Grassmannians

Hilbert's Fifteenth Problem, Incidence Geometry, and Algebraic Combinatorics

Ryan Bansal

July 2026

The Counting Puzzle

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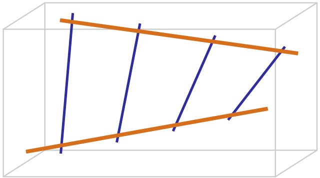

four fixed lines in 3-space, two transversals

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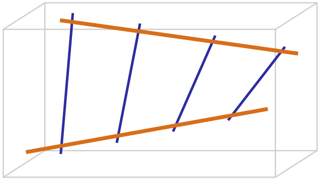

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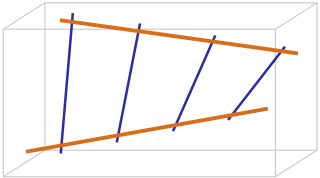

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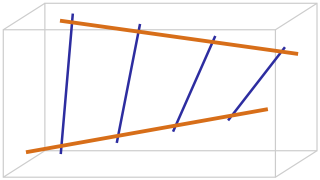
This is the world of Hilbert's fifteenth problem: making Schubert's geometric counting rules precise.



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A Game That Knows the Answer

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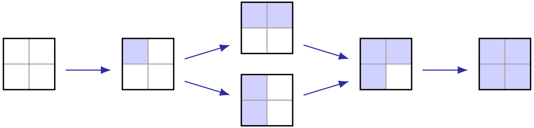
Box game. Each move adds one box to a row. After four conditions, fill the square.



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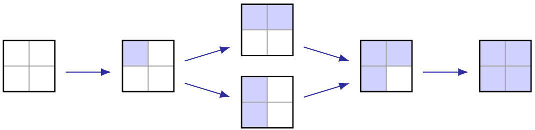

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Question for the talk

Why does this game know geometry, and why are these the allowed moves?

Grassmannians

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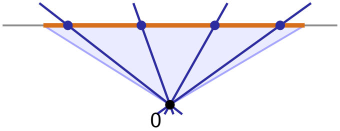
So $\mathrm{Gr}(2, 4)$ means:

$$\{2\text{-dimensional subspaces of } \mathbb{C}^4\}.$$

Why Lines Become Planes



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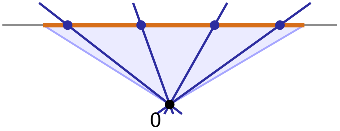

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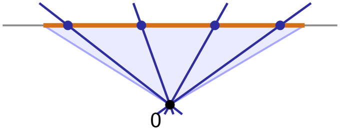

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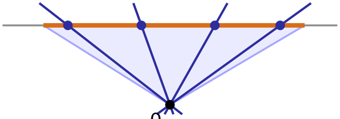

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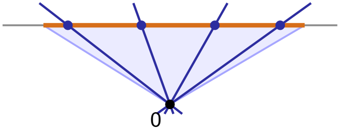

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Concretely: a direction is written in *homogeneous coordinates* $[x_1 : x_2 : x_3 : x_4]$, where scaling does not matter. The plane $\{(a,b,0,0)\} \subset \mathbb{C}^4$ gives the points $[a : b : 0 : 0]$ — one number b/a, plus one point for $a = 0$: a projective line.

Coordinates for Planes

To work with this space, represent a plane by two spanning rows.

A 2-plane in $\mathbb{C}^4$ can be described by two spanning rows. Row operations change the basis, but not the plane.

$$\begin{pmatrix} 1 & 3 & 1 & 6 \\ 1 & 3 & 2 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 3 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 6 & 8 \\ 1 & 2 & 8 & 11 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \end{pmatrix}$$

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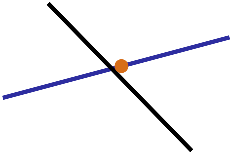
We choose the form where the leading 1's move **right-to-left** as the rows go down. The free entries are local coordinates on the Grassmannian.

