A Reflected Stochastic-Volatility Model

Computing option Greeks with Malliavin calculus

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The Greeks, and why they are hard here

An option's price depends on the stock price, the volatility, and interest rates. The Greeks measure how it reacts when these move: Delta, Gamma, Vega.

In Black–Scholes there is a price formula, so you get the Greeks by differentiating it. But that model assumes:

- volatility is constant,
- interest rates are constant,
- prices move smoothly.

Once the model is realistic, there is no formula left to differentiate.

A way around the missing formula

Malliavin calculus lets us differentiate random variables directly. It rewrites each Greek as an average,

$$\text{Greek} = \mathbb{E}\big[\,\text{payoff} \times \text{weight}\,\big],$$

which we can estimate by simulation.

Two things have to be true:

- the price is Malliavin differentiable;
- the Malliavin covariance is non-degenerate.

The second one is the hard part, and it is what this project is about.

Starting point

The simplest stock model is geometric Brownian motion,

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

The first term is steady growth, the second is random noise. The problem is that the drift μ and the volatility σ are constants.

Making volatility and rates random

Give volatility its own equation, as Heston does:

$$d\sigma_t = \kappa(\theta - \sigma_t)\, dt + \xi\, b(\sigma_t)\, dW_t^\sigma.$$

It drifts back toward a long-run level θ, but wobbles on the way.

The short rate gets the same treatment (Vasicek). And price and volatility are negatively correlated: when prices fall, volatility tends to rise.

Why bound the volatility?

Two reasons, independent of each other.

Mathematical. A floor $\sigma_{\text{min}} > 0$ keeps the volatility away from zero, which is exactly where the Heston square root degenerates and where the Feller condition is needed. A ceiling $\sigma_{\text{max}} < \infty$ stops the moment explosion the unreflected model can have.

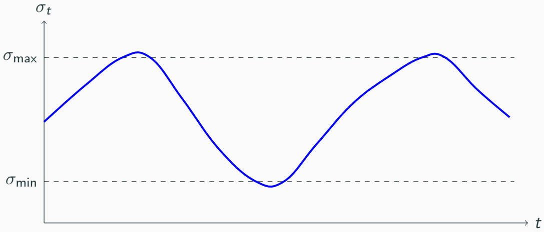
Markets. Equity markets limit how far a price can move in a session — price bands, cooling-off rules, market-wide halts — which caps realised volatility over short horizons.

The second is motivation only. No result below uses any particular market rule.

Reflection

Keep the volatility inside a band, pushing it back at each wall:

$$d\sigma_t = \kappa(\theta - \sigma_t)\,dt + \xi b(\sigma_t)\,dW_t^\sigma + dL_t^{\min} - dL_t^{\max}.$$



The two push terms act only when σ is sitting on a wall.

The full model

$$\begin{aligned} dS_t &= r_t S_t \, dt + \sigma_t S_t \, dW_t^S, \\ d\sigma_t &= \kappa(\theta - \sigma_t) \, dt + \xi b(\sigma_t) \, dW_t^\sigma + dL_t^{\min} - dL_t^{\max}, \\ dr_t &= \kappa_r(\theta_r - r_t) \, dt + \eta \, dW_t^r. \end{aligned}$$

The one thing to hold onto: the volatility can never drop below $\sigma_{\min}$. That floor is what the main result runs on.

What a Malliavin derivative is

A random variable is really a function of the whole random path. The Malliavin derivative $D_s F$ asks: if I nudge the path near time s, how much does F change?

The simplest case:

$$D_s W_t = 1 \text{ if } s \leq t, \qquad D_s W_t = 0 \text{ if } s > t.$$

A nudge now shifts everything after it by the same amount, but it cannot change the past.

The quantity that decides everything

From these derivatives, build the Malliavin covariance of the final price,

$$\gamma_{S_T} = \sum_j \int_0^T \left(D_s^j S_T\right)^2 ds.$$

If γ is positive, and $1/\gamma$ has finite averages of every order, then the price has a smooth density and the Greek weights exist.

So everything reduces to showing γ does not collapse to zero.

