Stone-Weierstrass Theorem

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Sets

Two fundamental kinds of sets are open sets and closed sets

- Complement of a closed set is open
- Complement of an open set is closed

Example (closed set)

Closed intervals denoted $[a, b]$ for $a, b \in \mathbb{R}$

Example (open set)

Open intervals denoted (a, b) for $a, b \in \mathbb{R}$

Continuity

A function $f : (a, b) \to \mathbb{R}$ is continuous if and only if for every $\epsilon > 0$ and every $x \in (a, b)$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

for $y \in (a, b)$.

Boundedness and Density

- Set A is dense in set B if for every $b \in B$ and every $\epsilon > 0$, there is a $a \in A$ such that $|a - b| < \epsilon$
- Set A is bounded if there exists a finite number M such that the distance between any two points in A is strictly less than or equal to M
- $C^0[a, b]$ denotes the set of all continuous functions defined on the closed and bounded interval $[a, b]$

Polynomial Approximation

Why approximate real-valued continuous functions by polynomials?

The significance of polynomial approximation:

- Polynomials are smooth
- Sum, product, and difference of two polynomials is a polynomial
- Easy to evaluate for calculators

Weierstrass Approximation Theorem

- Proved by Karl Weierstrass in 1885
- Central result in approximation theory
- The most important special case of Stone-Weierstrass theorem

Theorem (Weierstrass approximation theorem)

The set of all polynomials is dense in $C^0[a, b]$ *for* $a, b \in \mathbb{R}$.

Understanding Weierstrass Approximation Theorem

- Says that given a continuous real-valued function f and given a $\epsilon > 0$, there exists a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$
- $$\lim_{n \to \infty} p_n(x) = f(x)$$

Pictorial Representation

Red area is the ϵ-tube.

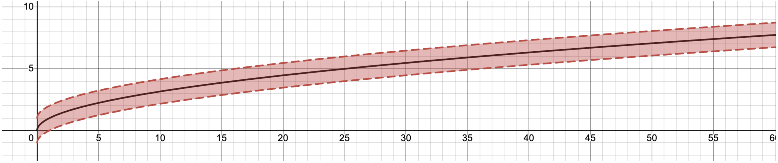


Figure: ϵ-tube of $y = \sqrt{x}$

Bernstein's Proof: Background

For each $n \in \mathbb{N}$, n^{th} Bernstein polynomial of f is

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Shows that any continuous function f can be approximated by the Bernstein polynomials of f to any given degree.

Graphical representation of Bernstein polynomials

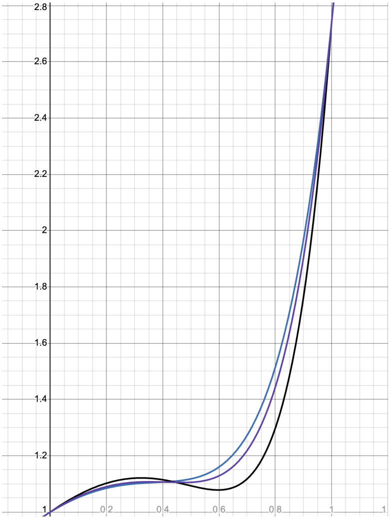


Figure: Approximation of $f(x) = 2^x e^{-x^2} + 3^x x^5 - x^4$

Bernstein Polynomials

- Influenced by binomial coefficients
- Proof depends on two nontrivial identities

We state these two identities as lemmas. For the following lemmas let

$$r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

First Lemma

Lemma

$\sum_{k=0}^{n} r_k(x) = 1$

Proof.

By the binomial theorem,

$$\sum_{k=0}^{n} r_k(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1.$$

■

Second Lemma

Lemma

$\sum_{k=0}^{n}(k - nx)^2 r_k(x) = nx(1 - x)$

Main Idea of Bernstein's Proof

The main idea of Bernstein's proof is to split

$$p_n(x) - f(x) = \sum_{k=0}^{n} (c_k - f(x))\, r_k(x)$$

into terms for which $\frac{k}{n}$ is close to x and terms for which $\frac{k}{n}$ is far from x, and then find upper bounds for both.

Some Topological Definitions

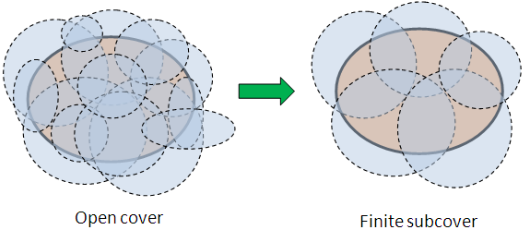
- Closure of set A is the smallest closed set containing A
  - Closure of set A contains all of its limit points
  - Closure of A is denoted $\bar{A}$
- Topological space X is Hausdorff if and only if for each pair of distinct points a and b in X, there exist disjoint open sets A and B in X such that $a \in A$ and $b \in B$.
  - Unique limits
  - Any space with the notion of distance

Density in General Topology

A is dense in B if and only if the closure of A is equal to B.

Compactness

- An open cover of set A in topological space X is a collection of open sets in X such that A is a subset of the union of that collection
- Subset A of topological space X is compact if for every open cover of A, there exists a finite sub-cover of A.



Stone-Weierstrass Theorem

Theorem (Stone-Weierstrass)

Let X be a compact Hausdorff space. If A is a subset of C^0X with the following properties,

1. *A is closed under addition, multiplication, and scalar multiplication*
2. *For every $x, y \in X$, there exists a function $f \in A$ such that $f(x) \neq f(y)$*
3. *A contains a nonzero constant function*

then $\bar{A} = C^0X$.

Trigonometric Approximation Theorem

A trigonometric polynomial is of the form

$$\sum_{k=0}^{n} a_k \cos(kx) + \sum_{k=0}^{n} b_k \sin(kx) + a_0$$

where the a_k's, the b_k's, and a_0 are real constants.

Theorem (Trigonometric Approximation Theorem)

The set of all trigonometric polynomials is dense in the set of all continuous 2π-periodic functions.

Complex Stone-Weierstrass Theorem

Denote the set of all continuous complex-valued functions defined on the set X by $C^0(X, \mathbb{C})$.

Theorem

Let X be a compact Hausdorff space. If A is a subset of $C^0(X, \mathbb{C})$ with the following properties,

1. *A is closed under addition, multiplication, and scalar multiplication.*
2. *For every $x, y \in X$, there exists a function $f \in A$ such that $f(x) \neq f(y)$.*
3. *A contains a nonzero constant function.*
4. *A contains the conjugate of each of its functions.*

then $\bar{A} = C^0(X, \mathbb{C})$.

Thank You!