Integral Geometry and Geometric Probability

A measure theoretic set of solutions to a class of problems

Abstract

Integral geometry studies averages of geometric quantities with respect to measures invariant under the action of a transformation group. Following the classical expositions of Santaló [1], Blaschke [3], and the undergraduate overview of Treiberg [2], we develop a measure-theoretic axiomatic framework, prove the principal planar formulas (Poincaré, Cauchy, Crofton, Santaló, Blaschke), characterize intrinsic volumes via Hadwiger's theorem, and indicate modern extensions to tomography, stereology, and microlocal sheaf theory. Every theorem in the core planar theory is proved from first principles; abstract extensions are stated with proof sketches and references.

Contents

1 Introduction

Integral geometry, often (colloquially) called *geometric probability* in applied contexts, originated in problems of geometric probability: Buffon's needle, Sylvester's problem, and the averaging of geometric data over random positions of lines, planes, and rigid motions. The subject was systematized by Blaschke [3], Poincaré, Crofton, and especially Luis Santaló [1], whose treatise remains the standard reference. Treiberg's expository slides [2] provide an accessible entry point emphasizing the planar kinematic formulas.

Modern descendants (primarily in applied contexts) include geometric measure theory, stereology, computed tomography, stochastic geometry, and microlocal sheaf theory. This paper presents a unified measure-theoretic treatment: we state explicit axioms for measures and invariance, prove uniqueness of kinematic densities, derive the classical planar formulas with complete proofs, extend to valuations and Hadwiger's theorem, and survey applications from materials science to category-theoretic Radon transforms.

1.1 Outline

1. Section 2 establishes σ-algebras, product measures, Haar measure, and the integral-geometric axioms.
2. Section 3 defines line and motion spaces and proves invariance of kinematic measure.
3. Section 4 proves Poincaré, Cauchy, and Crofton formulas.
4. Section 5 treats Santaló, Blaschke, and isoperimetric applications.
5. Section 6 develops valuations and the principal kinematic formula.

6. Section 7 discusses tomography, stereology, and random sets.
7. Section 8 sketches Radon transforms, sheaves, and categorical formulations.

1.2 Notation

We write $\mathbb{R}^n$ for Euclidean space, $\mathcal{H}^n$ for n-dimensional Hausdorff measure, λ^n for Lebesgue measure, and $\mathrm{SE}(n)$ for the special Euclidean group. The space of compact convex bodies in $\mathbb{R}^n$ is denoted $\mathcal{K}^n$. For a measurable set A, $\mathbf{1}_A$ is its indicator function.

2 Measure-Theoretic Foundations

We begin with an axiomatic foundation. All integral-geometric formulas reduce to integrating with respect to a measure that is invariant under a Lie group action.

2.1 Axioms for measures

Axiom 2.1 (M1: σ-algebra)**.** *A* measurable space *is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra on X, i.e., $\emptyset \in \mathcal{A}$, complements and countable unions of elements of $\mathcal{A}$ belong to $\mathcal{A}$.*

Axiom 2.2 (M2: Measure)**.** *A* measure *on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$ and countable additivity: if $A_i \in \mathcal{A}$ are pairwise disjoint, then*

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i).$$

Axiom 2.3 (M3: σ-finiteness)**.** *The measure μ is* σ-finite *if $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty$ for each n.*

Definition 2.4. *A measurable function $f : X \to \mathbb{R}$ is* integrable *if $\int_X |f| \, \mathrm{d}\mu < \infty$. We write $L^1(X, \mu)$ for the space of equivalence classes of integrable functions.*

Theorem 2.5 (Fubini–Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $\mu \times \nu$ the product measure on $X \times Y$. If $f : X \times Y \to \mathbb{R}$ is measurable and $f \geq 0$, then*

$$\int_{X \times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \Big(\int_Y f(x, y) \, \mathrm{d}\nu(y)\Big) \, \mathrm{d}\mu(x) = \int_Y \Big(\int_X f(x, y) \, \mathrm{d}\mu(x)\Big) \, \mathrm{d}\nu(y).$$

If $f \in L^1(X \times Y, \mu \times \nu)$, the same identity holds with absolute convergence of the inner integrals.

