Pell Equations

Shiven Bhargava
bhargava.shiven@gmail.com

June 10, 2026

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Introduction

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Generalized Pell equations are of the form $x^2 - dy^2 = n$, where $n \in \mathbb{Z}$.

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Nontrivial solutions are ones which are not trivial, and positive solutions are ones with positive x, y. In the rest of the talk, we assume d is not a perfect square.

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Euler made an irreversible blunder and attributed the equation to John Pell (1611–1685).

Background

Lagrange used continued fractions, which we will discuss shortly.

One can also use some approximation theorems and clever inequalities to solve Pell equations.

Continued Fractions in Pell Equations

Definition of Continued Fractions

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Definition 2.1

Let $a_0, a_1, a_2, a_3, \ldots, a_m$ be a sequence of real numbers. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots + \cfrac{1}{a_m}}}}}}$$

is a **finite continued fraction**, and is shorthanded by $[a_0; a_1, a_2, \ldots, a_m]$. If the fraction never ends, its an **infinite continued fraction** and is written as $[a_0; a_1, a_2, \ldots]$. A **simple continued fraction** is a continued fraction under the constraint that $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i > 0$.

Definition of Convergents and $(p_n), (q_n)$

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Definition 2.2

Let m, n be nonnegative integers such that $m \geq n$. Take the continued fraction $x = [a_0; a_1, a_2, a_3, \ldots, a_{n-1}, a_n, \ldots, a_m]$. The ***n*th convergent** to x is defined as $[a_0; a_1, a_2, a_3, \ldots, a_n]$. Define two sequences of real numbers (p_n) and (q_n) in relation to the continued fraction of x above where

$$p_{-1} = 1, p_0 = a_0, \text{ and } p_n = a_n p_{n-1} + p_{n-2},$$

and analogously,

$$q_{-1} = 0, q_0 = 1, \text{ and } q_n = a_n q_{n-1} + q_{n-2}.$$

Relations between (p_n) and (q_n)

Theorem 2.3

$\frac{p_n}{q_n}$ is the nth convergent to $[a_0; a_1, a_2, a_3, \ldots, a_m]$.

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Lemma 1

The following is true for all $1 \leq n \leq m$*:*

1. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$
2. $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$

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Corollary 2.4

If $[a_0; a_1, a_2, \ldots, a_m]$ is a simple continued fraction, then p_n, q_n are relatively prime integers for all $0 \leq n \leq m$.

Periodic Continued Fractions and Quadratic Irrationals

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Definition 2.5

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Theorem 2.7 (Lagrange's Theorem)

A real number α is a quadratic irrational iff its continued fraction is periodic.

Pell Solutions from Convergents

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Theorem 2.8

All positive solutions to a Pell equation are given by

$$(x, y) = \begin{cases} (p_{km-1}, q_{km-1}) & \text{if } k \text{ is even} \\ (p_{2km-1}, q_{2km-1}) & \text{if } k \text{ is odd} \end{cases}$$

where m is a positive integer, k is the minimal period of the continued fraction of $\sqrt{d}$, and $\frac{p_n}{q_n}$ is a convergent.

Finding Pell Solutions through Alternative Methods

Dirichlet's Approximation Theorem

Dirichlet's Approximation Theorem

Theorem 3.1 (Dirichlet's Approximation Theorem)

For any $\alpha \in \mathbb{R}$ and integer $N > 1$, there exists an integer p and a positive integer q such that $q \leq N$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN},$$

or equivalently,

$$|q\alpha - p| < \frac{1}{N} \leq \frac{1}{q}.$$

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$$\frac{k-1}{N} \leq a_i, a_j < \frac{k}{N} \implies |a_i - a_j| < \frac{1}{N}.$$

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Assume $i > j$. Substituting, we have

$$a_i - a_j = (i-j)\,\alpha + (\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor).$$

Let $p = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor, q = i - j$.

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Assume $i > j$. Substituting, we have

$$a_i - a_j = (i - j)\,\alpha + (\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor)\,.$$

Let $p = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor, q = i - j$. Notice that $q \in \mathbb{N}$. Hence,

$$|a_i - a_j| = |q\alpha - p| < \frac{1}{N}.$$

■

Existence of a Nontrivial Solution

Existence of a Nontrivial Solution

Theorem 3.2

A Pell equation always has a nontrivial solution. Furthermore, it has a positive solution.

Composition Principle

Composition Principle

Theorem 3.3 (Composition Principle)

Let $x_1, y_1, x_2,$ *and* y_2 *be integers such that* $x_1^2 - dy_1^2 = n_1$ *and* $x_2^2 - dy_2^2 = n_2$. *Define* $x_3 = x_1x_2 + dy_1y_2$ *and* $y_3 = x_1y_2 + y_1x_2$. *Then*

$$x_3^2 - dy_3^2 = n_1n_2 = \left(x_1^2 - dy_1^2\right)\left(x_2^2 - dy_2^2\right).$$

Notice that x_3, y_3 *are also the unique integers that satisfy* $x_3 + y_3\sqrt{d} = (x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d})$.