First result: the price has finite variance

Proposition (proved)

S_t is square integrable for every t, and $\sup_{t \leq T} \mathbb{E}[S_t^2] < \infty$.

Reflection makes this easy. The volatility is bounded, and boundedness on its own is enough.

The local time does not enter the proof at all. And there is a bonus: the unreflected model can have infinite moments past a threshold, which reflection rules out.

Second result: differentiating the price

The price depends on the volatility, so we need the derivative of the *reflected* volatility. The useful observation:

nudging the path is the same as adding a drift to the volatility equation.

Away from the walls, the nudge is carried forward by a factor J:

$$D_s^j \sigma_u = \xi\, b(\sigma_s)\, \ell_j^\sigma\, J_{s,u} \qquad \text{(proved).}$$

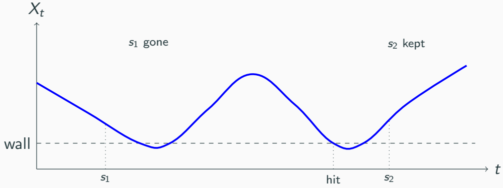
Nothing unusual happens inside the band.

At the wall, the derivative forgets

At a wall it is different. If the volatility touches a wall, any nudge made before that gets wiped out. Only nudges after the last contact still count.

For plain reflected Brownian motion I can prove this exactly:

$$D_s X_t = \mathbf{1}_{\{\text{no wall contact on } (s,t]\}}.$$



What I have, and what I don't

I can prove the derivative inside the band, and the wall behaviour for plain reflected Brownian motion.

For the full volatility, the exact formula at the wall,

$$D_s^j \sigma_t = \xi b(\sigma_s)\, \ell_j^\sigma\, J_{s,t}\, \mathbf{1}_{\{\text{no wall contact on } (s,t]\}},$$

I state as a conjecture. Its full proof leans on a theorem of Lipshutz and Ramanan, and one step there I have not finished.

The main result does not use this conjecture.

Main result: the idea

Write the derivative of the log-price as a main part plus a leftover:

$$D_s^1 Y_T = \sigma_s + R_s.$$

- σ_s is at least $\sigma_{\min}$, and it is the direct effect of volatility on the price. Reflection does not touch it.
- R_s is small when s is close to maturity T.

So just before maturity the derivative stays above the floor, give or take a small error. That is the whole engine.

Main result: the statement

On a short window before T, the covariance is bounded below by the floor minus a small remainder. Controlling the remainder gives:

Theorem (proved)

For every q, and small ε, $\quad \mathbb{P}\big(\gamma_{Y_T} < \varepsilon\big) \le C_q\, \varepsilon^q$.

So γ is almost never small. That gives $1/\gamma$ finite averages of every order, a smooth density, and the Greek weights.

The only input is $\sigma \ge \sigma_{\min}$. No Feller condition is needed.

The Greeks

With that in hand, Clark–Ocone and integration by parts give the weights, so each Greek is an average of the payoff times a weight.

As a check, in the Black–Scholes case my Delta weight reduces to the familiar $W_T^S/(S_0\,\sigma\,T)$, which is a good sign the setup is right.

One consequence I like, though it rests on the open conjecture: sensitivity to the starting volatility is erased at the first wall contact, so Vega only accumulates before the volatility first reaches a boundary.

Where things stand

Result	Status
Well-posed model; finite variance	proved
Malliavin differentiability	proved
Derivative inside the band; simple wall case	proved
Exact wall formula (full volatility)	open (conjecture)
Non-degeneracy of the covariance	proved
Smooth density; Greek weights exist	proved
Delta; Black–Scholes check	proved
Explicit reflected weights; Vega absorption	open

Conclusion

I took a Heston-type volatility, reflected it into a band, and added a Vasicek rate.

I proved the price is square integrable and Malliavin differentiable, and that the Malliavin covariance stays non-degenerate. That last one is what makes the Greeks exist, and it holds under weaker assumptions than the usual model needs, because the floor does the work.

Still open: the exact form of the derivative at the wall, and the explicit weights built on it.

Thank you. Questions?