Malliavin Calculus for a Reflected Stochastic-Volatility Model: Differentiability, Non-Degeneracy, and Greeks

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Abstract

We look at a stock price whose volatility is a diffusion reflected inside a band $[\sigma_{\min}, \sigma_{\max}]$, with a Vasicek short rate, and with correlated driving Brownian motions. We prove the price is square integrable and Malliavin differentiable. We then compute its Malliavin derivative, and show how reflection changes the chain rule at the boundary. The main result is that the floor $\sigma_{\min}$ forces the Malliavin covariance to be non-degenerate, with inverse moments of every order. Because of this the Greeks exist as Malliavin weights, and no Feller condition is needed. We finish with the Clark–Ocone representation and expressions for Delta, Gamma and Vega.

Contents

1 Introduction **2**

2 The model **2**

3 Background **4**
3.1 The Skorokhod problem and reflection 4
3.2 The Malliavin derivative 4
3.3 Non-degeneracy and the Greeks 5

4 Square integrability **5**

5 The Malliavin derivative **6**
5.1 Cameron–Martin perturbations act as drift perturbations 6
5.2 Differentiability 6
5.3 The interior flow and a size bound 7
5.4 The boundary 8
5.5 The derivative of the price 9

6 Non-degeneracy of the Malliavin covariance **9**

7 Clark–Ocone representation and the Greeks **11**

8 Status of the results **13**

9 Conclusion **13**

1 Introduction

The Greeks tell us how an option price responds to its inputs. If the model has a closed pricing formula, we just differentiate it. Most realistic models do not have one. The method of Fournié et al. [2] gets around this. It writes each Greek as the expected value of the payoff times a random weight, so no formula is needed. Two things have to be true for it to work. The random variable must be Malliavin differentiable, and its Malliavin covariance must be non-degenerate, meaning the inverse of that covariance has moments of every order.

The second condition is the hard one. In the Heston model it is hard because the volatility can get close to zero, and there the square-root diffusion degenerates. The Feller condition is the usual way of controlling this, and proving Malliavin regularity for the Heston volatility is the main work of [1].

In this paper the volatility is instead *reflected* inside a band $[\sigma_{\min}, \sigma_{\max}]$ with $\sigma_{\min} > 0$. It cannot fall to zero, and it cannot run off to infinity. Our point is that reflection is not just an obstacle to get past. It is a regularisation. The floor $\sigma_{\min}$ is a uniform ellipticity condition on the diffusion coefficient of the log-price, and the model hands it to us instead of our having to assume it. From the floor alone we prove that the Malliavin covariance is non-degenerate, with inverse moments of every order, by an elementary small-ball argument (Theorem 6.2). No Feller-type condition appears anywhere. The ceiling $\sigma_{\max}$ does a second job. It rules out the moment explosion that the unreflected model can have, and it gives square integrability at once.

The individual pieces are known. Malliavin Greeks for stochastic volatility are standard [2, 1]. Malliavin differentiability of reflected stochastic differential equations is known in one dimension [6]. Pathwise differentiability of reflected diffusions in their parameters is known, through a constrained linear equation with jumps [5]. What is new is putting these together, and the consequence just described. We are careful to separate what we prove from what we assume. We compute the derivative of the reflected volatility in the interior, and we prove the boundary behaviour in a prototype case, but the exact form at the boundary is only a conjecture (Conjecture 5.6). The main theorem is set up so that it does not need that conjecture.

2 The model

Fix a horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ carrying a standard three-dimensional Brownian motion $B = (B^1, B^2, B^3)$, with $\{\mathcal{F}_t\}$ the augmented filtration it generates and $\mathbb{P}$ the pricing measure. Correlated drivers are built from B by Cholesky factorisation of the instantaneous correlation matrix: with $\alpha = \sqrt{1-\rho_{S\sigma}^2}$, $\beta = (\rho_{\sigma r} - \rho_{S\sigma}\rho_{Sr})/\alpha$ and $\gamma = \sqrt{1-\rho_{Sr}^2 - \beta^2}$,

$$W_t^S = B_t^1, \qquad W_t^\sigma = \rho_{S\sigma} B_t^1 + \alpha B_t^2, \qquad W_t^r = \rho_{Sr} B_t^1 + \beta B_t^2 + \gamma B_t^3. \tag{1}$$

Write $\ell^S = (1,0,0)$, $\ell^\sigma = (\rho_{S\sigma}, \alpha, 0)$ and $\ell^r = (\rho_{Sr}, \beta, \gamma)$ for the rows of the Cholesky factor, so that ℓ^X_j is the coefficient of B^j in the driver W^X.

The model is

$$dS_t = r_t S_t\,dt + \sigma_t S_t\,dW_t^S, \qquad S_0 > 0, \tag{2}$$

$$d\sigma_t = \kappa(\theta - \sigma_t)\,dt + \xi\,b(\sigma_t)\,dW_t^\sigma + dL_t^{\min} - dL_t^{\max}, \qquad \sigma_0 \in (\sigma_{\min}, \sigma_{\max}), \tag{3}$$

$$dr_t = \kappa_r(\theta_r - r_t)\,dt + \eta\,dW_t^r, \qquad r_0 \in \mathbb{R}. \tag{4}$$

Equation (3) is a reflected stochastic differential equation on $[\sigma_{\min}, \sigma_{\max}]$, in the sense of the Skorokhod problem of Section 3.1. The processes $L^{\min}, L^{\max}$ are the regulators: both are continuous, nondecreasing, and null at 0; $L^{\min}$ increases only on $\{t : \sigma_t = \sigma_{\min}\}$ and pushes upward, while $L^{\max}$ increases only on $\{t : \sigma_t = \sigma_{\max}\}$ and pushes downward. Equation (4) is the Vasicek model [10], and (3) without the regulators is of the type introduced by Heston [3]. Standard choices of the volatility shape are $b(\sigma) = \sigma$ and $b(\sigma) = \sqrt{\sigma}$; the latter is the Heston square root, whose singular point 0 the reflection excludes.