Infinite Solutions

Infinite Solutions

Corollary 3.4

If $(x, y) = (x_1, y_1)$ *is a positive solution to* $x^2 - dy^2 = 1$*, then the coefficients of* $\pm(x_1 + y_1\sqrt{d})^k$ *are a solution for all* $k \in \mathbb{Z}$*. The trivial solutions have* $k = 0$ *and the positive ones have* $k > 0$ *and* $+$ *out front. In particular, the equation* $x^2 - dy^2 = 1$ *has infinitely many nontrivial solutions.*

Infinite Solutions

Corollary 3.4

If $(x, y) = (x_1, y_1)$ is a positive solution to $x^2 - dy^2 = 1$, then the coefficients of $\pm(x_1 + y_1\sqrt{d})^k$ are a solution for all $k \in \mathbb{Z}$. The trivial solutions have $k = 0$ and the positive ones have $k > 0$ and $+$ out front. In particular, the equation $x^2 - dy^2 = 1$ has infinitely many nontrivial solutions.

Corollary 3.5

If the equation $x^2 - dy^2 = n \neq 0$ has a solution, then it has infinitely many solutions.

All Pell Solutions

All Pell Solutions

Theorem 3.6

There exists a solution (x_1, y_1) *to the equation* $x^2 - dy^2 = 1$ *with* $x_1, y_1 \in \mathbb{N}$ *such that* y_1 *is minimal by Theorem 3.2. Then all solutions to* $x^2 - dy^2 = 1$ *where* $x, y \in \mathbb{Z}$ *are generated by the following expression:*

$$x + y\sqrt{d} = \pm\left(x_1 + y_1\sqrt{d}\right)^k.$$

In this equation, $k \in \mathbb{Z}$*. Additionally, the solutions where* x, y *are positive are generated by taking* $k > 0$ *and the* $+$ *sign out front.*

Applications

Rational Approximations to $\sqrt{d}$

Pell solutions provide accurate approximations to $\sqrt{d}$.

$$x^2 - dy^2 = 1 \implies \left(\frac{x}{y}\right)^2 - \frac{1}{y^2} = d.$$

Hence, for large y,

$$\frac{x}{y} \approx \sqrt{d}.$$

Simultaneous Polygonal Numbers

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$S_n = 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

$$P_n = 1 + 4 + 7 + 10 + \cdots + (3n-2) = \frac{3n^2 - n}{2}$$

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Triangular and square:

$$T_n = S_m \implies \frac{n(n+1)}{2} = m^2 \implies 2m^2 = \left(n + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Let $x = 2n+1$, $y = 2m$. We want $x^2 - 2y^2 = 1$. The conditions that x is odd and y is even are true for all solutions to the previous Pell equation. Hence all positive solutions yield a unique pair (m, n). One solution is $(x, y) = (17, 12), (n, m) = (8, 6)$.

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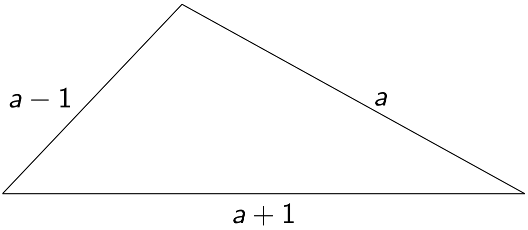
$$P_n = 1 + 4 + 7 + 10 + \cdots + (3n-2) = \frac{3n^2 - n}{2}$$

Pentagonal and square:

$$P_n = S_m \implies \frac{3n^2 - n}{2} = m^2 \implies 2m^2 = 3\left(n - \frac{1}{6}\right)^2 - \frac{1}{12}.$$

Let $x = 6n - 1$, $y = 2m$. We want $x^2 - 6y^2 = 1$. One can show that in the given Pell equation x is always of the form $6n \pm 1$ (not $x = 6n - 1$ necessarily) and y is even. One solution is $(x, y) = (485, 198), (n, m) = (81, 99)$.

Consecutive Heronian Triangles



$$A = \sqrt{s(s-(a-1))(s-a)(s-(a+1))}, s = \frac{3a}{2}$$

$$s(s-a+1)(s-a)(s-a-1) = A^2 \implies \frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a+2}{2} \cdot \frac{a-2}{2} = A^2.$$

Then $3a^2(a^2-4) = (4A)^2$, so let $a = 2x$, giving $3x^2(x^2-1) = A^2$, so $x^2 - 1 = 3y^2$ for some y. Hence $a = 2x, A = 3xy$ where $x^2 - 3y^2 = 1$ with no constraints on x, y. This includes $(3,4,5), (13,14,15), (51,52,53)$.

Thank you!

Thanks for listening!