Characterization of Graphical Matroids

From Independence Axioms to Tutte's Excluded-Minor Theorem

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Euler Circle

July 12, 2026

Outline

Two Notions of Independence

Linear Algebra

- Vectors $v_1, \ldots, v_n$
- Independent if no nontrivial linear combination is zero
- Bases, dimension

Graph Theory

- Edges of a graph G
- Independent if the edge set contains no cycle
- Spanning trees, forests

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Whitney's Insight (1935)

These two axiomatic systems are, structurally, *the same*. A **matroid** abstracts the common logic of independence.

A Brief History

- **1935:** Hassler Whitney, *On the abstract properties of linear dependence*
- **1935:** Takeo Nakasawa, independent axiomatization
- Garrett Birkhoff: lattice-theoretic (flats) perspective
- **1936:** Saunders Mac Lane connects matroids to projective geometry
- Applications today: optimization, coding theory, cryptography, algebraic geometry

Goal of this talk: understand *graphical (cycle) matroids* and the theorem that characterizes exactly which matroids are graphic.

Definition: Matroid

Independence Axioms

A matroid $M = (E, \mathcal{I})$ consists of a finite ground set E and a collection $\mathcal{I}$ of *independent sets* satisfying:

I-1 $\emptyset \in \mathcal{I}$

I-2 (Hereditary) $A \in \mathcal{I},\ B \subseteq A \implies B \in \mathcal{I}$

I-3 (Exchange) $A, B \in \mathcal{I},\ |B| > |A| \implies \exists x \in B \setminus A$ with $A \cup \{x\} \in \mathcal{I}$

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A subset not in $\mathcal{I}$ is called **dependent**. Axiom **I**-3 is the key axiom distinguishing matroids from general set systems.

Example: The Vector Matroid

Theorem (Vector Matroid $M[A]$)

Let A be a matrix over a field $\mathbb{F}$, E its columns, and $\mathcal{I}$ the linearly independent subsets. Then $(E, \mathcal{I})$ is a matroid.

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- Columns $\{1, 3, 5\}$: independent
- All five columns: dependent

The Cycle Matroid of a Graph

Theorem (Cycle Matroid $M(G)$)

Let $G = (V, E)$ be a graph, and let $\mathcal{I}$ be the edge sets containing no cycle. Then $(E, \mathcal{I})$ is a matroid.

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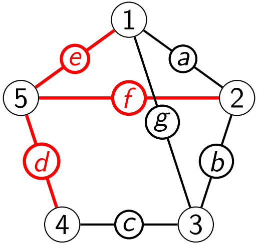
Let $G = (V, E)$ be a graph, and let $\mathcal{I}$ be the edge sets containing no cycle. Then $(E, \mathcal{I})$ is a matroid.

Proof idea:

- I-1, I-2: trivial (removing edges can't create cycles)
- I-3: an acyclic edge set on $|V|$ vertices with $|A|$ edges has $|V| - |A|$ components; the larger forest B must connect two components of A

Example: A Graph and Its Cycle Matroid



- $\{d, e, f\}$ is independent: no cycle formed
- $\{a, b, c\}$ is dependent if they form a triangle

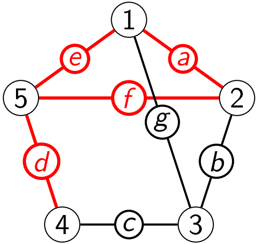
A matroid is **graphic** if it is isomorphic to $M(G)$ for some graph G.

Bases

Definition

A **base** is a maximal independent set. All bases have the same cardinality $r(M)$, the **rank** of M.

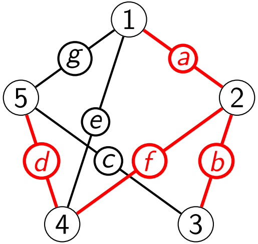
- Vector matroid: bases = column bases of A
- Cycle matroid: bases = spanning forests (spanning trees if G connected)



Definition

A **circuit** is a minimal dependent set.

- Vector matroid: minimal linearly dependent column sets
- Cycle matroid: **simple cycles** of G
- A **loop** is an element x with $\{x\}$ dependent



The Rank Function

$$r(A) = \max\{\, |I| : I \subseteq A,\ I \in \mathcal{I}\,\}$$

Rank Axioms

R-1: $0 \leq r(A) \leq |A|$ **R-2:** monotone **R-3:** submodular:
$r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

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Cycle matroid formula:

$$r(S) = |V| - c(S)$$

where $c(S)$ is the number of connected components of (V, S).

Duality

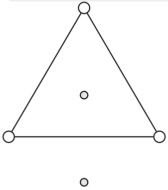
Dual Matroid

Given bases $\mathcal{B}$ of M, the sets $\{E \setminus B : B \in \mathcal{B}\}$ form the bases of the **dual matroid** M^*.

- $(M^*)^* = M$
- Circuits of M^* are called **cocircuits** of M

Theorem (Whitney's Planarity Theorem)

A connected graph G is planar $\iff$ $M(G)^*$ is graphic, and then $M(G)^* \cong M(G^*)$ for the planar dual G^*.



