CHARACTERIZATION OF GRAPHICAL MATROIDS

SHALINI SIVAKUMAR

Abstract. In this paper, we present an expository treatment of graphical matroids—also called *cycle matroids*—and their place within the broader landscape of matroid theory. Starting from the foundational axioms of a matroid, we construct the cycle matroid $M(G)$ of an undirected graph G and verify that it satisfies all three independence axioms. We then develop the structural terminology of bases, circuits, and the rank function, and show how classical objects in graph theory—spanning trees, simple cycles, and cocycles—translate perfectly into matroid language. Our principal result is **Tutte's Excluded-Minor Theorem**: a matroid is graphic if and only if it is regular and contains no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$. We build toward this theorem by establishing the theory of matroid duality, minors, and regularity. Throughout, concrete examples and TikZ diagrams guide the reader who may be encountering matroid theory for the first time.

1. Introduction

The concept of *independence* appears throughout modern mathematics. In linear algebra, a set of vectors is independent if none of them can be expressed as a linear combination of the others. In graph theory, a set of edges is independent—if it contains no cycle. At first glance these two notions seem to belong to entirely different worlds. Yet when one writes down their axiomatic descriptions side by side, the rules are strikingly identical.

It was **Hassler Whitney** who first made this observation precise. In his landmark 1935 paper *On the abstract properties of linear dependence* [12], Whitney introduced the notion of a *matroid*: a finite combinatorial structure that captures the essential logic of independence, free from any particular choice of coordinates, field, or graph. The term "matroid" is itself a play on "matrix," reflecting the linear-algebraic origins of the concept.

Whitney was not alone in this pursuit. In 1935, Nakasawa independently axiomatized linear dependence along similar lines. Garrett Birkhoff [2] approached the structure through lattice theory, focusing on the order-theoretic properties of "flats" or closed sets. By 1936, Saunders Mac Lane had drawn a deep connection between matroids and projective geometry. This remarkable convergence of ideas from algebra, combinatorics, and geometry in the mid-1930s established matroid theory as a powerful unifying framework that has since found applications in optimization, cryptography, coding theory, and algebraic geometry.

Date: July 12, 2026.

This paper focuses on a central and historically important class of matroids: *graphical matroids*, also called *cycle matroids*. A graphical matroid is one that arises from a graph by declaring a set of edges to be independent whenever it contains no cycle. The class of graphical matroids is rich enough to encode all the combinatorial structure of a graph, yet abstract enough to reveal deep connections to other parts of mathematics.

Our main theorem is the following characterization, due to Tutte [10, 11]:

Theorem 1.1 (Tutte's Excluded-Minor Theorem)**.** *A matroid $\mathcal{M}$ is graphic if and only if it is regular and contains no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$.*

We will explain and prove (or carefully sketch) each ingredient of this theorem over the course of the paper.

2. Background

We collect the prerequisite definitions from linear algebra and graph theory that motivate and support our study of matroids.

2.1. Linear Algebra.

We work over an arbitrary field $\mathbb{F}$. The reader may safely take $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$ on a first reading.

Definition 2.1. A set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors over a field $\mathbb{F}$ is *linearly independent* if, for scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$, the equation

$$\sum_{i=1}^{n} a_i v_i = 0$$

has only the trivial solution $a_1 = a_2 = \cdots = a_n = 0$. Equivalently, no vector in S can be expressed as a linear combination of the others.

Definition 2.2. The *span* of a set $S = \{v_1, \ldots, v_n\}$ of vectors over $\mathbb{F}$ is

$$\operatorname{span}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_1, \ldots, a_n \in \mathbb{F} \right\}.$$

Definition 2.3. A *basis* of a vector space V is a linearly independent subset of V whose span equals V. Equivalently, a basis is a linearly independent set of maximum cardinality.

Remark 2.4. All bases of a finite-dimensional vector space have the same cardinality, called the *dimension* of V. This fact is the prototype of the analogous statement for matroids.

Definition 2.5. The *rank* of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the column space of A, i.e., the maximum number of linearly independent columns.

2.2. **Graph Theory.** We recall the standard terminology for undirected graphs.

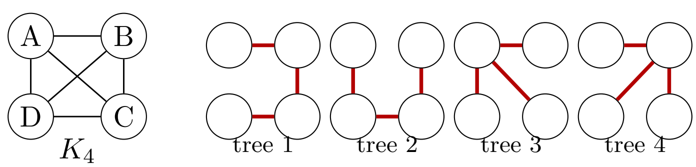
Definition 2.6. A *graph* $G = (V, E)$ consists of a finite set V of *vertices* and a finite set E of *edges*, where each edge is an unordered pair of vertices. We allow *loops* (edges from a vertex to itself) and *multiple edges* between the same pair of vertices; such a graph is called a *multigraph*.

Definition 2.7. A *walk* in G is a finite sequence of edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$. A walk is a *trail* if all edges are distinct. A walk is a *path* if all vertices $v_0, v_1, \ldots, v_n$ are distinct. A *cycle* is a closed path, i.e., a trail where $v_0 = v_n$ and all internal vertices are distinct.

Definition 2.8. A graph G is *connected* if there exists a path between every pair of vertices. A *forest* is an acyclic graph (a graph with no cycles). A *tree* is a connected forest.

Definition 2.9. A *spanning tree* of a connected graph G on n vertices is a subgraph of G that is a tree containing all n vertices. A spanning tree of G has exactly $n - 1$ edges.

Example 2.10. Example: Spanning Trees of K_4



The complete graph K_4 has 16 spanning trees. Four of them are illustrated above. Note that no spanning tree contains a cycle.

Definition 2.11. A *cocycle* (or *bond*) of a graph G is a minimal set of edges whose removal disconnects G (or, if G is disconnected, disconnects one component from the rest). Cocycles are the graph-theoretic duals of cycles.

3. Matroids: Definition and First Examples

There are many equivalent ways to define a matroid. We begin with the most intuitive: the axioms for independent sets.

3.1. **The Axioms of a Matroid.**

Definition 3.1. A *matroid* is an ordered pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set called the *ground set* and $\mathcal{I}$ is a collection of subsets of E called *independent sets*, satisfying the following three axioms:

- **I-1.** (*Nontriviality*) $\emptyset \in \mathcal{I}$.
- **I-2.** (*Hereditary property*) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- **I-3.** (*Independence augmentation*) If $A, B \in \mathcal{I}$ and $|B| > |A|$, then there exists an element $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{I}$.

A subset of E that is *not* in $\mathcal{I}$ is called a *dependent set*.

Notation 3.2. We write $E(\mathcal{M})$ for the ground set of a matroid $\mathcal{M}$. When no confusion arises we abbreviate $E(\mathcal{M})$ to E.

Axiom I-3 is sometimes called the *exchange property* or the *augmentation property*. It is the key axiom that distinguishes matroids from arbitrary simplicial complexes, and it encodes the "uniform" nature of independence.

3.2. **Vector Matroids.** The motivating example of a matroid comes from linear algebra.

Theorem 3.3 (Vector Matroid)**.** *Let A be a matrix over a field $\mathbb{F}$, and let E be the set of column vectors of A. Let $\mathcal{I}$ be the collection of subsets of E that are linearly independent over $\mathbb{F}$. Then $(E, \mathcal{I})$ is a matroid, called the* vector matroid *of A and denoted $M[A]$.*

Proof. We verify the three axioms of Definition 6.1.