A Condition Becomes a Subset

Now translate the projective condition into the subspace language.



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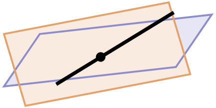

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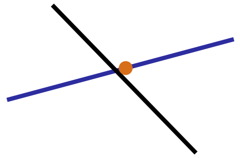

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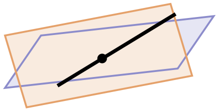

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A fixed projective line L has a fixed 2-plane above it. A moving line ℓ gives a moving 2-plane V. Then

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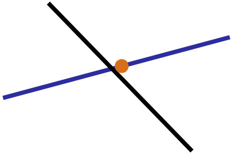

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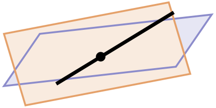

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So "meet L" selects a subset of $\mathrm{Gr}(2,4)$ where the moving 2-plane contains some direction from the fixed 2-plane.

Flags

So far we compared the moving plane against a single fixed 2-plane. A flag completes the measuring device with one fixed reference subspace of every dimension, nested inside each other.

Definition

A **flag** is a chain $F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$ with $\dim F_q = q$.

For this talk we use the flag that reveals coordinates from the right:

$$F_1 = (0,0,0,*), \quad F_2 = (0,0,*,*), \quad F_3 = (0,*,*,*), \quad F_4 = \mathbb{C}^4.$$

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The visibility number is

$$d(q) = \dim(V \cap F_q).$$

Think of F_q as a growing window: $d(q)$ says how much of V is visible at stage q.

A Visibility Computation

Use the plane

$$V = \operatorname{span}\left((0,0,1,2),\ (1,3,0,4)\right).$$

Its general vector is

$$(b,\ 3b,\ a,\ 2a+4b).$$

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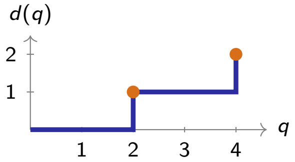
- In $F_3 = (0,*,*,*)$: still need $b = 0$, so $d(3) = 1$.
- In $F_4 = \mathbb{C}^4$: everything is visible, so $d(4) = 2$.

$$d(q): \quad 0,\ 1,\ 1,\ 2 \qquad \text{jumps at } q = 2, 4.$$

Lemma

As q increases, $d(q)$ can rise only by 0 or 1.



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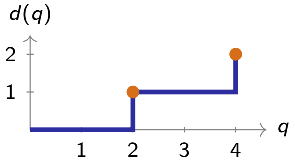

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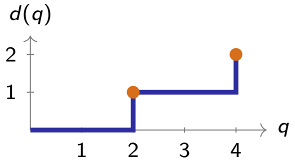

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Our plane has jump pattern $(2, 4)$: one direction became visible one step early.

Partitions

Fix a partition λ in a *Young Diagram*, which records a chosen visibility pattern.

$(3,4)$ $(2,4)$ $(1,4)$ $(2,3)$ $(1,3)$ $(1,2)$

Definition (Schubert cell)

The **Schubert cell** C_λ is the set of planes whose visibility pattern is **exactly** the pattern recorded by λ.

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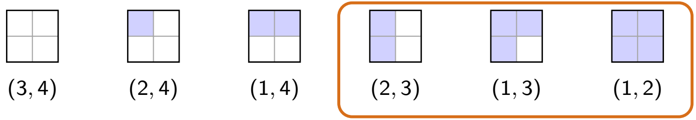

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Example: for jumps $(2,3)$, the Schubert variety is the union of the three circled cells.

Names for Conditions

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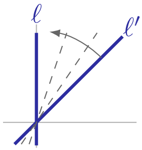

$$M_t = \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix}, \qquad M_0\,\ell = \ell', \quad M_1\,\ell = \ell.$$

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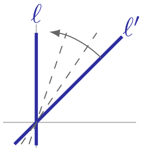

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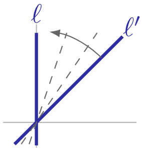

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One name. The slide moves the subset "meet ℓ" onto "meet ℓ'," and the count survives the move. For counting, they are the *same* condition. So we name only the shape $\sigma_{(1)}$. In general, σ_λ names Ω_λ up to sliding.