Proof. See any standard text on measure theory (e.g., Folland). We use this repeatedly to change the order of integration on *flag manifolds.* □

2.2 Haar measure and invariance

Axiom 2.6 (G1: Topological group)**.** *A* locally compact Hausdorff group *G carries a Borel σ-algebra $\mathcal{B}(G)$ generated by open sets.*

Axiom 2.7 (G2: Left-invariance)**.** *A Borel measure μ on G is* left-invariant *if $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and Borel $A \subset G$.*

Theorem 2.8 (Existence and uniqueness of Haar measure)**.** *On every locally compact Hausdorff group G there exists a left-invariant Borel measure μ_G with $\mu_G(U) > 0$ for every nonempty open U, unique up to a positive multiplicative constant. If G is σ-compact, we may normalize μ_G so that $\mu_G(K) = 1$ for a fixed compact K with nonempty interior.*

Proof. For $G = \mathrm{SE}(2) \cong \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, we construct explicitly below. The general existence theorem is classical (Haar, Weil, Cartan); uniqueness on σ-compact groups follows from regularity. □

2.3 Integral-geometric axioms

Axiom 2.9 (IG1: Homogeneous space)**.** *Let G be a Lie group acting transitively and smoothly on a manifold M. Fix $x_0 \in M$ with stabilizer $H = \{g \in G : g \cdot x_0 = x_0\}$. Then $M \cong G/H$ as smooth manifolds.*

Axiom 2.10 (IG2: Invariant measure on M)**.** *There exists a unique (up to scaling) G-invariant σ-finite Borel measure μ_M on M: $\mu_M(gA) = \mu_M(A)$ for all $g \in G$.*

Theorem 2.11 (Uniqueness on homogeneous spaces)**.** *If H is compact and $M = G/H$ is σ-compact, any two nonzero G-invariant σ-finite Borel measures on M are proportional.*

Proof. Let μ, ν be such measures. For a compact $K \subset M$, choose $f \in C_c(M)$ with $f \equiv 1$ on K. Then

$$\mu(K) = \int f \, \mathrm{d}\mu = \int_G \int_H f(gh \cdot x_0) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_G(g)$$

by Weil's formula, and similarly for ν. Since μ_H is finite, $\mu(K)/\nu(K)$ is constant independent of K, proving proportionality. □

2.4 Change of variables

Theorem 2.12 (Sard–Fubini for submersions)**.** *Let $\Phi : U \subset \mathbb{R}^k \to \mathbb{R}^m$ be a C^1 submersion and μ a measure on U with $\mathrm{d}\mu = \rho(x) \, \mathrm{d}\lambda^k(x)$. If $y = \Phi(x)$, then for integrable f,*

$$\int_U f(x) \, \mathrm{d}\mu(x) = \int_{\Phi(U)} \Big(\int_{\Phi^{-1}(y)} \frac{f(x)\, \rho(x)}{\sqrt{\det J_\Phi J_\Phi^T}} \, \mathrm{d}\mathcal{H}^{k-m}(x) \Big) \, \mathrm{d}\lambda^m(y),$$

where J_Φ is the Jacobian matrix. In the case $k = m$, this reduces to $\int f(x) \, \mathrm{d}\mu(x) = \int f(\Phi^{-1}(y)) \, |\det D\Phi^{-1}(y)| \, \rho(\Phi^{-}$

Proof. The $k = m$ case is the standard Jacobian formula. The submersion case follows from the coarea formula (Federer). □

Remark 2.13 (Disintegration viewpoint). Integrating over a flag manifold in two different coordinate systems—the central technique of integral geometry—is precisely an application of Fubini's theorem (Axiom 2.5) after a change of variables (Axiom 2.12). The "slicing" in Poincaré's formula is disintegration of product measure along fibers of a submersion.