I-1. The empty set contains no vectors, and the empty linear combination trivially sums to zero with no nontrivial coefficients, so $\emptyset \in \mathcal{I}$.

I-2. Any subset of a linearly independent set is linearly independent, since dropping vectors cannot create a linear dependency.

I-3. Let $A, B \in \mathcal{I}$ with $|B| > |A|$. Let V be the vector space spanned by $A \cup B$, so $\dim V \geq |B|$. If $A \cup \{x\}$ were linearly dependent for every $x \in B \setminus A$, then V would be contained in the span of A, giving $\dim V \leq |A| < |B|$, a contradiction. Therefore there exists $x \in B \setminus A$ with $A \cup \{x\}$ linearly independent. $\square$

Example 3.4. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The columns of A form the ground set E of the vector matroid $M[A]$. For instance, the set of the first, third, and fifth columns is an independent set of $M[A]$, since those three vectors are linearly independent. The set of all five columns is dependent.

3.3. **The Cycle Matroid of a Graph.** The second fundamental source of matroids is graph theory.

Theorem 3.5 (Cycle Matroid)**.** *Let $G = (V, E)$ be a graph. Let $\mathcal{I}$ be the collection of subsets of E that contain no cycle. Then $(E, \mathcal{I})$ is a matroid, called the* cycle matroid *of G and denoted $M(G)$.*

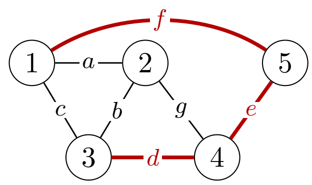
Proof. We verify the three axioms.

I-1. The empty set of edges contains no cycle, so $\emptyset \in \mathcal{I}$.

I-2. If a set of edges A contains no cycle, then any subset $B \subseteq A$ also contains no cycle (removing edges cannot create cycles), so $B \in \mathcal{I}$.

I-3. Let $A, B \in \mathcal{I}$ with $|B| > |A|$. A set of edges F with no cycle is a forest, and a forest on a connected component with k vertices has at most $k-1$ edges. More precisely, the subgraph induced by A has $|V|$ vertices and $|A|$ edges and is a forest, so it has $|V| - |A|$ connected components; similarly the subgraph for B has $|V| - |B|$ components. Since $|B| > |A|$, the forest B spans more edges and thus has *fewer* components than A. Therefore some edge $e \in B \setminus A$ connects two different components of A, so $A \cup \{e\}$ is still acyclic, i.e., $A \cup \{e\} \in \mathcal{I}$. $\square$

Example 3.6. Example: An Independent Edge Set in H



independent set $\{d, e, f\}$

Consider the graph H with edges a, b, c, d, e, f, g. The set $\{d, e, f\}$ is an independent set of $M(H)$ because the subgraph formed by these three edges contains no cycle. On the other hand, the set $\{a, b, c\}$ is dependent if those three edges form a triangle.

Definition 3.7. A matroid $\mathcal{M}$ is called *graphic* if it is isomorphic to the cycle matroid $M(G)$ for some graph G. (Two matroids are *isomorphic* if there is a bijection between their ground sets that preserves independence.)

Remark 3.8. Not every matroid is graphic. As we will see in Section 12, the Fano matroid F_7 and the Vámos matroid V_8 are canonical examples of matroids that cannot arise as the cycle matroid of any graph.

3.4. **Other Classes of Matroids.** For completeness we mention two additional families.

Proposition 3.9 (Uniform Matroid)**.** *Let E be a set with n elements, and let $0 \le r \le n$. Let $\mathcal{I}$ be the collection of all subsets $A \subseteq E$ with $|A| \le r$. Then $(E, \mathcal{I})$ is a matroid, called the* uniform matroid *$U_{r,n}$.*

Proof. **I-1**: $|\emptyset| = 0 \le r$, so $\emptyset \in \mathcal{I}$. **I-2**: If $|A| \le r$ and $B \subseteq A$, then $|B| \le |A| \le r$. **I-3**: If $|A| < |B| \le r$, pick any $x \in B \setminus A$; then $|A \cup \{x\}| = |A| + 1 \le |B| \le r$. $\square$

Proposition 3.10 (Transversal Matroid)**.** *Let $A_1, A_2, \ldots, A_m$ be (not necessarily distinct) subsets of a finite set E. Let $\mathcal{I}$ be the collection of all subsets $X \subseteq E$ such that X is a* partial transversal *of the family $(A_1, \ldots, A_m)$, meaning there exists an injection $\phi : X \to \{1, \ldots, m\}$*

with $x \in A_{\phi(x)}$ *for all* $x \in X$*. Then* $(E, \mathcal{I})$ *is a matroid, called the* transversal matroid *of the family.*

The proof of Proposition 6.10 uses Hall's Marriage Theorem and is omitted; see [6] for a complete proof.

4. Bases and Circuits

We now develop two additional structural perspectives on matroids: *bases* and *circuits*. These are analogues of the familiar notions from linear algebra and graph theory.

4.1. Bases.

Definition 4.1. A *base* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is an independent set of maximum cardinality. The collection of all bases is denoted $\mathcal{B}(\mathcal{M})$.

In a vector matroid $M[A]$, the bases are the maximal sets of linearly independent columns, i.e., the *column bases* of the matrix A. In the cycle matroid $M(G)$, the bases are the *spanning forests* of G; when G is connected, these are exactly the spanning trees.

Proposition 4.2. *All bases of a matroid* $\mathcal{M}$ *have the same cardinality. This common cardinality is called the* rank *of* $\mathcal{M}$*, denoted* $r(\mathcal{M})$*.*

Proof. Suppose B_1 and B_2 are bases of $\mathcal{M}$ with $|B_2| > |B_1|$. Since both are independent, by property I-3 there exists $x \in B_2 \setminus B_1$ such that $B_1 \cup \{x\}$ is independent. But B_1 is a base (maximal independent set), so no proper superset of B_1 can be independent. This is a contradiction. Therefore all bases have the same cardinality. □

The following lemma characterizes the base family of any matroid, and shows that bases *alone* determine the matroid.

Lemma 4.3 (Base Axioms)**.** *The set of bases* $\mathcal{B}$ *of a matroid with ground set* E *satisfies:*

- ***B-1.*** $\mathcal{B}$ *is nonempty.*
- ***B-2.*** *(*Exchange property*) If* $B_1, B_2 \in \mathcal{B}$ *and* $x_1 \in B_1 \setminus B_2$*, there exists* $x_2 \in B_2 \setminus B_1$ *such that* $B_1 \setminus \{x_1\} \cup \{x_2\}$ *is also a base.*

Proof. **B-1.** Since $\emptyset \in \mathcal{I}$, the family $\mathcal{I}$ is nonempty, and any maximal element of $\mathcal{I}$ (with respect to inclusion) is a base.