Multiplying Conditions

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So we slide L' into the most convenient position: touching L.

$$L \cap L' = \{p\}, \qquad L \text{ and } L' \text{ span a plane } P.$$

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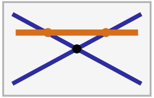

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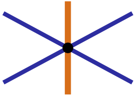

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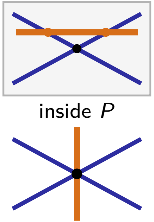

In $\mathrm{Gr}(2,4)$ this means:

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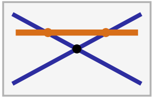

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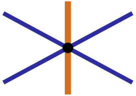

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Thus one product splits into the two limiting conditions:

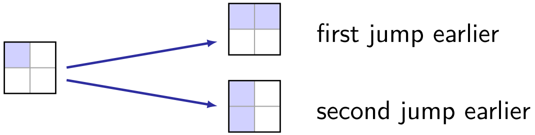
$$\sigma_{(1)}\,\sigma_{(1)} = \sigma_{(2)} + \sigma_{(1,1)}.$$

The Box Rule

One more “meet a line” condition means one more unit of early visibility.

In the partition picture, that means:

add one box.

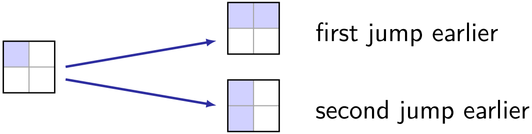


The Box Rule

One more “meet a line” condition means one more unit of early visibility.

In the partition picture, that means:

add one box.



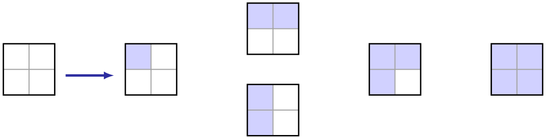
The only restriction is that the result must still be a Young diagram. This is the one-box case of **Pieri’s rule**.

In symbols:

$$\sigma_\lambda \cdot \sigma_{(1)} = \sum_{\text{legal one-box } \mu} \sigma_\mu .$$

Four Lines Revisited

Each fixed line asks for one box. Four fixed lines ask us to fill the whole 2×2 rectangle.



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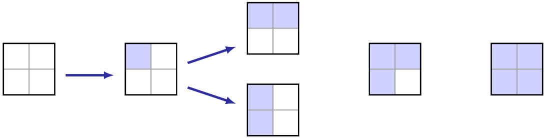

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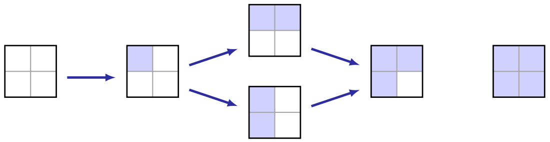

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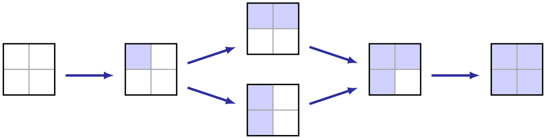

The full rectangle means everything is visible as early as possible: that pins the plane down completely, a single point of $\mathrm{Gr}(2,4)$.

There are **two** legal paths, so the answer to the opening puzzle is **two** lines:

$$\sigma_{(1)}^4 = 2\,\sigma_{(2,2)}.$$

The Point

geometry	**combinatorics**
a subspace, measured by a flag	staircase $\rightsquigarrow$ partition
degeneration of conditions	jumps move earlier, boxes are added
one extra incidence condition	add one box
four line conditions	two transversals

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Pieri's rule is not an arbitrary combinatorial game. It is what incidence conditions look like after geometry has been measured, encoded, and multiplied.

The story continues in the paper with complete flags, permutations, and Monk's rule.

Thank You!

Questions?