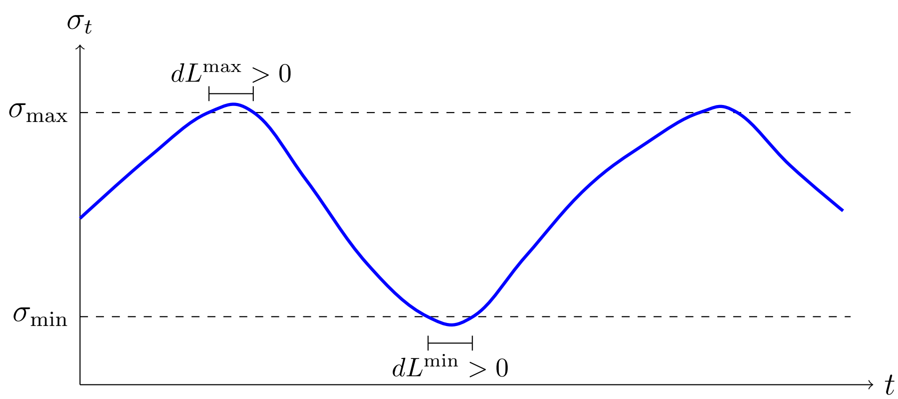

Figure 1: A path of the reflected volatility. The regulators act only at the two boundaries, supplying the minimal push that keeps the path in $[\sigma_{\min}, \sigma_{\max}]$.

Assumption 1. Throughout:

(A1) $0 < \sigma_{\min} < \sigma_{\max} < \infty$, and $\kappa, \xi, \kappa_r, \eta > 0$ with $\theta \in (\sigma_{\min}, \sigma_{\max})$.

(A2) $b \in C^1(U)$ with $b > 0$ on U, for some open $U \supset [\sigma_{\min}, \sigma_{\max}]$; set $b_{\max} = \sup_U |b|$ and $b'_{\max} = \sup_U |b'|$.

(A3) The Skorokhod problem for (3) on $[\sigma_{\min}, \sigma_{\max}]$ with normal reflection at both boundaries is well posed, so that (2)–(4) admits a unique strong solution [4].

(A4) The correlation matrix is positive definite, so $\alpha, \gamma > 0$.

Remark 2.1 (Why a bounded volatility). There are two reasons to keep σ inside a band, and they are independent of each other. The first is mathematical. A floor $\sigma_{\min} > 0$ keeps the volatility away from zero, which is the degenerate region where the Heston square root loses ellipticity and where the Feller condition is needed. A ceiling $\sigma_{\max} < \infty$ stops the moment explosion that unreflected stochastic-volatility models can have. So bounding the volatility is a regularisation, and reflection is a natural way to impose it while leaving the dynamics diffusive inside the band. The second reason is that equity markets limit how far a price can move within a session, through price bands, cooling-off rules and market-wide trading halts, and any such limit caps realised volatility over a short horizon. We want to be clear that this second reason is motivation only. Nothing below depends on any particular market mechanism. The hypotheses are exactly (A1)–(A4), and the only structural fact the proofs use is that σ_t stays in $[\sigma_{\min}, \sigma_{\max}]$.

3 Background

We collect the tools we use. Proofs of the standard facts are in [8] for stochastic calculus and [7] for Malliavin calculus.

3.1 The Skorokhod problem and reflection

Definition 3.1 (Skorokhod problem on a half-line)**.** Given a continuous path ψ with $\psi_0 \geq 0$, the pair (φ, ζ) solves the Skorokhod problem for ψ on $[0, \infty)$ if $\varphi = \psi + \zeta \geq 0$, and ζ is continuous, nondecreasing, $\zeta_0 = 0$, and increases only on $\{t : \varphi_t = 0\}$.

Lemma 3.2 (explicit solution)**.** *The Skorokhod problem has the unique solution*

$$\zeta_t = \max\Big(0,\ -\inf_{u \leq t} \psi_u\Big), \qquad \varphi_t = \psi_t + \zeta_t.$$

The map $\psi \mapsto \varphi$ is Lipschitz in the supremum norm.

Proof. Nonnegativity, monotonicity and the support property are immediate from the formula, and minimality forces uniqueness; the Lipschitz property follows since $\psi \mapsto \inf_{u \leq \cdot} \psi_u$ is 1-Lipschitz in $\| \cdot \|_\infty$. □

On the band $[\sigma_{\min}, \sigma_{\max}]$ the construction is the two-sided analogue, producing the two regulators of (3); existence and uniqueness for reflected SDEs with Lipschitz coefficients is due to Lions and Sznitman [4]. The regulator is a boundary local time: it increases only on a set of times of Lebesgue measure zero, yet increases to a strictly positive value.

3.2 The Malliavin derivative

Let $H = L^2([0, T]; \mathbb{R}^3)$ be the Cameron–Martin space. For $h \in H$, perturbing the driving path by $\varepsilon \int_0^{\cdot} h_r \, dr$ and differentiating at $\varepsilon = 0$ defines the directional derivative of a random variable F; when this is represented as $\int_0^T \langle D_s F, h_s \rangle \, ds$, the kernel $D_s F$ is the Malliavin derivative. We write D_s^j for the component in the direction B^j, and $\mathbb{D}^{1,p}$ for the associated Sobolev–Watanabe spaces [7].