B-2. Let $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$. Set $B_1' = B_1 \setminus \{x_1\}$, so $B_1' \in \mathcal{I}$ by I-2, and $|B_2| = |B_1| > |B_1'|$. By I-3, there exists $x_2 \in B_2 \setminus B_1'$ with $B_1' \cup \{x_2\} \in \mathcal{I}$. Since $x_2 \in B_2 \setminus B_1'$ and $x_1 \notin B_2$, we have $x_2 \neq x_1$, so $x_2 \in B_2 \setminus B_1$. Furthermore $|B_1' \cup \{x_2\}| = |B_1'| + 1 = |B_1| = r(\mathcal{M})$, so it is a base. □

Theorem 4.4. *Let E be a finite set and $\mathcal{B}$ a nonempty collection of subsets of E satisfying **B-1** and **B-2**. Then the family $\mathcal{I}$ of all subsets of elements of $\mathcal{B}$ forms the independent sets of a matroid on E whose base family is exactly $\mathcal{B}$.*

Corollary 4.5. *A collection $\mathcal{B}$ of subsets of a finite ground set E is the set of bases of some matroid if and only if it satisfies the properties stated in Lemma 7.3.*

4.2. **Circuits.**

Definition 4.6. A *circuit* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a minimal dependent set, i.e., a subset $C \subseteq E$ such that $C \notin \mathcal{I}$ but $C \setminus \{x\} \in \mathcal{I}$ for every $x \in C$. The collection of all circuits is denoted $\mathcal{C}(\mathcal{M})$.

In a vector matroid $M[A]$, the circuits are the minimal linearly dependent sets of column vectors. In the cycle matroid $M(G)$, the circuits are precisely the simple cycles of G (closed walks with no repeated vertex).

Definition 4.7. An element $x \in E$ is a *loop* of $\mathcal{M}$ if $\{x\}$ is a circuit, i.e., if the singleton $\{x\}$ is already dependent. In a cycle matroid, a loop corresponds to a loop-edge (an edge from a vertex to itself).

Lemma 4.8 (Circuit Axioms)**.** *The set of circuits $\mathcal{C}$ of a matroid with ground set E satisfies:*

- ***C-1.*** $\emptyset \notin \mathcal{C}$.
- ***C-2.*** *If $C \in \mathcal{C}$, then no proper subset of C is in $\mathcal{C}$.*
- ***C-3.*** *(*Circuit elimination*) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $x \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{x\}$ contains a circuit.*

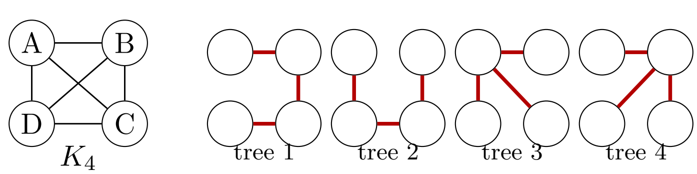
5. Minors

Given a matroid M on a set E and a subset T of E, there exists a matroid $M \backslash T$ on $E - T$ whose independent sets are those independent sets in M that are contained in $E - T$. This operation, more formally known as the operation of contraction, or the dual of the operation of deletion. These contracted sets are known as *minors*.

5.1. **Deletion.**

In the graph-theoretical space, deletion is basic property. For a graph G and a subset T of $E(G)$, $G \backslash T$ is the graph obtained by deleting the edges in T

Example 5.1. Example: Deleting Edges from K_4



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Axiom I-3 is sometimes called the *exchange property* or the *augmentation property*. It is the key axiom that distinguishes matroids from arbitrary simplicial complexes, and it encodes the "uniform" nature of independence.

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Proof. We verify the three axioms of Definition 6.1.

I-1. The empty set contains no vectors, and the empty linear combination trivially sums to zero with no nontrivial coefficients, so $\emptyset \in \mathcal{I}$.

I-2. Any subset of a linearly independent set is linearly independent, since dropping vectors cannot create a linear dependency.

I-3. Let $A, B \in \mathcal{I}$ with $|B| > |A|$. Let V be the vector space spanned by $A \cup B$, so $\dim V \geq |B|$. If $A \cup \{x\}$ were linearly dependent for every $x \in B \setminus A$, then V would be contained in the span of A, giving $\dim V \leq |A| < |B|$, a contradiction. Therefore there exists $x \in B \setminus A$ with $A \cup \{x\}$ linearly independent. $\square$

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Theorem 6.5 (Cycle Matroid)**.** *Let $G = (V, E)$ be a graph. Let $\mathcal{I}$ be the collection of subsets of E that contain no cycle. Then $(E, \mathcal{I})$ is a matroid, called the* cycle matroid *of G and denoted $M(G)$.*

Proof. We verify the three axioms.

I-1. The empty set of edges contains no cycle, so $\emptyset \in \mathcal{I}$.

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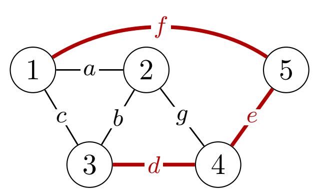
I-3. Let $A, B \in \mathcal{I}$ with $|B| > |A|$. A set of edges F with no cycle is a forest, and a forest on a connected component with k vertices has at most $k - 1$ edges. More precisely, the subgraph induced by A has $|V|$ vertices and $|A|$ edges and is a forest, so it has $|V| - |A|$ connected components; similarly the subgraph for B has $|V| - |B|$ components. Since $|B| > |A|$, the forest B spans more edges and thus has *fewer* components than A. Therefore some edge $e \in B \setminus A$ connects two different components of A, so $A \cup \{e\}$ is still acyclic, i.e., $A \cup \{e\} \in \mathcal{I}$. $\square$

Example 6.6. Example: A Vector Matroid Representation



independent set $\{d, e, f\}$

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6.4. Other Classes of Matroids.

For completeness we mention two additional families.

Proposition 6.9 (Uniform Matroid)**.** *Let E be a set with n elements, and let $0 \leq r \leq n$. Let $\mathcal{I}$ be the collection of all subsets $A \subseteq E$ with $|A| \leq r$. Then $(E, \mathcal{I})$ is a matroid, called the* uniform matroid $U_{r,n}$.

Proof. **I-1**: $|\emptyset| = 0 \leq r$, so $\emptyset \in \mathcal{I}$. **I-2**: If $|A| \leq r$ and $B \subseteq A$, then $|B| \leq |A| \leq r$. **I-3**: If $|A| < |B| \leq r$, pick any $x \in B \setminus A$; then $|A \cup \{x\}| = |A| + 1 \leq |B| \leq r$. □

Proposition 6.10 (Transversal Matroid)**.** *Let $A_1, A_2, \ldots, A_m$ be (not necessarily distinct) subsets of a finite set E. Let $\mathcal{I}$ be the collection of all subsets $X \subseteq E$ such that X is a* partial transversal *of the family $(A_1, \ldots, A_m)$, meaning there exists an injection $\phi : X \to \{1, \ldots, m\}$ with $x \in A_{\phi(x)}$ for all $x \in X$. Then $(E, \mathcal{I})$ is a matroid, called the* transversal matroid *of the family.*

The proof of Proposition 6.10 uses Hall's Marriage Theorem and is omitted; see [6] for a complete treatment.

7. Bases and Circuits

We now develop two additional structural perspectives on matroids: *bases* and *circuits*. These are analogues of the familiar notions from linear algebra and graph theory.

