$ tracked explicitly
- Drives gap to zero along central path

Unified View

Both algorithms search for a primal-dual pair satisfying **complementary slackness**.

The Central Claim

The Simplex Method and Interior Point Methods are not independent algorithms. They are **two different routes to the same destination** — satisfying complementary slackness — and **Duality is the map**.

Connection 1: Simplex Constructs Dual Certificate Implicitly

At optimal basis B:

$$y^* = (c_B^T A_B^{-1})^T$$

- Computed from final tableau quantities
- Dual feasible: $\bar{c}_j = c_j - (A^T y^*)_j \geq 0$
- Objective match: $(y^*)^T b = c^T x^*$

Key Insight

Simplex stopping condition ($\bar{c}_j \geq 0$) is **precisely** dual feasibility. Simplex terminates when it has found a dual certificate.

Connection 2: Interior Point Makes It Explicit

At every iterate, interior point maintains:

- Primal-feasible $x > 0$ (satisfying $Ax = b$)
- Dual-feasible (y, s) (satisfying $A^T y + s = c$, $s > 0$)
- Duality gap $x^T s = n\mu > 0$ decreasing each step

The algorithm terminates when the **duality gap falls below threshold**. Following the central path to $\mu = 0$ is following the duality gap to zero.

Connection 3: Both Are Correct for the Same Reason

Aspect	Simplex	Interior Point
Search space	Boundary: vertex to vertex	Interior: smooth curve
Step type	Combinatorial pivots	Continuous Newton steps
Dual solution	Only at termination	At every step
Stopping	All $\bar{c} \geq 0$	Gap $x^T s < \varepsilon$
Worst case	Exponential	Polynomial

Both search for the **unique certificate of optimality** Strong Duality guarantees. They are **two constructive proofs of the same theorem**.

Why This Matters Beyond Elegance

1. **Mutual Verifiability:** Two unrelated algorithms always agree. Not coincidence — consequence of strong duality.
2. **Unified Theory:** Duality explains why **both** work, not which is better.
3. **Algorithm Design:** Guides new methods (primal-dual simplex, predictor-corrector).
4. **Extensions:** Same framework extends to SDP, conic optimization, and beyond.

Summary

Three Pillars of Linear Programming

1. **Simplex Method:** Vertex search, excellent in practice, dual certificate implicit
2. **Interior Point:** Polynomial guarantee, smooth central path, explicit primal-dual
3. **Duality Theory:** Weak (bound), Strong (tightness), Complementary Slackness (characterization)

The Synthesis

Duality underlies **both** algorithms. Simplex and interior point are **two constructive proofs of strong duality**.

Future Directions

Semidefinite Programming (SDP)

- Matrix variable replaces x
- Duality extends elegantly
- Control theory, combinatorics

Integer Programming

- Variables constrained to integers
- Breaks LP duality theory
- Branch-and-bound, cutting planes

Modern Applications

- Machine learning (SVMs)
- Power grid optimization
- Portfolio optimization
- Supply chain logistics

Theoretical Questions

- Smoothed complexity of simplex
- Quantum algorithms for LP
- Online and robust optimization

Thank You!

Questions?

This presentation advertises the full paper.
See the paper for complete proofs and references.