

Integral Geometry

Invariant measures, kinematic formulas, and the Radon transform

Integral Geometry

Geometric probability: the founding problems

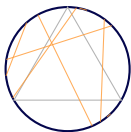
Integral geometry began as *geometric probability*; simply stated, assigning geometric objects a probability and computing an average.

- **Buffon (1733/1777).** A needle of length $\ell < d$ dropped across ruled lines a distance d apart meets a line with probability $\mathbb{P} = \frac{2\ell}{\pi d}$.
- **Barbier (1860), Crofton (1868).** Length and perimeter appear as *averages* over random lines; the measure of lines meeting a convex body is its perimeter.
- **Sylvester, Czuber, Deltheil.** A century of “random line,” “random triangle,” “random chord” problems.

However, the recurring difficulty remained... what does “random” mean?

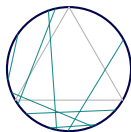
Bertrand's paradox (1889)

Draw a “random” chord of a circle. Is it longer than the side $\sqrt{3} R$ of the inscribed equilateral triangle? These all disagree:



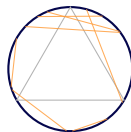
Random endpoints

$$\mathbb{P} = \frac{1}{3}$$



Random radius

$$\mathbb{P} = \frac{1}{2}$$



Random midpoint

$$\mathbb{P} = \frac{1}{4}$$

The question is ill-posed until one fixes a **measure** on the set of lines. Only the middle law is invariant under the rigid motions of the plane (Poincaré; Jaynes 1973).

Building on Herglotz's Göttingen lectures, [Wilhelm Blaschke](#) coined *Integralgeometrie* in Hamburg in 1934. In a single year his students found famous proofs:

"At the end of 1934 Luis Santaló found his now famous proofs of the isoperimetric inequality . . . and Blaschke himself found the fundamental kinematic formula."
— S. S. Chern

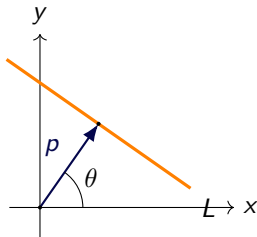
In parallel, [Radon \(1917\)](#) and [Funk \(1913\)](#) had already posed the inverse

problem that would become tomography.

The space of lines and its invariant measure

A line: $\cos \theta x + \sin \theta y = p$, with $(p, \theta) \in \mathbb{R} \times [0, \pi)$. The set of lines is the homogeneous space $\text{SE}(2)/(\text{SO}(2) \ltimes \mathbb{R})$.

$$dK = dp d\theta.$$



Under a motion (rotate α , translate (x_0, y_0)), $\theta' = \theta + \alpha$,
 $p' = p + x_0 \cos \theta' + y_0 \sin \theta'$, with
 Jacobian 1, so $dp' d\theta' = dp d\theta$.

Uniqueness

On G/H with H compact there is a unique (up to scale) G -invariant measure. So dK is forced, and “random line” means integration against dK — resolving Bertrand.

Crofton's formula

Theorem (Crofton, 1868). For a bounded domain $D \subset \mathbb{R}^2$,

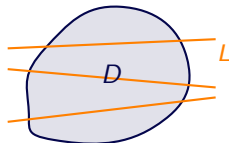
$$\int \text{length}(L \cap D) dK = \pi \text{Area}(D).$$

Proof. Write $\text{length}(L \cap D) = \int \mathbf{1}_D dq$ (q arclength along L); the left side is a triple integral over the flag $\{(L, Z) : Z \in L \cap D\}$. For fixed θ , $(p, q) \mapsto (x, y)$ is a rotation, so $dp dq = dx dy$ and

$$\int_0^\pi \left(\int_{\mathbb{R}^2} \mathbf{1}_D dx dy \right) d\theta = \pi \text{Area}(D). \quad \square$$

Counting rather than measuring gives

Poincaré $\int n(L \cap C) dK = 2 \text{length}(C)$; for convex K the measure of lines meeting K is $\text{per}(\partial K)$ (**Cauchy**).