7.1. Bases.

Definition 7.1. A *base* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is an independent set of maximum cardinality. The collection of all bases is denoted $\mathcal{B}(\mathcal{M})$.

In a vector matroid $M[A]$, the bases are the maximal sets of linearly independent columns, i.e., the *column bases* of the matrix A. In the cycle matroid $M(G)$, the bases are the *spanning forests* of G; when G is connected, these are exactly the spanning trees.

Proposition 7.2. *All bases of a matroid $\mathcal{M}$ have the same cardinality. This common cardinality is called the* rank *of $\mathcal{M}$, denoted $r(\mathcal{M})$.*

Proof. Suppose B_1 and B_2 are bases of $\mathcal{M}$ with $|B_2| > |B_1|$. Since both are independent, by property I-3 there exists $x \in B_2 \setminus B_1$ such that $B_1 \cup \{x\}$ is independent. But B_1 is a base (maximal independent set), so no proper superset of B_1 can be independent. This is a contradiction. Therefore all bases have the same cardinality. □

The following lemma characterizes the base family of any matroid, and shows that bases *alone* determine the matroid.

Lemma 7.3 (Base Axioms)**.** *The set of bases $\mathcal{B}$ of a matroid with ground set E satisfies:*

B-1. *$\mathcal{B}$ is nonempty.*

B-2. *(*Exchange property*) If $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$, there exists $x_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{x_1\} \cup \{x_2\}$ is also a base.*

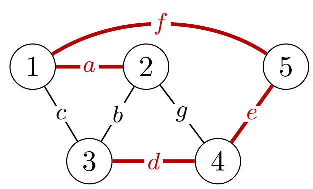
Proof. **B-1.** Since $\emptyset \in \mathcal{I}$, the family $\mathcal{I}$ is nonempty, and any maximal element of $\mathcal{I}$ (with respect to inclusion) is a base.

B-2. Let $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$. Set $B_1' = B_1 \setminus \{x_1\}$, so $B_1' \in \mathcal{I}$ by I-2, and $|B_2| = |B_1| > |B_1'|$. By I-3, there exists $x_2 \in B_2 \setminus B_1'$ with $B_1' \cup \{x_2\} \in \mathcal{I}$. Since $x_2 \in B_2 \setminus B_1'$ and $x_1 \notin B_2$, we have $x_2 \neq x_1$, so $x_2 \in B_2 \setminus B_1$. Furthermore $|B_1' \cup \{x_2\}| = |B_1'| + 1 = |B_1| = r(\mathcal{M})$, so it is a base. □

Theorem 7.4. *Let E be a finite set and $\mathcal{B}$ a nonempty collection of subsets of E satisfying* ***B-1*** *and* ***B-2****. Then the family $\mathcal{I}$ of all subsets of elements of $\mathcal{B}$ forms the independent sets of a matroid on E whose base family is exactly $\mathcal{B}$.*

Corollary 7.5. *A collection $\mathcal{B}$ of subsets of a finite ground set E is the set of bases of some matroid if and only if it satisfies the properties stated in Lemma 7.3.*

Example 7.6. Example: Bases in a Cycle Matroid



spanning tree $\{a, d, e, f\}$

In the cycle matroid $M(H)$ of graph H (Example 6.6), the bases are the spanning trees of H. For instance, $\{a, d, e, f\}$ is a base of $M(H)$ since it is a spanning tree of H. By Proposition 7.2, every spanning tree of a connected n-vertex graph has exactly $n - 1$ edges, confirming that all bases have the same cardinality.

7.2. **Circuits.**

Definition 7.7. A *circuit* of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a minimal dependent set, i.e., a subset $C \subseteq E$ such that $C \notin \mathcal{I}$ but $C \setminus \{x\} \in \mathcal{I}$ for every $x \in C$. The collection of all circuits is denoted $\mathcal{C}(\mathcal{M})$.

In a vector matroid $M[A]$, the circuits are the minimal linearly dependent sets of column vectors. In the cycle matroid $M(G)$, the circuits are precisely the simple cycles of G (closed walks with no repeated vertex).

Definition 7.8. An element $x \in E$ is a *loop* of $\mathcal{M}$ if $\{x\}$ is a circuit, i.e., if the singleton $\{x\}$ is already dependent. In a cycle matroid, a loop corresponds to a loop-edge (an edge from a vertex to itself).

Lemma 7.9 (Circuit Axioms)**.** *The set of circuits $\mathcal{C}$ of a matroid with ground set E satisfies:*

- ***C-1.*** $\emptyset \notin \mathcal{C}$.
- ***C-2.*** *If $C \in \mathcal{C}$, then no proper subset of C is in $\mathcal{C}$.*
- ***C-3.*** *(*Circuit elimination*) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $x \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{x\}$ contains a circuit.*

Proof. **C-1.** By I-1, $\emptyset \in \mathcal{I}$, so $\emptyset$ is not dependent and cannot be a circuit.

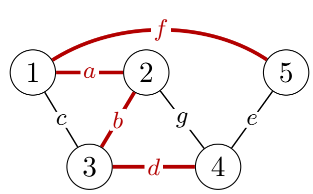
C-2. If C is minimal dependent and $C' \subsetneq C$, then C' is independent (by minimality of C), hence not a circuit.

C-3. Assume for contradiction that $(C_1 \cup C_2) \setminus \{x\}$ contains no circuit. Then it is independent. Since C_2 is a circuit, $C_2 \setminus \{x\}$ is independent, so by I-2 any subset of $C_2 \setminus \{x\}$ is in $\mathcal{I}$. Let I be a maximal independent subset of $C_1 \cup C_2$ that contains $C_2 \setminus \{x\}$. Since C_2 is a circuit, $x \notin I$. Also, $I \not\supseteq C_1$, so there is some $f \in C_1 \setminus I$. Note $e \in C_2 \setminus C_1$ and $f \in C_1 \setminus I$, so $e \neq f$ and in particular $|I| \leq |(C_1 \cup C_2) \setminus \{e, f\}|$. By the maximality assumption, $I \cup \{g\}$ is independent for some $g \in (C_1 \cup C_2) \setminus \{x\} \setminus I$. This contradicts the maximality of I. ☐

Theorem 7.10. *Let E be a finite set and $\mathcal{C}$ a collection of subsets of E satisfying **C-1**, **C-2**, and **C-3**. Then the family $\mathcal{I}$ of all subsets of E that contain no element of $\mathcal{C}$ forms the independent sets of a matroid on E whose circuit family is exactly $\mathcal{C}$.*

Corollary 7.11. *A collection $\mathcal{C}$ of subsets of a finite ground set E is the set of circuits of some matroid if and only if it satisfies the properties in Lemma 7.9.*

Example 7.12. Example: Circuits in a Cycle Matroid



circuit $\{a, b, d, f\}$

In the cycle matroid $M(H)$, the circuits are the simple cycles of H. For example, if $\{a, b, d, f\}$ forms a cycle in H, then it is a circuit of $M(H)$. Removing any single edge from this set yields an acyclic subgraph, confirming minimality.

8. The Rank Function

Whitney's original definition of a matroid used the rank function. We now show that the rank function encodes the same information as the independent sets.