2.5 Conditional probability

Definition 2.14. *If μ is a probability measure and $\mu(B) > 0$, the* conditional probability *of A given B is*

$$\mathbb{P}(A \mid B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

For a σ-finite kinematic measure μ_K on lines, we normalize on a bounded region to obtain probabilities.

This formalism underlies Sylvester's problem and all "expected number of intersections" formulas in Sections 4 and 5.

3 Kinematic Spaces and Invariant Measures

3.1 Lines in the plane

An unoriented line in $\mathbb{R}^2$ is determined by coordinates (p, θ) with equation

$$\cos\theta\, x + \sin\theta\, y = p, \qquad p \in [0, \infty), \quad \theta \in [0, \pi),$$

or equivalently $(\tilde{p}, \eta)$ with $\tilde{p} \in \mathbb{R}$, $\eta \in [0, \pi)$. The space of lines is denoted $\mathcal{L}_2$.

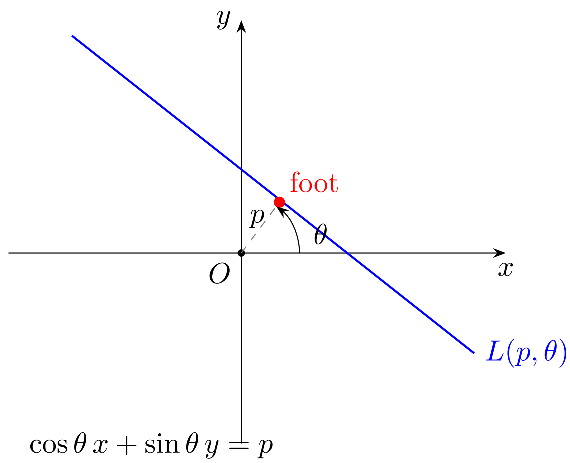

Figure 1: Line coordinates (p, θ): p is distance from origin to the foot of the perpendicular, θ is its direction.

Definition 3.1 (Kinematic measure on lines)**.** *The* kinematic measure *on* $\mathcal{L}_2$ *is*

$$\mathrm{d}K = \mathrm{d}p \wedge \mathrm{d}\theta$$

in the coordinates $p \in [0, \infty)$*,* $\theta \in [0, 2\pi)$ *(or* $\mathrm{d}\tilde{p} \wedge \mathrm{d}\eta$ *on* $\mathbb{R} \times [0, \pi)$*).*

Theorem 3.2 (Invariance of line kinematic measure)**.** *The measure* $\mathrm{d}K = \mathrm{d}p \wedge \mathrm{d}\theta$ *is invariant under* $\mathrm{SE}(2)$*. Moreover, any* σ*-finite* $\mathrm{SE}(2)$*-invariant measure on* $\mathcal{L}_2$ *is a constant multiple of* $\mathrm{d}K$*.*

Proof. A rigid motion M acts by rotation R_α and translation (x_0, y_0):

$$p' = p + \cos(\theta + \alpha)\, x_0 + \sin(\theta + \alpha)\, y_0, \qquad \theta' = \theta + \alpha.$$

The Jacobian is

$$\frac{\partial(p', \theta')}{\partial(p, \theta)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $\mathrm{d}p' \wedge \mathrm{d}\theta' = \mathrm{d}p \wedge \mathrm{d}\theta$. Alternatively, using differential forms,

$$\mathrm{d}p' = \mathrm{d}p + (-\sin(\theta + \alpha)\, x_0 + \cos(\theta + \alpha)\, y_0)\, \mathrm{d}\theta, \quad \mathrm{d}\theta' = \mathrm{d}\theta,$$

and $\mathrm{d}p' \wedge \mathrm{d}\theta' = \mathrm{d}p \wedge \mathrm{d}\theta$ since $\mathrm{d}\theta \wedge \mathrm{d}\theta = 0$. Uniqueness follows from Axiom 2.11 since $\mathcal{L}_2 \cong \mathbb{R} \times \mathbb{P}^1$ with $\mathrm{SE}(2)$ acting transitively. $\square$