Example 3.3. $D^j_s B^j_t = \mathbf{1}_{\{s \le t\}}$: a perturbation at time s shifts B^j_t by the same amount for $t \ge s$ and has no effect for $t < s$. More generally, for an Itô integral with adapted, Malliavin differentiable integrand,

$$D^j_s \int_0^T H_r \, dW_r \;=\; H_s \, \ell_j \;+\; \int_s^T D^j_s H_r \, dW_r, \tag{5}$$

the first term being the direct effect of the perturbation on the integrand at time s.

We use the chain rule $D^j_s f(F) = f'(F) D^j_s F$ for Lipschitz f, the duality (integration by parts) formula $\mathbb{E}[F \, \delta(u)] = \mathbb{E}[\int_0^T D_t F \, u_t \, dt]$ defining the Skorokhod integral δ, and the following criterion.

Lemma 3.4 (Sugita's criterion)**.** *If $F \in L^2$ and there is a deterministic constant C with $\|F(B + \int_0^\cdot h) - F(B)\|_{L^2} \le C\|h\|_H$ for all $h \in H$, then $F \in \mathbb{D}^{1,2}$ and $\|DF\|_H \le C$ almost surely. See [9] and [7, Ch. 1].*

3.3 Non-degeneracy and the Greeks

For $F \in \mathbb{D}^{1,2}$ real valued, the Malliavin covariance is $\gamma_F = \sum_{j=1}^3 \int_0^T (D^j_s F)^2 \, ds$. If $\gamma_F > 0$ almost surely then F has a density; if in addition $F \in \mathbb{D}^\infty$ and $\gamma_F^{-1} \in \bigcap_{p \ge 1} L^p$, then the density is C^∞ and integration by parts may be iterated [7, Ch. 2]. The latter condition is what produces Malliavin weights: for a parameter λ,

$$\partial_\lambda \, \mathbb{E}[\Phi(F)] = \mathbb{E}\big[\Phi(F) \, \pi_\lambda\big], \qquad \pi_\lambda = \delta(u_\lambda), \tag{6}$$

with no derivative on Φ [2]. Establishing $\gamma^{-1} \in \bigcap_p L^p$ for our model is the object of Section 6.

4 Square integrability

Lemma 4.1 (a priori bounds)**.** *Under Assumption 1, $\sigma_{\min} \le \sigma_t \le \sigma_{\max}$ for all $t \in [0, T]$, almost surely.*

Proof. This is the defining property of the solution of the Skorokhod problem on $[\sigma_{\min}, \sigma_{\max}]$: the regulators act precisely so that the constrained path does not leave the band [4]. □

Lemma 4.2 (rate moments)**.** *For each t, r_t is Gaussian, $\sup_{t \le T} \mathbb{E}[r_t^2] < \infty$, and $\mathbb{E}\big[\exp(\lambda \int_0^T r_u \, du)\big] < \infty$ for every $\lambda \in \mathbb{R}$.*

Proof. Equation (4) is linear with deterministic coefficients, so r_t and $\int_0^T r_u \, du$ are jointly Gaussian with finite mean and variance. A Gaussian random variable has finite exponential moments of every order. □

Proposition 4.3 (L^2 membership)**.** *Under Assumption 1, $S_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for every $t \in [0, T]$, and $\sup_{t \le T} \mathbb{E}[S_t^2] < \infty$.*

Proof. Write $S_t = S_0 e^{Y_t}$, where

$$Y_t = \int_0^t \Big(r_u - \tfrac{1}{2}\sigma_u^2\Big)\,du + \int_0^t \sigma_u\,dW_u^S.$$

Then

$$\mathbb{E}[S_t^2] = S_0^2\,\mathbb{E}\Big[\exp\Big(2\int_0^t r_u\,du - \int_0^t \sigma_u^2\,du + 2\int_0^t \sigma_u\,dW_u^S\Big)\Big] \le S_0^2\,A^{1/2}B^{1/2},$$

by the Cauchy–Schwarz inequality, where

$$A = \mathbb{E}\Big[\exp\Big(4\int_0^t r_u\,du\Big)\Big], \qquad B = \mathbb{E}\Big[\exp\Big(-2\int_0^t \sigma_u^2\,du + 4\int_0^t \sigma_u\,dW_u^S\Big)\Big].$$

Lemma 4.2 gives $A < \infty$. For B, set $M_t = \int_0^t \sigma_u\,dW_u^S$. By Lemma 4.1 its quadratic variation obeys $\langle M\rangle_t = \int_0^t \sigma_u^2\,du \le \sigma_{\max}^2 T$ almost surely, so for a continuous martingale with uniformly bounded quadratic variation, $\mathbb{E}[e^{4M_t}] \le e^{8\sigma_{\max}^2 T}$. Since $-2\int_0^t \sigma_u^2\,du \le 0$, we obtain $B \le e^{8\sigma_{\max}^2 T} < \infty$. The bound is uniform in $t \le T$. □

Remark 4.4. The regulators never appear in the proof of Proposition 4.3. Reflection makes σ bounded, and boundedness on its own gives square integrability. This is worth saying out loud, because it is the opposite of what one expects. An unreflected stochastic-volatility price can have $\mathbb{E}[S_t^p] = \infty$ once p passes a threshold. Here every moment is finite. The boundary local time only becomes essential in Section 5, where we differentiate the volatility itself.

5 The Malliavin derivative

5.1 Cameron–Martin perturbations act as drift perturbations

Perturbing B in the direction $h \in H$ shifts W^σ by the drift $\hat{h}_r^\sigma = \rho_{S\sigma} h_r^1 + \alpha h_r^2$, by (1). Hence the noise term of (3) contributes an additional drift, and the perturbed volatility solves the reflected equation

$$d\sigma_t^\varepsilon = \Big[\kappa(\theta - \sigma_t^\varepsilon) + \varepsilon\,\xi\,b(\sigma_t^\varepsilon)\,\hat{h}_t^\sigma\Big]dt + \xi\,b(\sigma_t^\varepsilon)\,dW_t^\sigma + dL_t^{\varepsilon,\min} - dL_t^{\varepsilon,\max}. \tag{7}$$

The Malliavin derivative of σ is therefore the response of a reflected diffusion to an added drift $v_t = \xi b(\sigma_t)\hat{h}_t^\sigma$, which is exactly the perturbation analysed in [5].