8.1. **Definition and Axioms.**

Definition 8.1. The *rank function* $r_{\mathcal{M}} : 2^E \to \mathbb{N} \cup \{0\}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ is defined by

$$r(A) = \max\{|I| : I \subseteq A,\ I \in \mathcal{I}\}$$

for each $A \subseteq E$. In other words, $r(A)$ is the cardinality of the largest independent set contained in A. When $\mathcal{M}$ is clear from context, we write r for $r_{\mathcal{M}}$.

Lemma 8.2 (Rank Axioms)**.** *The rank function r of a matroid with ground set E satisfies:*

R-1. *For every $A \subseteq E$, $0 \le r(A) \le |A|$.*
R-2. *(*Monotonicity*) If $A \subseteq B \subseteq E$, then $r(A) \le r(B)$.*
R-3. *(*Submodularity*) For all $A, B \subseteq E$,*

$$r(A \cup B) + r(A \cap B) \le r(A) + r(B).$$

Proof. **R-1.** The rank is the size of a subset of A, so $0 \le r(A) \le |A|$.

R-2. Any independent subset of A is also a subset of B, so the maximum over subsets of A cannot exceed the maximum over subsets of B.

R-3. Let $X_\cap$ and $X_\cup$ be maximum independent subsets of $A \cap B$ and $A \cup B$, respectively, with $X_\cap \subseteq X_\cup$. By I-2, since $X_\cup \cap A \subseteq X_\cup$, it is independent, so $r(A) \ge |X_\cup \cap A|$. Similarly $r(B) \ge |X_\cup \cap B|$. Therefore

$$\begin{aligned}
r(A) + r(B) &\ge |X_\cup \cap A| + |X_\cup \cap B| \\
&= |(X_\cup \cap A) \cup (X_\cup \cap B)| + |(X_\cup \cap A) \cap (X_\cup \cap B)| \\
&= |X_\cup \cap (A \cup B)| + |X_\cup \cap (A \cap B)| \\
&= |X_\cup| + |X_\cup \cap (A \cap B)| \\
&= r(A \cup B) + r(A \cap B). here
\end{aligned}$$

□

Theorem 8.3. *Let E be a finite set, and let $r : 2^E \to \mathbb{N} \cup \{0\}$ satisfy **R-1**, **R-2**, and **R-3**. Then the collection $\mathcal{I} = \{A \subseteq E : r(A) = |A|\}$ forms the independent sets of a matroid on E with rank function r.*

Proof sketch. We verify the three independence axioms. **I-1**: $r(\emptyset) = 0 = |\emptyset|$, so $\emptyset \in \mathcal{I}$. **I-2**: Suppose $A \in \mathcal{I}$ and $B \subseteq A$. Then $r(A) = |A|$, and by R-2, $r(B) \leq r(A) = |A|$. Applying R-3 with the pair $(B, A \setminus B)$ shows $r(B) = |B|$, so $B \in \mathcal{I}$. **I-3**: Let $A, B \in \mathcal{I}$ with $|B| > |A|$. Suppose for contradiction that $A \cup \{x\} \notin \mathcal{I}$ for all $x \in B \setminus A$, meaning $r(A \cup \{x\}) = |A|$ for all such x. By Lemma 8.4 (stated below), $r(A \cup B) = r(A) = |A| < |B| = r(B)$, contradicting R-2 since $B \subseteq A \cup B$. $\square$

Lemma 8.4. *Let E be a finite set, r satisfy **R-1**–**R-3**, and let $A, B \subseteq E$ with $A \subseteq B$. If $r(A \cup \{x\}) = r(A)$ for every $x \in B \setminus A$, then $r(A \cup B) = r(A)$.*

Proof. Induction on $|B \setminus A|$. The base case $|B \setminus A| = 1$ is hypothesis. For the inductive step, let $B = A \cup \{x_1, \ldots, x_k\}$. By hypothesis, $r(A \cup \{x_k\}) = r(A)$, and by induction $r((A \cup \{x_k\}) \cup \{x_1, \ldots, x_{k-1}\}) = r(A \cup \{x_k\}) = r(A)$. $\square$

Corollary 8.5. *A function $r : 2^E \to \mathbb{N} \cup \{0\}$ is the rank function of a matroid on E if and only if it satisfies **R-1**, **R-2**, and **R-3**.*

8.2. Characterizing Independent Sets, Bases, and Circuits via Rank.

Proposition 8.6. *Let $\mathcal{M}$ be a matroid with rank function r on ground set E. For any subset $A \subseteq E$:*

(1) A is independent if and only if $r(A) = |A|$.
(2) A is a base if and only if $|A| = r(A) = r(\mathcal{M})$.
(3) A is a circuit if and only if $A \neq \emptyset$ and $r(A \setminus \{x\}) = |A| - 1 = r(A)$ for every $x \in A$.

Proof. Parts (1) and (2) follow immediately from the definitions of rank and base. For part (3): if A is a circuit, then A is dependent so $r(A) < |A|$, and every proper subset $A \setminus \{x\}$ is independent so $r(A \setminus \{x\}) = |A| - 1$. Since $r(A \setminus \{x\}) \leq r(A) \leq r(A \setminus \{x\}) + 1$ (adding one element changes rank by at most 1), we get $r(A) = |A| - 1$. $\square$

8.3. Rank Functions of Vector and Cycle Matroids.

Proposition 8.7. *Let A be a matrix over a field $\mathbb{F}$ and E its set of columns. The rank function of the vector matroid $M[A]$ satisfies $r(S) = \operatorname{rank}(A_S)$ for each $S \subseteq E$, where A_S is the submatrix of A with column set S.*

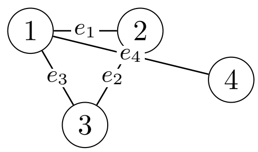
Proposition 8.8. *Let $G = (V, E)$ be a graph. The rank function of the cycle matroid $M(G)$ satisfies*

$$r(S) = |V| - c(S)$$

for each $S \subseteq E$, where $c(S)$ is the number of connected components of the spanning subgraph (V, S) (the graph on all vertices of G using only the edges in S).

Proof. A maximum acyclic subgraph of (V, S) is a spanning forest of (V, S). A forest on a graph with $|V|$ vertices and c connected components has exactly $|V| - c$ edges. Therefore $r(S)$ equals the number of edges in a spanning forest of (V, S), which is $|V| - c(S)$. $\square$

Example 8.9. Example: Rank Computation in a Graph



$r(E) = 4 - 1 = 3$
$r(\emptyset) = 4 - 4 = 0$
$r(\{e_1, e_2, e_3\}) = 4 - 2 = 2$

Let G be a triangle on vertices $\{1, 2, 3\}$ with edges $e_1 = 12$, $e_2 = 23$, $e_3 = 13$, plus a pendant edge $e_4 = 14$. Then $|V| = 4$ and $c(\emptyset) = 4$, so $r(\emptyset) = 0$. Also $c(\{e_1, e_2, e_3\}) = 2$ (vertices $\{1, 2, 3\}$ are connected and vertex 4 is isolated), giving $r(\{e_1, e_2, e_3\}) = 4 - 2 = 2$.

9. Duality

One of the most elegant features of matroid theory is the notion of *duality*. Every matroid has a natural dual, and the dual of a cycle matroid of a planar graph is again a cycle matroid—of the *planar dual graph*.