3.2 Motions in the plane

A rigid motion is $M_{a,b,\varphi}(x, y) = (x\cos\varphi - y\sin\varphi + a, x\sin\varphi + y\cos\varphi + b)$. The kinematic density on SE(2) is

$$\mathrm{d}K_{\mathrm{mot}} = \mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}\varphi.$$

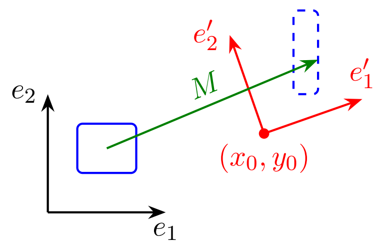

Figure 2: Rigid motion $M \in \mathrm{SE}(2)$: rotation followed by translation.

Theorem 3.3 (Invariance of motion kinematic measure)**.** $\mathrm{d}K_{\mathrm{mot}} = \mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}\varphi$ *is a left-invariant Haar measure on* SE(2)*, unique up to scaling.*

Proof. $\mathrm{SE}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. Haar measure is the product of Lebesgue measure on $\mathbb{R}^2$ and normalized Haar on SO(2). In coordinates (a, b, φ), left translation corresponds to composing motions, which preserves $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}\varphi$. □

3.3 Flag manifolds

Definition 3.4. *The* flag manifold *for lines and points is*

$$\mathrm{Flag} = \{(L, Z) : L \in \mathcal{L}_2,\ Z \in L\}.$$

For a curve C, the subset $S = \{(L, Z) : Z \in L \cap C,\ L \cap C \neq \emptyset\}$ carries two natural parameterizations used in all classical proofs.

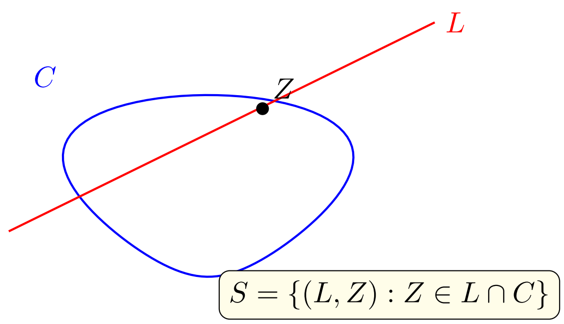

Figure 3: Flag point (L, Z) with $Z \in L \cap C$.

The integral-geometric method (*double counting*) integrates $\mathbf{1}_S$ over Flag in two coordinate systems and applies Fubini.

4 Classical Planar Formulas

Throughout, C denotes a piecewise C^1 curve in $\mathbb{R}^2$, length(C) its total length, and $n(L \cap C)$ the number of intersection points (finite $\mathrm{d}K$-a.e.).

4.1 Poincaré's formula for lines

Theorem 4.1 (Poincaré, 1896). *Let C be a piecewise C^1 curve. Then*

$$\int_{\{L:L\cap C\neq\emptyset\}} n(L\cap C)\ \mathrm{d}K(L) = 2\,\mathrm{length}(C).$$

Proof. Define $S = \{(L,Z) : Z \in L\cap C\}$. Then

$$\int_{\{L:L\cap C\neq\emptyset\}} n(L\cap C)\ \mathrm{d}K = \int_S \mathrm{d}K$$

by counting intersection points.

First coordinates: Parameterize C by arc length: $Z(s) = (x(s), y(s))$, $s \in [0, s_0]$, with $|\dot{Z}| = 1$. For fixed $Z \in C$, lines through Z are parameterized by $\eta \in [0,\pi)$ with $\tilde{p} = x(s)\cos\eta + y(s)\sin\eta$. Using tangent angle $\varphi(s)$ with $(\cos\varphi, \sin\varphi) = \dot{Z}(s)$,

$$\mathrm{d}\tilde{p}\wedge\mathrm{d}\eta = |\cos(\varphi(s)-\eta)|\ \mathrm{d}s\wedge\mathrm{d}\eta.$$

(See Section A.1 for the Jacobian computation.)