5.2 Differentiability

Proposition 5.1. *Under Assumption 1, $\sigma_t \in \mathbb{D}^{1,p}$ for every $p \ge 1$ and every t, and $\|D\sigma_t\|_H \le C$ almost surely for a deterministic constant C.*

Proof. Let $\Delta_t = \sigma_t(B + \int_0^{\cdot} h) - \sigma_t(B)$. By Lemma 3.2 the Skorokhod map is Lipschitz in the supremum norm, and b is Lipschitz on U by (A2), so combining the two, using the Itô isometry on the difference of the stochastic integrals and Hölder on the drift difference, gives constants C_1 with

$$\mathbb{E}\Big[\sup_{r\le t}|\Delta_r|^2\Big] \le C_1\int_0^t \mathbb{E}\Big[\sup_{u\le s}|\Delta_u|^2\Big]ds + C_1\,\xi^2 b_{\max}^2\,T\,\|h\|_H^2.$$

Gronwall's inequality yields $\mathbb{E}[\sup_{r\le T}|\Delta_r|^2] \le C^2\|h\|_H^2$ with C deterministic, so $\|\sigma_t(B+\int h)-\sigma_t(B)\|_{L^2}\le C\|h\|_H$. Lemma 3.4 gives $\sigma_t\in\mathbb{D}^{1,2}$ with $\|D\sigma_t\|_H\le C$ almost surely; since the bound is deterministic, $\sigma_t\in\mathbb{D}^{1,p}$ for all p. □

Proposition 5.2 (the rate derivative)**.** *For $s\le t$ and $j\in\{1,2,3\}$,*

$$D_s^j r_t = \ell_j^r\,\eta\,e^{-\kappa_r(t-s)}, \qquad \text{and } D_s^j r_t = 0 \text{ for } s>t.$$

Proof. Differentiating (4) in the direction B^j gives, for $t>s$, the deterministic linear equation $d_t(D_s^j r_t) = -\kappa_r\,D_s^j r_t\,dt$, with the injection $D_s^j r_s = \eta\,\ell_j^r$ arising from (5) applied to the noise term. Solving gives the exponential. □

5.3 The interior flow and a size bound

Definition 5.3. The interior first-variation flow is

$$J_{s,t} = \exp\Big(\int_s^t\Big[-\kappa-\tfrac12\xi^2 b'(\sigma_r)^2\Big]dr + \int_s^t \xi\,b'(\sigma_r)\,dW_r^\sigma\Big), \qquad s\le t.$$

Lemma 5.4. *For $s\le u$ in $[0,T]$ and $j\in\{1,2,3\}$:*

(i) on the event $\{\sigma_r\in(\sigma_{\min},\sigma_{\max})$ for all $r\in[s,u]\}$,

$$D_s^j\sigma_u = \xi\,b(\sigma_s)\,\ell_j^\sigma\,J_{s,u};$$

(ii) unconditionally, $|D_s^j\sigma_u|\le \xi\,b_{\max}\,J_{s,u}$;

(iii) for every $m\ge 0$, $\mathbb{E}[J_{s,u}^m]\le \exp\big(\tfrac12 m^2\xi^2(b'_{\max})^2T\big)$, uniformly in $s,u\in[0,T]$.

Proof. (i) On the stated event the regulators are constant on $[s,u]$, so (3) is an ordinary SDE there. Differentiating gives the linear equation

$$d_u\big(D_s^j\sigma_u\big) = -\kappa\,D_s^j\sigma_u\,du + \xi\,b'(\sigma_u)\,D_s^j\sigma_u\,dW_u^\sigma,$$

with injection $D_s^j\sigma_s = \xi\,b(\sigma_s)\,\ell_j^\sigma$ from (5). Its solution is the stochastic exponential of Definition 5.3 multiplied by the injection.

(iii) Write $J_{s,u} = e^A e^M$ with $A=\int_s^u[-\kappa-\frac12\xi^2 b'(\sigma_r)^2]dr\le 0$ and $M=\int_s^u \xi b'(\sigma_r)\,dW_r^\sigma$. Then $e^{mA}\le 1$, and M is a continuous martingale with $\langle M\rangle_u-\langle M\rangle_s = \int_s^u\xi^2 b'(\sigma_r)^2dr \le \xi^2(b'_{\max})^2T$ almost surely, whence $\mathbb{E}[e^{mM}]\le\exp(\frac12 m^2\xi^2(b'_{\max})^2T)$. Multiplying gives the bound.