9.1. **Dual Matroids.**

Theorem 9.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with base family $\mathcal{B}$. Define*

$$\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}.$$

Then $\mathcal{B}^*$ is the base family of a matroid on E, called the dual matroid $\mathcal{M}^* = (E, \mathcal{I}^*)$.

Proof. We verify axioms B-1 and B-2 for $\mathcal{B}^*$. Axiom B-1 holds because $\mathcal{B} \neq \emptyset$ implies $\mathcal{B}^* \neq \emptyset$. For B-2: let $B_1^*, B_2^* \in \mathcal{B}^*$ with $x \in B_1^* \setminus B_2^*$. Then $B_i = E \setminus B_i^*$ are bases of $\mathcal{M}$ with $x \notin B_1$ and $x \in B_2$. Since $x \in B_2 \setminus B_1$, by the exchange property for $\mathcal{B}$ (B-2) there exists $y \in B_1 \setminus B_2$ with $(B_1 \setminus \{y\}) \cup \{x\} \in \mathcal{B}$. Equivalently, $y \notin B_2$ and $y \in B_1$, so $y \in B_1^*$ and $y \notin B_2^*$, and $(B_1^* \setminus \{x\}) \cup \{y\} = E \setminus ((B_1 \setminus \{y\}) \cup \{x\}) \in \mathcal{B}^*$. $\square$

Definition 9.2. The bases of $\mathcal{M}^*$ are called *cobases* and the circuits of $\mathcal{M}^*$ are called *cocircuits* of $\mathcal{M}$. The rank function of $\mathcal{M}^*$ is called the *corank function* and is denoted r^*.

Proposition 9.3. *For any matroid $\mathcal{M}$, $(\mathcal{M}^*)^* = \mathcal{M}$.*

Proof. The bases of $(\mathcal{M}^*)^*$ are $\{E \setminus B^* : B^* \in \mathcal{B}^*\} = \{E \setminus (E \setminus B) : B \in \mathcal{B}\} = \mathcal{B}$. $\square$

Proposition 9.4. *Let $\mathcal{M}$ be a matroid with ground set E and rank function r. Then for every $A \subseteq E$,*

$$r^*(A) = |A| - r(\mathcal{M}) + r(E \setminus A).$$

Proof. The corank $r^*(A)$ equals the cardinality of the largest independent set of $\mathcal{M}^*$ contained in A. A set $I \subseteq A$ is independent in $\mathcal{M}^*$ if and only if it is a subset of some cobase, i.e., the complement $E \setminus I$ contains a base of $\mathcal{M}$. This holds if and only if $r(E \setminus I) = r(\mathcal{M})$. One can show (see [6]) that the maximum such $|I|$ equals $|A| - r(\mathcal{M}) + r(E \setminus A)$. □

9.2. **Cocircuits and the Graph-Theoretic Dual.**

Theorem 9.5. *Let $\mathcal{M}$ be a matroid.*

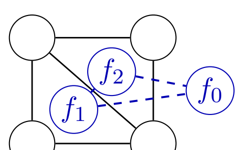
(1) A set C^* is a cocircuit of $\mathcal{M}$ if and only if C^* has nonempty intersection with every base of $\mathcal{M}$, and is minimal with this property.
(2) A set B is a base of $\mathcal{M}$ if and only if B has nonempty intersection with every cocircuit of $\mathcal{M}$, and is maximal with this property.

The following is a landmark result connecting planar graph duality and matroid duality.

Theorem 9.6 (Whitney's Planarity Theorem)**.** *Let G be a connected graph. Then G is planar if and only if the dual matroid $M(G)^*$ is graphic. Moreover, when G is planar with planar dual G^*, we have $M(G)^* \cong M(G^*)$.*

Proof sketch. If G is planar, the planar dual G^* is defined by placing a vertex in each face of G and connecting two such vertices by an edge for each edge of G separating the corresponding faces. The spanning trees of G are in bijection with the complements of spanning trees of G^* (a classical fact in planar graph theory), so the bases of $M(G)$ are exactly the complements of the bases of $M(G^*)$, giving $M(G)^* = M(G^*)$. The converse direction requires more work; see [6], Chapter 5. □

Example 9.7. Example: Planar Dual Matroids



H in black, H^* in blue

The cycle matroid $M(H)$ has a dual $M(H)^* \cong M(H^*)$. The cocircuits of $M(H)$ (i.e., the circuits of $M(H)^*$) correspond to the minimal edge cuts (bonds) of H.

10. Minors

We now introduce the operations of *deletion* and *contraction*, which allow us to build new matroids from old ones. The resulting matroids are called *minors*.

10.1. **Deletion and Contraction.**

Definition 10.1. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $A \subseteq E$. The *deletion* of A from $\mathcal{M}$, denoted $\mathcal{M} \setminus A$ (or $\mathcal{M}|(E \setminus A)$), is the matroid on ground set $E \setminus A$ whose independent sets are $\{I \in \mathcal{I} : I \subseteq E \setminus A\}$.

Definition 10.2. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $A \subseteq E$. The *contraction* of A from $\mathcal{M}$, denoted $\mathcal{M}/A$, is the matroid on ground set $E \setminus A$ defined by

$$\mathcal{M}/A = (\mathcal{M}^* \setminus A)^*.$$

Equivalently, its rank function satisfies $r_{\mathcal{M}/A}(X) = r_{\mathcal{M}}(X \cup A) - r_{\mathcal{M}}(A)$ for all $X \subseteq E \setminus A$.

Both deletion and contraction are naturally interpretable in the graph-theoretic setting.

Proposition 10.3. *Let $G = (V, E)$ be a graph and $e \in E$.*

(1) $M(G) \setminus \{e\} = M(G \setminus e)$, where $G \setminus e$ is the graph obtained by removing edge e.
(2) $M(G)/\{e\} = M(G/e)$, where G/e is the graph obtained by contracting edge e (identifying its endpoints).

Proof. For part (1): the edges of $G \setminus e$ that contain no cycle are exactly the subsets of $E \setminus \{e\}$ containing no cycle in G, which are the independent sets of $M(G) \setminus \{e\}$.

For part (2): contracting $e = uv$ identifies u and v into a single vertex $\overline{v}$, and the remaining edges are in bijection with $E \setminus \{e\}$. A subset $S \subseteq E \setminus \{e\}$ is acyclic in G/e if and only if $S \cup \{e\}$ has rank $|S| + 1$ in $M(G)$, which matches the contraction rank formula $r_{M(G)/\{e\}}(S) = r_{M(G)}(S \cup \{e\}) - r_{M(G)}(\{e\})$. □

Lemma 10.4. *Let $\mathcal{M}$ be a matroid with ground set E, and let A and B be disjoint subsets of E. Then:*

(1) $(\mathcal{M} \setminus A) \setminus B = \mathcal{M} \setminus (A \cup B) = (\mathcal{M} \setminus B) \setminus A$.
(2) $(\mathcal{M}/A)/B = \mathcal{M}/(A \cup B) = (\mathcal{M}/B)/A$.
(3) $(\mathcal{M}/A) \setminus B = (\mathcal{M} \setminus B)/A$.

Lemma 10.4 shows that any sequence of deletions and contractions can be reduced to a single deletion and a single contraction, applied in either order.