Fubini:

$$\int_S \mathrm{d}K = \int_0^{s_0}\int_0^{\pi} |\cos(\varphi(s)-\eta)|\ \mathrm{d}\eta\ \mathrm{d}s = \int_0^{s_0} 2\ \mathrm{d}s = 2\,\mathrm{length}(C).$$

□

Corollary 4.2 (Lines meeting a convex body). *If $\Omega \subset \mathbb{R}^2$ is compact convex, then*

$$\int_{\{L:L\cap\Omega\neq\emptyset\}} \mathrm{d}K = \mathrm{length}(\partial\Omega).$$

Proof. For convex Ω, $n(L\cap\partial\Omega) \in \{0,2\}$ for $\mathrm{d}K$-a.e. L. Apply Axiom 4.1 with $C = \partial\Omega$. □

4.2 Sylvester's problem

Theorem 4.3 (Sylvester, 1889). *Let $\omega \subset \Omega$ be bounded convex sets. The probability that a random line meets ω, given that it meets Ω, is*

$$\mathbb{P}(\textit{meets } \omega \mid \textit{meets } \Omega) = \frac{\mathrm{length}(\partial\omega)}{\mathrm{length}(\partial\Omega)}.$$

Proof. Normalize $\mathrm{d}K$ to a probability measure on $\{L : L\cap\Omega \neq \emptyset\}$ by dividing by $\mathrm{length}(\partial\Omega)$ (Axiom 4.2). The event "meets ω" has measure $\mathrm{length}(\partial\omega)$ by the same corollary applied to ω. □

4.3 Cauchy's formula

Definition 4.4 (Support function). *For compact convex Ω and $\theta \in [0, 2\pi)$, the* support function *is*

$$h(\theta) = \max\{p : L(p,\theta)\cap\Omega \neq \emptyset\}.$$

The width *in direction θ is $w(\theta) = h(\theta) + h(\theta+\pi)$.*

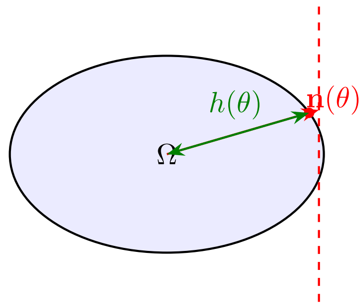

Figure 4: Support function $h(\theta)$ and supporting line $L(h(\theta), \theta)$.

Theorem 4.5 (Cauchy, 1841)**.** *For bounded convex Ω,*

$$\text{length}(\partial\Omega) = \int_0^{2\pi} h(\theta)\ \mathrm{d}\theta = \int_0^{\pi} w(\theta)\ \mathrm{d}\theta.$$

Proof. By Axiom 4.2,

$$\text{length}(\partial\Omega) = \int_{\{L : L\cap\Omega\neq\emptyset\}} \mathrm{d}K = \int_0^{2\pi}\int_0^{h(\theta)} \mathrm{d}p\ \mathrm{d}\theta = \int_0^{2\pi} h(\theta)\ \mathrm{d}\theta.$$

Since $h(\theta+\pi)$ corresponds to the same line with opposite orientation, $\int_0^{2\pi} h = \int_0^{\pi}(h(\theta) + h(\theta + \pi))\ \mathrm{d}\theta = \int_0^{\pi} w$. □

Theorem 4.6 (Area via support function)**.** *If $\partial\Omega$ is C^2, then*

$$\text{area}(\Omega) = \frac{1}{2}\int_0^{2\pi} h(\theta)\big(h(\theta) + h''(\theta)\big)\ \mathrm{d}\theta.$$

Proof. Let $Z(\theta) = L(h(\theta), \theta) \cap \partial\Omega$. The outer normal is $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$, so $h = \mathbf{n}\cdot Z$. Differentiating and using $\dot{\mathbf{n}} \perp Z$ gives $\dot{Z} = (h + h'')\dot{\mathbf{n}}$, hence $\mathrm{d}s/\,\mathrm{d}\theta = h + h''$. Thus

$$\text{area}(\Omega) = \frac{1}{2}\oint_{\partial\Omega} \mathbf{n}\cdot\mathbf{r}\ \mathrm{d}s = \frac{1}{2}\int_0^{2\pi} h\,(h + h'')\ \mathrm{d}\theta.$$