Kinematic density and the fundamental formula

Move a whole figure by $g = (\text{rotation } \varphi, \text{translation } (a, b))$. The invariant (Haar) measure on the motion group $\text{SE}(2)$ is

$$dg = da \, db \, d\varphi.$$

Theorem (Blaschke–Santaló, 1935). For convex Ω_1, Ω_2 with areas A_i and perimeters L_i ,

$$\int_{\{g: \Omega_1 \cap g\Omega_2 \neq \emptyset\}} dg = 2\pi(A_1 + A_2) + L_1 L_2,$$

and, counting boundary intersections,

$$\int n(\partial\Omega_1 \cap g \partial\Omega_2) \, dg = 4 L_1 L_2 \quad (\text{Poincaré, 1912}).$$

The isoperimetric inequality by counting

Take congruent copies $\Omega_1 \cong \Omega_2 \cong \Omega$ ($L = \text{per}$, $A = \text{Area}$). Let m_i be the measure of placements with exactly i boundary intersections ($m_i = 0$ for odd i). The two formulas read

$$\sum_i i m_i = 4L^2, \quad \sum_i m_i = 4\pi A + L^2.$$

Subtracting twice the second from the first,

$$\sum_i (i - 2) m_i = 2L^2 - 8\pi A = 2m_4 + 4m_6 + \cdots \geq 0,$$

hence

$$L^2 \geq 4\pi A,$$

with equality iff $m_4 = m_6 = \cdots = 0$, i.e. Ω is a disk. Santaló's refinement with a moving disk of radius s yields [Bonnesen's inequality](#)

$$L^2 - 4\pi A \geq \pi^2 (R_{\text{out}} - R_{\text{in}})^2.$$

Interlude — Questions for YOU!

Problem 1

Two lines are drawn independently, each conditioned to meet a convex Ω . Show that the probability they cross *inside* Ω is $\frac{2\pi A(\Omega)}{L(\partial\Omega)^2}$, and deduce it is $\leq \frac{1}{2}$, with equality only for the disk.

Problem 2

A convex body in $\mathbb{R}^3$ is sliced by planes with the invariant measure. Which classical quantity is recovered as the average area of a section? (Hint: it is one of the intrinsic volumes.)

Flats, sections, and intrinsic volumes

Let $A(n, k)$ be the affine Grassmannian of k -flats in $\mathbb{R}^n$; it has a unique motion-invariant measure μ_k . The **Steiner formula** defines the intrinsic volumes $V_0, \dots, V_n$:

$$\text{vol}(K \oplus \varepsilon B) = \sum_{j=0}^n \varepsilon^{n-j} \kappa_{n-j} V_j(K),$$

with $V_n = \text{vol}$, $2V_{n-1} = \text{surface area}$, $V_1 \propto \text{mean width}$, $V_0 = \chi$.

Crofton formula in $\mathbb{R}^n$

For $0 \leq j \leq k \leq n$,

$$V_j(K) = c_{n,k,j} \int_{A(n,k)} V_{j-n+k}(K \cap E) \, d\mu_k(E).$$

A global invariant is the average of a lower one over all flat sections; the plane case ($n=2, k=1, j=1$) is exactly Cauchy–Crofton.

Hadwiger's theorem and the principal kinematic formula

Theorem (Hadwiger, 1957). Every continuous, rigid-motion-invariant valuation on convex bodies in $\mathbb{R}^n$ is a linear combination of $V_0, \dots, V_n$ — an $(n+1)$ -dimensional space.

Because the space is finite dimensional, any invariant average must expand in the V_i . This forces the **principal kinematic formula**:

$$\int_{\text{SE}(n)} \chi(K \cap gM) \, dg = \sum_{i=0}^n c_{n,i} V_i(K) V_{n-i}(M).$$

- In the plane this is Cauchy, Crofton and Santaló.

Funk and Radon: the inverse problem

A second branch of integral geometry asks the *opposite* question: recover a function from its integrals.

- **Funk (1913).** An even function on S^2 is determined by its integrals over great circles (the Funk–Minkowski transform).
- **Radon (1917).** A function on $\mathbb{R}^n$ is determined by its integrals over hyperplanes:

$$\widehat{f}(\omega, p) = \int_{\langle x, \omega \rangle = p} f \, d\sigma, \quad \omega \in S^{n-1}, \, p \in \mathbb{R}.$$

The **dual (back-projection)** transform averages over the hyperplanes through a point,

$$\check{f}(x) = \int_{S^{n-1}} \varphi(\omega, \langle x, \omega \rangle) \, d\omega,$$

and $\widehat{}, \check{}$ are adjoint. Radon's paper was dormant until Fritz John revived it in the 1930s.

Inversion, and a parity result

Radon's inversion recovers f from $\widehat{f}$, but its *form* depends on the parity of n :

$$n \text{ odd: } f(x) = c_n \int_{S^{n-1}} \partial_p^{n-1} \widehat{f}(\omega, \langle \omega, x \rangle) d\omega$$

$$n \text{ even: } f(x) = c_n \int_{S^{n-1}} \int_{\mathbb{R}} \frac{\widehat{f}(\omega, p)}{(p - \langle \omega, x \rangle)^n} dp d\omega.$$

- **Odd n :** a purely *local* differential operator — $f(x)$ depends only on hyperplanes near x (an analogue of Huygens' principle).
- **Even n :** a *nonlocal* integral operator — reconstruction at x needs all the data.