(ii) Approximate the reflected volatility by the penalised diffusions σ^n solving

$$d\sigma_t^n = \Big[\kappa(\theta-\sigma_t^n) + n(\sigma_{\min}-\sigma_t^n)^+ - n(\sigma_t^n-\sigma_{\max})^+\Big]dt + \xi b(\sigma_t^n)\,dW_t^\sigma,$$

which are ordinary SDEs and satisfy $\sigma^n \to \sigma$ in L^2 as $n \to \infty$ (the penalisation method; see [4]). Differentiating, $|D^j_s \sigma^n_u| = \xi\, b(\sigma^n_s)\, |\ell^\sigma_j|\, J^n_{s,u}$, where J^n differs from J by the extra factor $\exp\big(-n \int_s^u \mathbf{1}_{\{\sigma^n_r \notin (\sigma_{\min},\sigma_{\max})\}}\, dr\big) \le 1$ in its exponent. Hence $|D^j_s \sigma^n_u| \le \xi b_{\max} J_{s,u}$ uniformly in n. The bounding function $\xi b_{\max} J_{s,u}$ lies in L^2 by (iii), and the set of elements of $L^2([0,T]\times\Omega)$ dominated by it is convex and strongly closed, hence weakly closed. Since D is closable and $\sigma^n \to \sigma$, the derivatives converge weakly, and the bound passes to the limit. □

5.4 The boundary

Lemma 5.5 (reflected Brownian motion)**.** *Let $X_t = x + W_t + \zeta_t$ be Brownian motion reflected at 0 on $[0,\infty)$, with ζ the regulator of Lemma 3.2, and let $g_t = \sup\{u \le t : X_u = 0\}$, with $g_t = 0$ if the set is empty. Then for $s \le t$,*

$$D_s X_t = \mathbf{1}_{\{g_t < s \le t\}} = \mathbf{1}_{\{X_u > 0 \text{ for all } u \in (s,t]\}}.$$

Proof. Let $\psi_u = x + W_u$ and perturb W by $\varepsilon\varphi$ with $\varphi_u = \int_0^u h_r\, dr$, writing $\psi^\varepsilon_u = \psi_u + \varepsilon\varphi_u$. By Lemma 3.2,

$$X^\varepsilon_t = \psi^\varepsilon_t + \max\Big(0, -\inf_{u\le t} \psi^\varepsilon_u\Big).$$

If $\inf_{u\le t}\psi_u > 0$, then for small ε the maximum is attained at 0, the regulator vanishes identically, and $\partial_\varepsilon X^\varepsilon_t|_{\varepsilon=0} = \varphi_t = \int_0^t h_r\, dr$, so that $D_s X_t = \mathbf{1}_{\{s\le t\}}$; and in this case $g_t = 0$, consistent with the claim. Otherwise the infimum is negative and is attained, almost surely uniquely, at $\tau = \arg\min_{u\le t}\psi_u$, which coincides with g_t. The envelope (Danskin) theorem applied to $\varepsilon \mapsto -\inf_{u\le t}\psi^\varepsilon_u$ gives derivative $-\varphi_\tau$ at $\varepsilon = 0$, so

$$\partial_\varepsilon X^\varepsilon_t\Big|_{\varepsilon=0} = \varphi_t - \varphi_{g_t} = \int_{g_t}^t h_r\, dr = \int_0^T \mathbf{1}_{\{g_t < r \le t\}}\, h_r\, dr.$$

Reading off the kernel gives $D_s X_t = \mathbf{1}_{\{g_t < s \le t\}}$. □

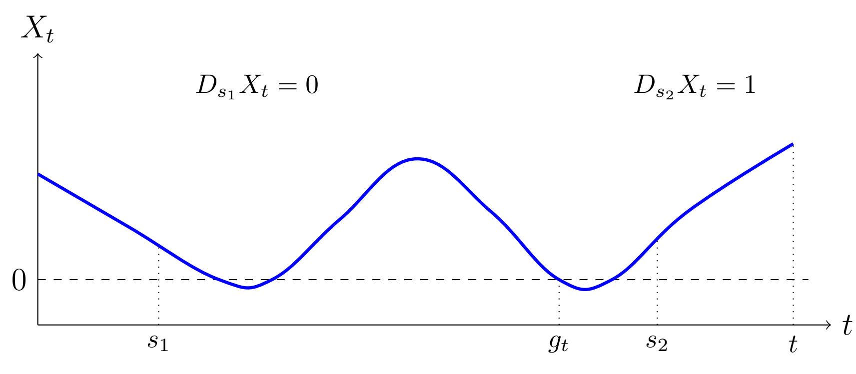

Figure 2: Reflection erases the effect of earlier perturbations. The perturbation at s_1 is followed by a boundary visit at g_t and is erased; the perturbation at $s_2 > g_t$ lies in the current excursion and survives.

Conjecture 5.6 (exact form at the boundary)**.** *Under Assumption 1, with flat boundaries and normal reflection, a version of the Malliavin derivative of the volatility is, for $s \le t$,*

$$D_s^j \sigma_t = \xi\, b(\sigma_s)\, \ell_j^\sigma\, J_{s,t}\, \mathbf{1}_{\{\sigma_u \in (\sigma_{\min}, \sigma_{\max}) \text{ for all } u \in (s,t]\}} :$$

the interior flow of Lemma 5.4(i), killed at the first boundary contact after s.

Remark 5.7. The interior form is proved in Lemma 5.4(i). The killing at the boundary is proved for the prototype in Lemma 5.5, where reflection erases the effect of any perturbation made before the last boundary visit. For the full equation (3), the derivative process is described in [5] as the solution of a constrained linear equation with jumps. When the boundary is flat and the reflection is normal, that jump becomes a reset of the derivative to zero, which is what the displayed formula says. Two things are missing before this is a theorem. First, we must identify the Cameron–Martin derivative with the drift-perturbation derivative of [5] for the coupled system, and equation (7) is the natural bridge. Second, we must justify the penalisation limit $\exp(-n \int_s^u \mathbf{1}_{\{\sigma^n \notin (\sigma_{\min}, \sigma_{\max})\}}) \to \mathbf{1}_{\{\text{no contact on } (s,u]\}}$. So we state the result as a conjecture. We stress that the main result, Theorem 6.2, uses only the size bound of Lemma 5.4(ii), and does not depend on Conjecture 5.6.

