# Expander Graphs

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# Motivation for Network Design

## Two Properties

- **Sparsity**: Have a low number of edges. This reduces needed infrastructure or computational costs, but would intuitively make the network more fragile and easily fractured.
- **Connectivity**: Not easily broken into isolated components. This increases efficiency, but would intuitively require the network to have a dense amount of edges.

Expander graphs are simultaneously very sparse and highly connected.

# Brief History

- **1973**: Bassalygo & Pinsker prove the existence of expanders using the Probabilistic Method.
- **1973**: Margulis achieves the first explicit, deterministic algebraic construction.
- **1987**: Ajtai, Komlós, and Szemerédi (AKS) formalize the "hitting property" of random walks.
- **1996**: Sipser and Spielman achieve strictly linear-time decoding for expander codes.
- **2002**: Reingold, Vadhan, & Wigderson introduce the combinatorial Zig-Zag product.
- Insights into expander graphs and their applications continue to this day.

# Definitions of Expansion

## Combinatorial Expansion

Measures "bottleneckedness". Define $\partial S = \left\{ \{u, v\} \in E \;\middle|\; u \in S,\ v \in \overline{S} \right\}$. For a graph $G$, the edge expansion ratio $h(G)$ is:

$$h(G) = \min_{0<|S|\leq \frac{n}{2}} \frac{|\partial S|}{|S|}$$

## Algebraic Expansion (Spectral Graph Theory)

Calculating $h(G)$ is NP-hard. Instead, we use the adjacency matrix $A$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Define the spectral gap as $\gamma = d - \lambda$.

The **Discrete Cheeger Inequality** bridges algebraic and combinatorial expansion:

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

# Probabilistic Expansion: Mixing & Random Walks

- **Expander Mixing Lemma**: A large spectral gap forces the distribution of edges between any two vertex subsets $S, T$ of a $d$-regular graph to mimic that of a completely random graph:

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}$$

- **Random Walks**: Take a random walk on an expander graph. From each vertex, you travel to any of its neighbors all with equal probability. Due to rapid mixing properties of expander graphs (as is proved in my paper), it turns out random walks will distribute evenly in just $O(\log n)$ steps, while for other graphs it would take $O(n^2)$ or more.

# Limits to Expansion & Constructions

- **Limit**: The *Alon-Boppana Bound* says that for large $n$, the absolute second eigenvalue is bounded:

$$\lambda \geq 2\sqrt{d-1} - o_n(1).$$

  Graphs that meet the lower bound are **Ramanujan graphs**.

- **Explicit Constructions**:
  - *Algebraic*: Margulis provided the very first explicit family of expander graphs. For any integer $m$, he defined a graph $G_m$ on the vertex set $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$. Each vertex $(x, y)$ is connected to eight neighbors modulo $m$:

$$(x \pm y, y), \quad (x, y \pm x), \quad (x \pm y + 1, y), \quad (x, y \pm x + 1)$$

  - *Combinatorial*: Inductively build expander families with a type of graph product (combining two graphs): zig-zag product, lifts.

# Applications: Derandomization

**Problem:** A randomized algorithm (RP/BPP) needs $r$ bits. To drop the error rate to $2^{-k}$, the naive approach runs it $k$ times, costing a large number of random bits: $\mathbf{k \cdot r}$.

**Solution (by Impagliazzo & Zuckerman):**

- Build an expander where every vertex is a possible $r$-bit random seed.
- Pick one random starting seed (costs $r$ bits).
- Take a random walk of $k$ steps. Each step only requires choosing a neighbor (costs $\log_2 d$ bits).
- **Result:** Because of the Hitting Property of expanders, the walk is highly unlikely to stay trapped walking among seeds with "bad" error. Then the exact same exponential $2^{-k}$ error reduction can be achieved using only $\mathbf{r + O(k)}$ random bits!

# Applications: Expander Codes and The PCP Theorem

**Expander Codes**: In coding theory, expander codes were constructed by Sipser and Spielman that allow for efficient linear-time decoding.

### Probabilistically Checkable Proofs (PCP)

States that every problem in NP has a proof that can be verified by examining only a *constant* number of bits, using a logarithmic number of random bits.

- **Dinur's Combinatorial Proof (2005)**: Replaced highly complex algebraic proofs with a purely combinatorial method using expander graphs.

# Current Research

- **High-Dimensional Expanders (HDXs)**:
  - Generalizing 1D graphs to higher-dimensional simplicial complexes (triangles, tetrahedrons, etc.).
  - Applications in topological data analysis.
- **Quantum Expander Codes**:
  - Using hypergraph products to build quantum low-density parity-check (qLDPC) codes.
  - Excellent for removing noise in large-scale quantum computing, where qubits are especially unstable.

# Conclusion

- Expander graphs prove that networks do not have to compromise between sparsity (efficiency) and connectivity (robustness).
- They connect topics across algebraic geometry, probabilistic random walks, and combinatorial structures.
- They are powerful tools in computer science: saving random bits, creating linear-time error correctors, etc.