10.2. **Minors.**

Definition 10.5. A matroid $\mathcal{N}$ is a *minor* of $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by a sequence of deletions and contractions. By Lemma 10.4, this is equivalent to saying $\mathcal{N} = \mathcal{M} \setminus A/B$ for some disjoint $A, B \subseteq E(\mathcal{M})$.

Proposition 10.6. *A matroid $\mathcal{N}$ is a minor of $\mathcal{M}$ if and only if $\mathcal{N}^*$ is a minor of $\mathcal{M}^*$.*

Definition 10.7. A class of matroids $\mathcal{K}$ is *minor-closed* if, whenever $\mathcal{M} \in \mathcal{K}$ and $\mathcal{N}$ is a minor of $\mathcal{M}$, then $\mathcal{N} \in \mathcal{K}$.

Proposition 10.8. *The class of graphic matroids is minor-closed.*

Proof. By Proposition 10.3, deleting or contracting an edge of a graph produces a new graph, and the cycle matroid of the new graph equals the corresponding deletion or contraction of the cycle matroid. Since any sequence of deletions and contractions of a graph G yields another graph G', we have $M(G) \setminus A/B = M(G')$, which is graphic. □

11. Representability and Regularity

We now study when a matroid can be represented by a matrix, and identify the key algebraic property that characterizes graphic matroids.

11.1. Representability.

Definition 11.1. A matroid $\mathcal{M}$ on ground set E with $|E| = n$ is $\mathbb{F}$-*representable* (or *representable over the field* $\mathbb{F}$) if there exists a matrix A with n columns over $\mathbb{F}$ such that $\mathcal{M} \cong M[A]$, i.e., a subset of columns of A is linearly independent if and only if the corresponding subset of E is independent in $\mathcal{M}$. The matrix A is called an $\mathbb{F}$-*representation* of $\mathcal{M}$.

Definition 11.2. A matroid is *binary* if it is representable over $GF(2) = \mathbb{Z}/2\mathbb{Z}$. A matroid is *ternary* if it is representable over $GF(3) = \mathbb{Z}/3\mathbb{Z}$. A matroid is *representable* (without qualification) if there exists *some* field $\mathbb{F}$ over which it is representable.

Definition 11.3. A matroid $\mathcal{M}$ is *regular* if it is representable over every field $\mathbb{F}$. Equivalently, $\mathcal{M}$ is regular if it has a *totally unimodular* representation: a real matrix A with the property that every square submatrix has determinant in $\{-1, 0, +1\}$.

Theorem 11.4. *Every graphic matroid is regular.*

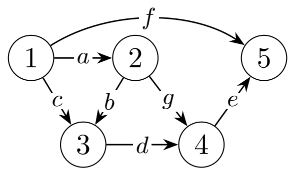
Proof. Let $G = (V, E)$ be a graph. We use the *vertex-edge incidence matrix* A_G. Orient each edge of G arbitrarily; then define A_G to be the matrix with rows indexed by V and columns indexed by E, where the entry (v, e) is $+1$ if v is the head of e, -1 if v is the tail of e, and 0 otherwise.

Claim 11.5. *A set $S \subseteq E$ of columns of A_G is linearly dependent over $\mathbb{R}$ (or any field) if and only if S contains a cycle of G.*

Indeed, if S contains a cycle $v_0, v_1, \ldots, v_k = v_0$, then the signed sum of the corresponding column vectors along the cycle is zero (each vertex is the head of exactly one edge and the tail of another), giving a linear dependency. Conversely, if S is linearly dependent, one can trace a dependency in the column space back to a cycle in G (a standard argument; see [6], Theorem 5.1.3).

Since each entry of A_G is in $\{-1, 0, 1\}$ and every square submatrix has determinant ± 1 or 0 (by the theory of totally unimodular matrices; see [8]), A_G is a totally unimodular representation of $M(G)$. Therefore $M(G)$ is regular. ☐

Example 11.6. Example: Incidence Matrix Representation



$$A_H = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

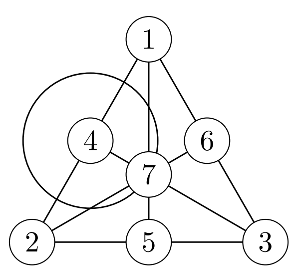
Consider graph H with vertices $\{1, 2, 3, 4, 5\}$ and edges $a = 12, b = 23, c = 13, d = 34, e = 45, f = 14, g = 15$. After orienting each edge, the vertex-edge incidence matrix A_H has the form shown above and is totally unimodular.

11.2. **The Fano Matroid and Non-Representability.**

Definition 11.7. The *Fano matroid* F_7 is defined on the ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$, where a set of three elements is a circuit if and only if the corresponding three points are collinear in the *Fano plane*—the projective plane over $GF(2)$. The seven lines of the Fano plane are $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{4, 5, 7\}$, $\{5, 6, 1\}$, $\{6, 7, 2\}$, $\{7, 1, 3\}$.

Proposition 11.8. *The Fano matroid F_7 is representable over $GF(2)$ but over no field of characteristic other than 2. In particular, F_7 is not regular.*

Example 11.9. Example: The Fano Plane



11.3. **The Uniform Matroid $U_{2,4}$ and Binary Matroids.**

Proposition 11.10. *The uniform matroid $U_{2,4}$ is not binary. Its circuits are all 3-element subsets of a 4-element ground set.*

Proof. Suppose $U_{2,4}$ were representable over $GF(2)$. Then we would need a 2×4 matrix over $GF(2)$ with no two columns proportional and with no three columns summing to zero. But over $GF(2)$, the only nonzero 2-vectors are $\binom{1}{0}, \binom{0}{1}, \binom{1}{1}$—only three such vectors exist, so we cannot choose four distinct nonzero columns. Contradiction. $\square$

Theorem 11.11 (Tutte, 1958)**.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

12. Excluded-Minor Characterization of Graphic Matroids

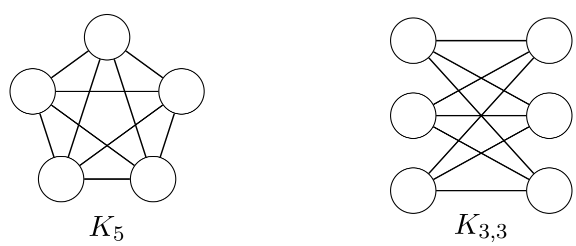
We now arrive at the main theorem of the paper. The five excluded minors for the class of graphic matroids are $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, and $M^*(K_{3,3})$.

12.1. The Five Excluded Minors.
We have already met $U_{2,4}$ (Section 11.3) and F_7 (Section 11.2). We now introduce the remaining three.

Definition 12.1. The *dual Fano matroid* F_7^* is the dual of F_7. Its circuits are the complements of the cobases of F_7, and it is representable over $GF(2)$ but not over any field of odd characteristic.

Definition 12.2. The matroids $M^*(K_5)$ and $M^*(K_{3,3})$ are the duals of the cycle matroids of the complete graph K_5 (5 vertices, 10 edges) and the complete bipartite graph $K_{3,3}$ (6 vertices, 9 edges), respectively. These are the dual cocycle matroids of the two Kuratowski graphs.