5.5 The derivative of the price

Proposition 5.8. *Let $Y_t = \log S_t$. Then $D_s^j S_t = S_t\, D_s^j Y_t$, and for $s \le t$,*

$$D_s^j Y_t = \ell_j^S\, \sigma_s + \int_s^t \Big(D_s^j r_u - \sigma_u\, D_s^j \sigma_u \Big) du + \int_s^t D_s^j \sigma_u\, dW_u^S.$$

Proof. Since $S_t = S_0 e^{Y_t}$ and exp is smooth, the chain rule [7, Prop. 1.2.3] gives $D_s^j S_t = S_t D_s^j Y_t$. Differentiating $Y_t = \int_0^t (r_u - \frac{1}{2}\sigma_u^2) du + \int_0^t \sigma_u\, dW_u^S$: the Lebesgue integral contributes $\int_s^t (D_s^j r_u - \sigma_u D_s^j \sigma_u) du$, and the Itô integral contributes, by (5), the injection $\sigma_s\, \ell_j^S$ together with $\int_s^t D_s^j \sigma_u\, dW_u^S$. □

6 Non-degeneracy of the Malliavin covariance

Let $\gamma_{Y_T} = \sum_{j=1}^3 \int_0^T (D_s^j Y_T)^2\, ds$. By Proposition 5.8 with $j = 1$, and since $\ell_1^S = 1$,

$$D_s^1 Y_T = \sigma_s + R_s, \qquad R_s = \int_s^T \Big(D_s^1 r_u - \sigma_u D_s^1 \sigma_u \Big) du + \int_s^T D_s^1 \sigma_u\, dW_u^S. \tag{8}$$

Remark 6.1 (the idea). The decomposition (8) contains the whole argument. The leading term σ_s is how a perturbation passes directly from the volatility to the price, and by Lemma 4.1 it is at least $\sigma_{\min}$ for every s. Reflection does not touch it, because reflection enters only through $D_s^1 \sigma_u$, which sits inside the remainder R_s. The remainder is an integral over $[s, T]$, so it is small when s is close to T. On a short window before maturity, then, the derivative is bounded below by a deterministic constant, up to an error we can control, and that is enough to force the covariance to be non-degenerate. In particular, the worry that reflection might destroy non-degeneracy, by pinning the volatility to a boundary and killing its derivative, does not arise. The lower bound is carried by the leading term, not by the derivative of the volatility.

Theorem 6.2 (small-ball estimate)**.** *Let* $\varepsilon_0 = \frac{1}{4}\sigma_{\min}^2 \min(1,T)$. *For every* $q \geq 1$ *there exists* $C_q < \infty$ *such that*

$$\mathbb{P}\big(\gamma_{Y_T} < \varepsilon\big) \leq C_q\, \varepsilon^q \qquad \textit{for all } \varepsilon \in (0, \varepsilon_0).$$

Proof. Fix $\delta \in (0, \min(1,T)]$. Discarding the nonnegative terms $j = 2, 3$ and restricting the integral to the window $[T-\delta, T]$,

$$\gamma_{Y_T} \;\geq\; \int_{T-\delta}^{T} \big(D_s^1 Y_T\big)^2 ds = \int_{T-\delta}^{T} \big(\sigma_s + R_s\big)^2 ds.$$

The elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, valid since $(a+b)^2 - \frac{1}{2}a^2 + b^2 = \frac{1}{2}(a+2b)^2 \geq 0$, together with $\sigma_s \geq \sigma_{\min}$ from Lemma 4.1, gives

$$\gamma_{Y_T} \;\geq\; \tfrac{1}{2}\sigma_{\min}^2\, \delta \;-\; I_\delta, \qquad I_\delta := \int_{T-\delta}^{T} R_s^2\, ds. \tag{9}$$

Step 1: moments of the remainder. We claim that for every $q \geq 1$ there is $K_q < \infty$ with

$$\mathbb{E}\big[R_s^{2q}\big] \leq K_q\, (T-s)^q \qquad \text{whenever } T - s \leq 1. \tag{10}$$

By Proposition 5.2, $|D_s^1 r_u| \leq |\rho_{Sr}|\,\eta =: c_r$, and by Lemma 5.4(ii), $|D_s^1 \sigma_u| \leq \xi b_{\max} J_{s,u}$. For the Lebesgue integral in (8), Hölder's inequality gives

$$\mathbb{E}\Big[\Big(\int_s^T \big(D_s^1 r_u - \sigma_u D_s^1 \sigma_u\big) du\Big)^{2q}\Big] \leq (T-s)^{2q-1} \int_s^T \mathbb{E}\Big[\big(c_r + \sigma_{\max}\, \xi b_{\max} J_{s,u}\big)^{2q}\Big] du \leq C(T-s)^{2q},$$

where C is finite by Lemma 5.4(iii) and the elementary inequality $(x+y)^{2q} \leq 2^{2q-1}(x^{2q} + y^{2q})$. For the Itô integral, the Burkholder–Davis–Gundy inequality followed by Hölder gives

$$\mathbb{E}\Big[\Big(\int_s^T D_s^1 \sigma_u\, dW_u^S\Big)^{2q}\Big] \leq b_q\, \mathbb{E}\Big[\Big(\int_s^T \big(D_s^1 \sigma_u\big)^2 du\Big)^q\Big] \leq b_q (\xi b_{\max})^{2q} (T-s)^{q-1} \int_s^T \mathbb{E}\big[J_{s,u}^{2q}\big] du \leq C'(T-s)^q.$$

Since $(T-s)^{2q} \leq (T-s)^q$ for $T - s \leq 1$, adding the two contributions and using $(x+y)^{2q} \leq 2^{2q-1}(x^{2q} + y^{2q})$ once more proves (10).