In unified form, $f = \frac{1}{2}(2\pi)^{1-n}(-\Delta)^{(n-1)/2} \overset{\times}{\widehat{f}}$.

Tomography: physical application

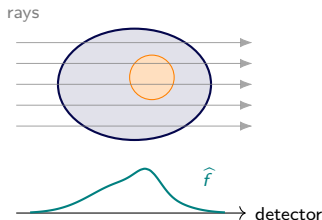
A CT scanner measures X-ray attenuation along lines: it samples the Radon transform, and the image is its inverse.

Two tools

Fourier-slice: the 1-D Fourier transform of a projection is a central slice of $\hat{f} \Rightarrow$ filtered back-projection.

Support theorem (Helgason): if f decays fast and $\hat{f} = 0$ on all hyperplanes missing a ball, then $f = 0$ outside it.

Cormack & Hounsfield: Nobel Prize 1979.



The X-ray transform and John's equation

Integrating over *lines* rather than hyperplanes gives the X-ray transform (Fritz John, 1938); in $\mathbb{R}^2$ it agrees with Radon, but for $n \geq 3$ it is *overdetermined* — lines in $\mathbb{R}^n$ form a $(2n - 2)$ -dimensional family, far more than n data.

Range characterization

In $\mathbb{R}^3$ the transform Xf of a rapidly decreasing f satisfies the ultrahyperbolic equation

$$\frac{\partial^2(Xf)}{\partial x_1 \partial y_2} - \frac{\partial^2(Xf)}{\partial x_2 \partial y_1} = 0 \quad (\text{John, 1938}),$$

and this equation characterizes the range.

Gelfand–Helgason: integral geometry as duality

Gelfand's school and Helgason
recast everything group-theoretically.
Take two homogeneous spaces of
one group G ,

$$X = G/K \quad (\text{"points"}), \quad \Xi = G/H \quad (\text{"planes"})$$

with an **incidence** relation $x \# \xi$
(cosets meet). Then

$$\widehat{f}(\xi) = \int_{\check{\xi}} f, \quad \check{\varphi}(x) = \int_{\hat{x}} \varphi,$$

are adjoint transforms w.r.t. the
invariant measures.

$$Z = \{(x, \xi) : x \# \xi\}$$

 π
 ρ
 $X = G/K$
 $\Xi = G/H$

The **double fibration**

(Gelfand–Graev–Shapiro, Helgason).

$G = \mathrm{SE}(n)$, planes = hyperplanes recovers
Radon; S^2 gives Funk; hyperbolic space
and horocycles give the noncompact
theory.

Complex integral geometry: an average Bézout theorem

Over $\mathbb{C}$ the same machinery meets algebraic geometry. For analytic manifolds C_h, C_r in $P_n(\mathbb{C})$ moving under the unitary group,

$$\int_{\mathcal{U}(n+1)} J_{r+h-n}(C_h \cap uC_r) \, du = m(\mathcal{U}(n+1)) J_h(C_h) J_r(C_r),$$

where J_h is the order (degree) invariant.

- For algebraic (compact) manifolds the intersection number is constant and this is [Bézout's theorem](#).
- For noncompact manifolds it is an [average Bézout theorem](#), linking integral geometry to Nevanlinna theory and value distribution (Chern, Griffiths).

Santaló also develops [symplectic](#) and hyperbolic integral geometry along the same lines.

The algebraic frontier

Hadwiger's theorem says invariant valuations form a finite-dimensional *vector space*. Alesker (2001–2004) showed they also carry a **product**, making them an algebra.

- This “algebraic integral geometry” (Alesker, Bernig, Fu) reproves and vastly extends the classical kinematic formulas.
- Kinematic formulas in **Hermitian** and other homogeneous spaces are recent (Alesker–Bernig–Fu); many groups remain **open**.
- Parallel modern life in **stochastic geometry** and **stereology**: Crofton-type formulas give unbiased 3-D estimates from 2-D sections.

A subject that began with a dropped needle is now an active area of representation theory and convex geometry.

Summary

- ① Invariance under a symmetry group selects a **unique** measure (Haar on G/H); this makes “random” well-posed.
- ② Integrating over a flag gave us formulas of Crofton, Cauchy, Poincaré and Santaló.
- ③ Counting intersections proves the **isoperimetric** and Bonnesen inequalities; Hadwiger’s theorem shows these are forced by finite dimensionality.
- ④ We covered the **Radon transform** and John’s equation.
- ⑤ Gelfand–Helgason duality.

Questions?

Thank you.

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