Example 12.3. Example: Kuratowski Graphs K_5 and $K_{3,3}$



The significance of $M^*(K_5)$ and $M^*(K_{3,3})$ comes from their connection to planar graphs via the following classical theorem:

Theorem 12.4 (Kuratowski's Theorem, 1930)**.** *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Theorem 12.5 (Whitney's 2-Isomorphism Theorem)**.** *Two graphs G and H have isomorphic cycle matroids, $M(G) \cong M(H)$, if and only if G and H are* 2-isomorphic*, meaning one can be obtained from the other by a sequence of operations that preserve 2-connectivity and cycle structure, specifically Whitney twists (reversing a 2-separation) and vertex identifications.*

Whitney's 2-Isomorphism Theorem shows that the cycle matroid loses vertex-connectivity information but captures edge-dependency exactly.

12.2. **Statement and Proof of Tutte's Theorem.** We are now ready to state and prove the main theorem.

Theorem 12.6 (Tutte's Excluded-Minor Theorem, [10, 11])**.** *A matroid $\mathcal{M}$ is graphic if and only if it is regular and contains no minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$.*

We break the proof into two directions.

Necessary Conditions (⇒).

Lemma 12.7. *No graphic matroid contains $U_{2,4}$ as a minor.*

Proof. Since graphic matroids are regular (Theorem 11.4) and binary (Theorem 11.11), and $U_{2,4}$ is not binary (Proposition 11.10), no graphic matroid can contain $U_{2,4}$ as a minor. □

Lemma 12.8. *No graphic matroid contains F_7 or F_7^* as a minor.*

Proof. Graphic matroids are representable over every field (Theorem 11.4). In particular, they are representable over $\mathbb{Q}$. However, F_7 is not representable over any field of characteristic $\neq 2$, and F_7^* likewise. Since $\mathbb{Q}$ has characteristic 0, neither F_7 nor F_7^* can be a minor of a graphic matroid (minors of representable matroids are representable over the same field). □

Lemma 12.9. *No graphic matroid contains $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor.*

Proof sketch. By Proposition 10.6, $M^*(K_5)$ is a minor of $\mathcal{M}$ if and only if $M(K_5)$ is a minor of $\mathcal{M}^*$. If $\mathcal{M} = M(G)$, then $\mathcal{M}^* = M(G)^*$, which is graphic (and equals $M(G^*)$) only when G is planar by Theorem 9.6. A non-planar graph G contains K_5 or $K_{3,3}$ as a minor (Wagner's Theorem; see [4]), so if G is planar, neither K_5 nor $K_{3,3}$ is a minor of G, hence $M^*(K_5)$ and $M^*(K_{3,3})$ are not minors of $M(G)$. □

Sufficient Conditions (⇐). The converse direction—showing that every regular matroid avoiding the five excluded minors is graphic—uses Seymour's Decomposition Theorem.

Theorem 12.10 (Seymour's Decomposition Theorem, [9])**.** *Every regular matroid is a 3-sum of graphic matroids, cographic matroids, and copies of the unique regular matroid R_{10} (the 10-element rank-5 matroid).*

Definition 12.11. A *cographic matroid* is the dual of a graphic matroid. Equivalently, $\mathcal{M}$ is cographic if $\mathcal{M} \cong M(G)^*$ for some graph G.

Proof of sufficiency in Theorem 12.6. Suppose $\mathcal{M}$ is regular and contains none of $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, $M^*(K_{3,3})$ as a minor. By Seymour's Theorem 12.10, $\mathcal{M}$ decomposes as a 3-sum of graphic, cographic, and R_{10} components. One shows:

(1) A cographic component $M(G)^*$ that is not graphic would force $M^*(K_5)$ or $M^*(K_{3,3})$ to be a minor (by Theorem 9.6 and Kuratowski's Theorem), contradicting our assumption.
(2) The matroid R_{10} has $M^*(K_5)$ as a minor, so it cannot appear.

Therefore $\mathcal{M}$ decomposes entirely into graphic components, and a 3-sum of graphic matroids is graphic (see [6], Chapter 9). Hence $\mathcal{M}$ is graphic. □

12.3. **Consequences and the Excluded-Minor Table.** We summarize the excluded-minor characterizations of the main classes of matroids in the following table.

Class	**Excluded Minors**	**Reference**
Binary	$U_{2,4}$	[10]
Ternary	$U_{2,5}, U_{3,5}, F_7, F_7^*$	[3]
Regular	$U_{2,4}, F_7, F_7^*$	[10]
Graphic	$U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})$	[10, 11]
Cographic	$U_{2,4}, F_7, F_7^*, M(K_5), M(K_{3,3})$	Dual of above

Corollary 12.12. *Every graphic matroid is both binary and regular. The class of graphic matroids is properly contained in the class of regular matroids, which is properly contained in the class of binary matroids.*

Proof. The inclusions follow from comparing excluded-minor lists: every excluded minor for "binary" ($U_{2,4}$) is already an excluded minor for "graphic", so graphic $\subseteq$ binary. Similarly graphic $\subseteq$ regular. The R_{10} matroid is regular but not graphic, and $U_{2,4}$ is binary but not regular, so both inclusions are strict. □

13. Further Questions

The theory of graphical matroids opens onto a wide panorama of research. We close with several directions for further exploration.

Rota's Conjecture. The central open problem in matroid representability is Rota's Conjecture (1970): for every prime power q, there are only finitely many excluded minors for representability over $GF(q)$. This was proved for $q = 2$ (one excluded minor: $U_{2,4}$), $q = 3$ (four excluded minors), and $q = 4$ (41 excluded minors), but remains open for $q \geq 5$. In 2014, Geelen, Gerards, and Whittle announced a proof of Rota's Conjecture in full generality, with details still being written; see [5].

The Graph Minor Theorem. Robertson and Seymour proved that every minor-closed class of graphs is characterized by a finite set of excluded minors [7]. The analogous problem for matroids—whether every minor-closed class of matroids has a finite excluded-minor characterization—is wide open. Only certain well-structured classes are known to have finite characterizations.

Algebraic Matroids. A matroid is *algebraic* over a field $\mathbb{F}$ if its elements can be assigned transcendental elements of a field extension $K/\mathbb{F}$ such that independence corresponds to algebraic independence. Every $\mathbb{F}$-representable matroid is algebraic over $\mathbb{F}$, but the converse fails: the Vámos matroid V_8 is algebraic over $\mathbb{Q}$ (in some conventions) but not representable over any field. The problem of characterizing algebraic matroids remains largely open.

Log-concavity and the Chow Ring. Recent spectacular results by Adiprasito, Huh, and Katz [1] resolved the long-standing Read–Heron–Rota conjectures by proving that the coefficients of the characteristic polynomial of any matroid form a *log-concave* sequence. This was achieved by associating to each matroid a "Chow ring" satisfying Poincaré duality, the Hard Lefschetz Theorem, and the Hodge–Riemann relations. These results demonstrate that matroids have a deep geometric nature extending far beyond their combinatorial definition.

Matroid Union and Intersection. The matroid union theorem (Nash-Williams, 1961) states that the union of k matroids on the same ground set is again a matroid. Matroid intersection—finding a common independent set of two matroids of maximum cardinality—is solvable in polynomial time and unifies a wide range of combinatorial optimization problems, from bipartite matching to arborescence packing.

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