Step 2: moments of I_δ. Applying Jensen's inequality to the convex function $t \mapsto t^q$ and the normalised measure ds/δ on $[T-\delta, T]$,

$$\mathbb{E}\big[I_\delta^q\big] \leq \delta^{q-1} \int_{T-\delta}^{T} \mathbb{E}\big[R_s^{2q}\big] ds \leq \delta^{q-1} K_q \int_{T-\delta}^{T} (T-s)^q\, ds = \frac{K_q}{q+1}\, \delta^{2q} \leq K_q\, \delta^{2q}.$$

Step 3: choice of window. Given $\varepsilon \in (0, \varepsilon_0)$, set $\delta = 4\varepsilon/\sigma_{\min}^2$; then $\delta \in (0, \min(1,T)]$ by the definition of ε_0, and $\frac{1}{2}\sigma_{\min}^2 \delta = 2\varepsilon$. So (9) reads $\gamma_{Y_T} \geq 2\varepsilon - I_\delta$, whence

$$\{\gamma_{Y_T} < \varepsilon\} \subseteq \{I_\delta > \varepsilon\}.$$

Markov's inequality applied with exponent q, together with Step 2, gives

$$\mathbb{P}\big(\gamma_{Y_T} < \varepsilon\big) \leq \mathbb{P}\big(I_\delta > \varepsilon\big) \leq \frac{\mathbb{E}[I_\delta^q]}{\varepsilon^q} \leq \frac{K_q\, \delta^{2q}}{\varepsilon^q} = K_q \Big(\frac{4}{\sigma_{\min}^2}\Big)^{2q} \varepsilon^q,$$

which is the assertion with $C_q = K_q (4/\sigma_{\min}^2)^{2q}$. $\square$

Corollary 6.3 (non-degeneracy). *$\gamma_{Y_T} > 0$ almost surely, and $\mathbb{E}[\gamma_{Y_T}^{-p}] < \infty$ for every $p > 0$. The same conclusions hold for $\gamma_{S_T} = S_T^2\, \gamma_{Y_T}$.*

Proof. Non-degeneracy is immediate: $\mathbb{P}(\gamma_{Y_T} = 0) \le \inf_{\varepsilon>0} \mathbb{P}(\gamma_{Y_T} < \varepsilon) = 0$ by Theorem 6.2. For the inverse moments, write

$$\mathbb{E}\big[\gamma_{Y_T}^{-p}\big] = \int_0^\infty p\, \varepsilon^{-p-1} \mathbb{P}\big(\gamma_{Y_T} < \varepsilon\big)\, d\varepsilon = \int_0^{\varepsilon_0} + \int_{\varepsilon_0}^\infty .$$

The second integral is at most $\int_{\varepsilon_0}^\infty p\varepsilon^{-p-1} d\varepsilon = \varepsilon_0^{-p} < \infty$. For the first, Theorem 6.2 with $q = p+1$ bounds the integrand by $p\, C_{p+1}\, \varepsilon^{-p-1+p+1} = p\, C_{p+1}$, which is integrable on $(0, \varepsilon_0)$. Hence $\mathbb{E}[\gamma_{Y_T}^{-p}] < \infty$.

For S_T: by the argument of Proposition 4.3 with the exponent of either sign, $\mathbb{E}[S_T^{-r}] < \infty$ for all $r > 0$. Since $\gamma_{S_T}^{-p} = S_T^{-2p} \gamma_{Y_T}^{-p}$, Cauchy–Schwarz gives $\mathbb{E}[\gamma_{S_T}^{-p}] \le \mathbb{E}[S_T^{-4p}]^{1/2}\, \mathbb{E}[\gamma_{Y_T}^{-2p}]^{1/2} < \infty$. □

Corollary 6.4 (smooth density and Malliavin weights). *S_T has a C^∞ density, and the integration-by-parts weights* (6) *for Delta, Gamma and Vega exist and have finite moments of every order.*

Proof. $S_T \in \mathbb{D}^\infty$ by Proposition 5.1 and smoothness of exp, and $\gamma_{S_T}^{-1} \in \bigcap_p L^p$ by Corollary 6.3. Apply the criterion for smoothness of densities and the integration-by-parts machinery [7, Ch. 2]. □

Remark 6.5 (the contribution). The only structural input to Theorem 6.2 is the deterministic bound $\sigma_s \ge \sigma_{\min} > 0$. There is no Feller-type condition, no assumption on the drift of the volatility, and no parameter has to be small. Reflection hands us, by construction, the uniform ellipticity that the non-degeneracy theory needs, and which the Feller condition is designed to approximate in the unreflected model. Together with Remark 4.4, where the ceiling ruled out moment explosion, this is the sense in which the band regularises rather than complicates. The lower boundary secures the density and the weights; the upper boundary secures the moments. Finally, the proof uses only the size bound of Lemma 5.4(ii), not the exact form of the derivative, so the result holds independently of Conjecture 5.6.

7 Clark–Ocone representation and the Greeks

Let Φ be a payoff, $F = e^{-\int_0^T r_u\, du} \Phi(S_T)$ the discounted value, and $P = \mathbb{E}[F]$ the price.