□

4.4 Crofton's formula

Theorem 4.7 (Crofton, 1868)**.** *Let $D \subset \mathbb{R}^2$ have compact closure. Then*

$$\pi\,\text{area}(D) = \int_{\{L : L\cap D\neq\emptyset\}} \text{length}(L\cap D)\ \mathrm{d}K(L).$$

Proof. Let $S = \{(L, Z) : Z \in L \cap D\}$. Then

$$I := \int_{\{L : L\cap D\neq\emptyset\}} \text{length}(L\cap D)\ \mathrm{d}K = \int_S \mathrm{d}q\ \mathrm{d}K,$$

where q is arc length along L from the foot point.

In coordinates $(\tilde{p}, \theta, q)$ with $(x, y) = (\tilde{p}\cos\theta - q\sin\theta, \tilde{p}\sin\theta + q\cos\theta)$, $\mathrm{d}\tilde{p} \wedge \mathrm{d}q = \mathrm{d}x \wedge \mathrm{d}y$ (rotation preserves area form). Hence

$$I = \frac{1}{2}\int_0^{2\pi}\int_{\mathbb{R}^2} \mathbf{1}_D(x, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \mathrm{area}(D)\,\mathrm{d}\theta = \pi\,\mathrm{area}(D).$$

□

Corollary 4.8 (Crofton's probability, 1885)**.** *For bounded convex Ω, the probability that two independent random lines both meeting Ω intersect inside Ω is*

$$\mathbb{P} = \frac{2\pi\,\mathrm{area}(\Omega)}{\mathrm{length}(\partial\Omega)^2} \leq \frac{1}{2},$$

with equality iff Ω is a disk.

Proof. By Axiom 4.7 and the expected intersection formula (cf. Treiberg slides), $\mathbb{P} = 2\pi\mathrm{area}/\mathrm{length}(\partial\Omega)^2$. The isoperimetric inequality $4\pi\mathrm{area} \leq \mathrm{length}^2$ gives $\mathbb{P} \leq 1/2$. □

4.5 Buffon's needle

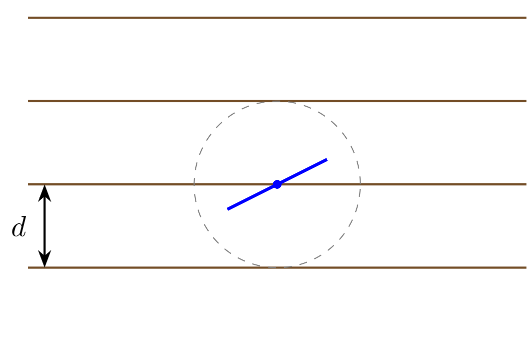

Figure 5: Buffon's needle: parallel cracks at distance d, needle length $\ell < d$.

Theorem 4.9 (Buffon, 1777)**.** *A needle of length $\ell < d$ dropped uniformly at random on a floor with parallel cracks distance d apart touches a crack with probability*

$$\mathbb{P} = \frac{2\ell}{\pi d}.$$

Proof. Only cracks with $|p| \leq d/2$ can meet the needle centered at the origin. By Axiom 4.1 with $C =$ needle, $n = 1$ when they meet, and Axiom 4.2 for the circle of radius $d/2$,

$$\mathbb{P} = \frac{2\ell}{2\pi \cdot (d/2)} = \frac{2\ell}{\pi d}.$$

□

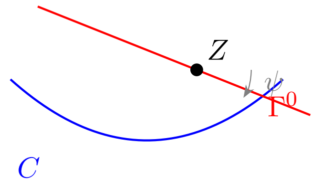

Figure 6: Intersection of curves C and Γ^0 at angle ψ.