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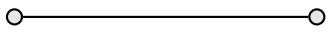


H^*

Minors: Deletion and Contraction

- **Deletion** $M \setminus A$: restrict to $E \setminus A$
- **Contraction** $M/A := (M^* \setminus A)^*$

Graph interpretation:

$$M(G) \setminus \{e\} = M(G \setminus e), \qquad M(G)/\{e\} = M(G/e)$$

Minor

N is a **minor** of M if $N = M \setminus A/B$ for disjoint A, B.

The class of graphic matroids is minor-closed.

Representability and Regularity

Definitions

- M is $\mathbb{F}$-**representable** if $M \cong M[A]$ for some matrix A over $\mathbb{F}$
- **Binary**: representable over $GF(2)$
- **Regular**: representable over *every* field (equivalently, totally unimodular representation)

Theorem

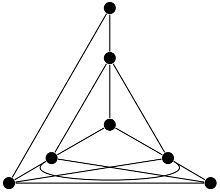
Every graphic matroid is regular.

Proof idea: the vertex–edge incidence matrix, suitably oriented, is totally unimodular; column dependence $\iff$ cycle in G.

Two Non-Graphic Matroids

Fano matroid F_7

- 7 points, 7 lines of the Fano plane $PG(2,2)$
- Representable over $GF(2)$ only
- Not regular



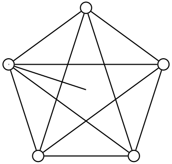
Uniform matroid $U_{2,4}$

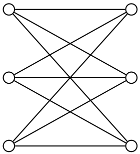
- 4 elements, rank 2
- Every 3-subset is a circuit
- Not binary (Tutte, 1958)

The Five Excluded Minors

Minor	Description
$U_{2,4}$	smallest non-binary matroid
F_7	Fano matroid
F_7^*	dual Fano matroid
$M^*(K_5)$	dual cycle matroid of K_5
$M^*(K_{3,3})$	dual cycle matroid of $K_{3,3}$

K_5 and $K_{3,3}$ are the **Kuratowski graphs** that characterize non-planarity.





Main Theorem

Theorem (Tutte's Excluded-Minor Theorem, 1959)

A matroid M is **graphic** *if and only if*

1. *M is* **regular**, *and*
2. *M contains* **no minor** *isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$.*

Main Theorem

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A matroid M is **graphic** *if and only if*

1. *M is* **regular***, and*
2. *M contains* **no minor** *isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$.*

Proof structure:

- ($\Rightarrow$) Each excluded minor fails a property graphic matroids must have (binary, representable over $\mathbb{Q}$, or planarity-linked)
- ($\Leftarrow$) Uses **Seymour's Decomposition Theorem**

Proof Sketch: Necessity

- Graphic $\Rightarrow$ regular $\Rightarrow$ binary
  $\Rightarrow$ no $U_{2,4}$ minor (Prop. 8.9)
- Graphic $\Rightarrow$ representable over $\mathbb{Q}$
  $\Rightarrow$ no F_7 or F_7^* minor (char. 2 only)
- If G planar, $M(G)^*$ graphic only via planar dual
  $\Rightarrow$ non-planarity (Wagner's Thm: K_5 or $K_{3,3}$ minor) is obstructed
  $\Rightarrow$ no $M^*(K_5)$, $M^*(K_{3,3})$ minor

Proof Sketch: Sufficiency

Theorem (Seymour, 1980)

Every regular matroid is a 3-sum of graphic matroids, cographic matroids, and copies of R_{10}.

- A cographic component that is not graphic forces $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor
- R_{10} itself has $M^*(K_5)$ as a minor
- Under our hypotheses, both are excluded $\Rightarrow$ all components are graphic $\Rightarrow$ M is graphic

The Hierarchy of Matroid Classes

Class	Excluded Minors
Binary	$U_{2,4}$
Regular	$U_{2,4},\ F_7,\ F_7^*$
Graphic	$U_{2,4},\ F_7,\ F_7^*,\ M^*(K_5),\ M^*(K_{3,3})$

$$\text{Graphic} \subsetneq \text{Regular} \subsetneq \text{Binary}$$

(R_{10}: regular but not graphic. $U_{2,4}$: binary but not regular.)

Open Problems and Extensions

- **Rota's Conjecture (1970):** finitely many excluded minors for representability over $GF(q)$ — proved for $q = 2, 3, 4$; open for $q \geq 5$
  (announced in full generality by Geelen–Gerards–Whittle, 2014)
- **Matroid minor theorem** for arbitrary minor-closed classes: open
- **Algebraic matroids** vs. representable matroids
- **Log-concavity** of the characteristic polynomial (Adiprasito–Huh–Katz, via Chow rings)
- **Matroid union/intersection** and combinatorial optimization

Summary

- Matroids unify linear and graph-theoretic independence
- Cycle matroids encode forests, spanning trees, and cycles of a graph
- Duality connects $M(G)^*$ to planar duality
- Every graphic matroid is regular and binary
- **Tutte's Theorem:** graphic $=$ regular $+$ five excluded minors

Thank you!