Theorem 7.1 (Clark–Ocone integrand). *Suppose $\Phi \in C_b^1$, so that $F \in \mathbb{D}^{1,2}$. Then*

$$F = \mathbb{E}[F] + \sum_{j=1}^{3} \int_0^T \mathbb{E}\big[D_t^j F \,\big|\, \mathcal{F}_t\big]\, dB_t^j,$$

where

$$D_t^j F = e^{-\int_0^T r_u\, du} \left[\Phi'(S_T)\, D_t^j S_T - \Phi(S_T)\, \ell_j^r\, \eta\, \frac{1 - e^{-\kappa_r (T-t)}}{\kappa_r} \right].$$

Proof. The Clark–Ocone formula in the Brownian setting is standard [7, Ch. 6]. The integrand follows from the product rule applied to $e^{-\int_0^T r}\Phi(S_T)$: the payoff contributes $\Phi'(S_T)D_t^j S_T$ by the chain rule, and the discount factor contributes through

$$D_t^j \int_t^T r_u\,du = \ell_j^r\,\eta \int_t^T e^{-\kappa_r(u-t)}du = \ell_j^r\,\eta\,\frac{1-e^{-\kappa_r(T-t)}}{\kappa_r},$$

by Proposition 5.2. □

Proposition 7.2 (Delta)**.** *Suppose a is adapted and satisfies the duality relation $\int_0^T a_t\,D_t^1 S_T\,dt = S_T$ almost surely. Then*

$$\Delta = \partial_{S_0} P = \frac{1}{S_0}\,\mathbb{E}\Big[e^{-\int_0^T r_u du}\,\Phi(S_T)\,\delta(a)\Big].$$

Proof. Since $S_T = S_0\,\Xi$ with Ξ independent of S_0, we have $\partial_{S_0} S_T = S_T/S_0$, so $\Delta = \frac{1}{S_0}\mathbb{E}[e^{-\int r}\Phi'(S_T)S_T]$. By the duality relation and the chain rule,

$$\Phi'(S_T)\,S_T = \int_0^T a_t\,\Phi'(S_T)\,D_t^1 S_T\,dt = \int_0^T a_t\,D_t^1\big(\Phi(S_T)\big)\,dt,$$

and the integration-by-parts formula for δ gives the stated expression. The weight exists by Corollary 6.4. □

Example 7.3 (Black–Scholes check)**.** If $r \equiv r_0$ and $\sigma \equiv \sigma_0$ are constant and the boundaries are never reached, then $D_t^1 S_T = \sigma_0 S_T$, so $a_t = 1/(\sigma_0 T)$ satisfies the duality relation and $\delta(a) = W_T^S/(\sigma_0 T)$. Proposition 7.2 then reduces to

$$\Delta = \mathbb{E}\Big[e^{-r_0 T}\,\Phi(S_T)\,\frac{W_T^S}{S_0\,\sigma_0\,T}\Big],$$

the classical Malliavin weight of [2].

Remark 7.4 (Gamma and Vega). As S_T is linear in S_0, a second differentiation falls entirely on the weight, producing $\Gamma = \mathbb{E}[e^{-\int r}\Phi(S_T)\pi_\Gamma]$ with an iterated Skorokhod integral π_Γ; in the setting of Example 7.3 this is $\pi_\Gamma = \frac{1}{S_0^2\sigma_0 T}\big(\frac{W_T^{S\,2}}{\sigma_0 T} - W_T^S - \frac{1}{\sigma_0}\big)$. For Vega we take $\mathcal{V} = \partial_{\sigma_0} P$; then $\partial_{\sigma_0} S_T = S_T\,\partial_{\sigma_0} Y_T$, where $\partial_{\sigma_0}\sigma_u$ solves the tangent equation of Lemma 5.4(i) with initial condition 1. Explicit closed forms for the reflected weights $\delta(a)$ and π_Γ depend on Conjecture 5.6 and are not derived here.

Conjecture 7.5 (absorption of Vega at the boundary)**.** *Under Conjecture 5.6, $\partial_{\sigma_0}\sigma_u = J_{0,u}\,\mathbf{1}_{\{\sigma\in(\sigma_{\min},\sigma_{\max})\ on\ (0,u]\}}$, so that the sensitivity of the price to the initial volatility accumulates only over the region before the first boundary contact.*

8 Status of the results

Statement	Status
Well-posedness (Assumption 1)	proved, via [4]
$\sigma_t \in [\sigma_{\min}, \sigma_{\max}]$; $S_t \in L^2$ (4.1, 4.3)	proved
$\sigma_t \in \mathbb{D}^{1,p}$ (Prop. 5.1)	proved, via [9]
$D_s r_t$; interior flow; size bound (5.2, 5.4)	proved
Boundary reset, prototype (Lem. 5.5)	proved
Boundary reset, general (Conj. 5.6)	open; backbone [5]
$D_s S_t$ assembled (Prop. 5.8)	proved
Small-ball estimate; inverse moments (6.2, 6.3)	proved
Smooth density; weights exist (Cor. 6.4)	proved
Clark–Ocone integrand (Thm. 7.1)	proved, for $\Phi \in C_b^1$
Delta; Black–Scholes limit (7.2, 7.3)	proved
Explicit reflected weights $\delta(a), \pi_\Gamma$	open
Absorption of Vega (Conj. 7.5)	open

9 Conclusion

We studied a stock price whose volatility is reflected inside a band, with a Vasicek short rate and correlated drivers. We proved the price is square integrable and Malliavin differentiable. We computed its Malliavin derivative, leaving one boundary term open. And we proved that the Malliavin covariance is non-degenerate, with inverse moments of every order. That last result is what the Malliavin approach to the Greeks needs, and it is our main result.

The mechanism is worth repeating. The lower boundary gives a deterministic floor on the volatility, and so uniform ellipticity for the log-price. The upper boundary gives boundedness, and so all moments. Reflection therefore supplies both of the hypotheses that the unreflected model has to work for, and it does so with no condition of Feller type. This is what we mean by saying that reflection regularises rather than obstructs.

Two problems are left. The first is to prove Conjecture 5.6. That means identifying the Cameron–Martin derivative with the drift-perturbation derivative of [5] for the coupled system, and justifying the penalisation limit. The second depends on the first: compute the explicit reflected weights, and show that the way they differ from the classical weights of Example 7.3 is supported on the paths that reach a boundary before maturity, and is governed by the boundary local time. That would turn Conjecture 7.5 into a theorem.

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