4.6 Poincaré's formula for curves

Theorem 4.10 (Poincaré, 1912)**.** *For piecewise C^1 curves C, Γ in $\mathbb{R}^2$,*

$$\int_{\Gamma^0 : C \cap \Gamma^0 \neq \emptyset} n(C \cap \Gamma^0)\ \mathrm{d}K_{\mathrm{mot}} = 4\,\mathrm{length}(C)\,\mathrm{length}(\Gamma).$$

Proof. On the flag $S = \{(\Gamma^0, Z) : Z \in C \cap \Gamma^0\}$, coordinates (s, t, ψ) (arc length on C, on Γ, and tangent angle difference) yield $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}\varphi = -\sin\psi\ \mathrm{d}s \wedge \mathrm{d}t \wedge \mathrm{d}\psi$ (Treiberg slides, Santaló Ch. 7). Integrating:

$$\int_S \mathrm{d}K = \int_C \int_\Gamma \int_0^{2\pi} |\sin\psi|\ \mathrm{d}\psi\ \mathrm{d}t\ \mathrm{d}s = 4\,\mathrm{length}(C)\,\mathrm{length}(\Gamma).$$

$\square$

5 Santaló, Blaschke, and Isoperimetric Inequalities

5.1 Santaló's formula

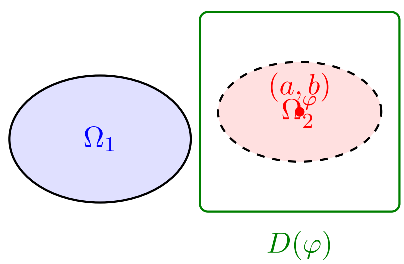

Figure 7: Overlap region $D(\varphi)$ of centers (a, b) for fixed rotation φ.

Theorem 5.1 (Santaló, 1935)**.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be bounded convex domains. If Ω_2' moves with kinematic density $\mathrm{d}K_{\mathrm{mot}}$, then*

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} \mathrm{d}K_{\mathrm{mot}} = 2\pi\big(\mathrm{area}(\Omega_1) + \mathrm{area}(\Omega_2)\big) + \mathrm{length}(\partial\Omega_1)\,\mathrm{length}(\partial\Omega_2).$$

Proof. Fix rotation φ. The set $D(\varphi)$ of centers (a, b) for which $\Omega_1 \cap M_{a,b,\varphi}(\Omega_2) \neq \emptyset$ is convex with support function $f(\alpha) = h_1(\alpha) + g_2(\alpha + \pi - \varphi)$, where h_1, g_2 are support functions of Ω_1, Ω_2. By Axiom 4.6,

$$\mathrm{area}(D(\varphi)) = \frac{1}{2} \int_0^{2\pi} f(\alpha)\big(f(\alpha) + f''(\alpha)\big)\ \mathrm{d}\alpha.$$

Integrating over $\varphi \in [0, 2\pi)$ and expanding:

$$J = \int_0^{2\pi} \mathrm{area}(D(\varphi))\, \mathrm{d}\varphi = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \big[h_1 + g_2(\cdot)\big]\big[h_1 + h_1'' + g_2(\cdot) + g_2''(\cdot)\big]\, \mathrm{d}\alpha\, \mathrm{d}\varphi.$$

Using Fubini, $\int h_1'' = 0$, and **??** 4.5**??** 4.6:

$$J = 4\pi \mathrm{area}(\Omega_1) + 4\pi \mathrm{area}(\Omega_2) + \mathrm{length}(\partial\Omega_1)\mathrm{length}(\partial\Omega_2).$$

Since $J = 2\int \mathrm{d}K_{\mathrm{mot}}$ over overlap positions, dividing by 2 gives the result. □

Corollary 5.2 (Expected boundary intersections)**.** *For bounded convex Ω_1, Ω_2,*

$$\mathbb{E}\big[n(\partial\Omega_1 \cap \partial\Omega_2') \mid \Omega_2' \cap \Omega_1 \neq \emptyset\big] = \frac{4\mathrm{length}(\partial\Omega_1)\mathrm{length}(\partial\Omega_2)}{2\pi(\mathrm{area}_1 + \mathrm{area}_2) + \mathrm{length}_1\mathrm{length}_2}.$$

For congruent copies, $\mathbb{E}(n) \geq 2$ with equality iff Ω_1 is a circle.

Proof. Apply Axiom 4.10 and Axiom 5.1 to the conditional expectation. For congruent bodies, isoperimetric inequality gives $4\pi\mathrm{area} \leq \mathrm{length}^2$, hence $\mathbb{E}(n) \geq 4\mathrm{length}^2/(2\mathrm{length}^2) = 2$. □

5.2 Blaschke's fundamental formula

Definition 5.3 (Total curvature)**.** *For a piecewise C^2 closed curve $\partial\Omega$ with exterior angles α_i at corners,*

$$c(\partial\Omega) = \int_{\partial\Omega} \kappa\, \mathrm{d}s + \sum_i \alpha_i.$$

By Gauss–Bonnet, $c(\partial\Omega) = 2\pi\chi(\Omega)$.

Theorem 5.4 (Blaschke, 1936)**.** *For domains Ω_1, Ω_2 bounded by finitely many piecewise C^2 simple closed curves, with Ω_2' moving kinematically,*

$$\int_{\{\Omega_2' : \Omega_2' \cap \Omega_1 \neq \emptyset\}} c(\partial(\Omega_1 \cap \Omega_2'))\, \mathrm{d}K_{\mathrm{mot}} = 2\pi \mathrm{area}(\Omega_1)\, c(\Omega_2) + 2\pi \mathrm{area}(\Omega_2)\, c(\Omega_1) + \mathrm{length}(\partial\Omega_1)\mathrm{length}(\partial\Omega_2).$$

Proof. Sketch: decompose $c(\partial(\Omega_1 \cap \Omega_2'))$ into contributions from $\partial\Omega_1$, $\partial\Omega_2'$, and corners. Each term integrates to a product of kinematic invariants; Santaló's formula handles the area terms. Full details in Santaló [1], Ch. 7. □

Corollary 5.5. *If both boundaries are simple closed curves ($c(\Omega_i) = 2\pi$), the number of connected components $\nu(\Omega_1 \cap \Omega_2')$ satisfies*

$$\int \nu\, \mathrm{d}K_{\mathrm{mot}} = 2\pi(\mathrm{area}_1 + \mathrm{area}_2) + \mathrm{length}_1\mathrm{length}_2.$$

If both are convex, $\nu \equiv 1$ and we recover Axiom 5.1.

5.3 Isoperimetric inequality via integral geometry

Theorem 5.6 (Isoperimetric inequality). *If C is a simple closed curve of length L enclosing area A, then*

$$4\pi A \leq L^2,$$

with equality iff C is a circle.

Santaló's proof for convex Ω. Let $\Omega_1 \cong \Omega_2 \cong \Omega$ be congruent convex copies. Define m_i = kinematic measure of positions with exactly i boundary intersection points. Odd i have measure zero. By **??** 4.10**??** 5.1:

$$4\text{length}^2 = 2m_2 + 4m_4 + 6m_6 + \cdots, \qquad 4\pi\text{area} + \text{length}^2 = m_2 + m_4 + m_6 + \cdots.$$

Subtracting: $\text{length}^2 - 4\pi\text{area} = m_4 + 2m_6 + \cdots \geq 0$. □

Theorem 5.7 (Bonnesen's inequality). *For convex Ω with inradius R_{in} and circumradius R_{out},*

$$L^2 - 4\pi A \geq \pi^2(R_{\text{out}} - R_{\text{in}})^2.$$

Proof. Let Ω_2' be a disk of radius s with $R_{\text{in}} \leq s \leq R_{\text{out}}$. Overlap of boundaries iff overlap of domains. The same m_i argument with a disk of circumference $2\pi s$ yields $sL \geq A + \pi s^2$, i.e. $f(s) = \pi s^2 - Ls + A \leq 0$ on $[R_{\text{in}}, R_{\text{out}}]$. Hence these endpoints lie between zeros of f, giving $R_{\text{out}} - R_{\text{in}} \leq \sqrt{L^2 - 4\pi A}/\pi$. □

6 Valuations and Hadwiger's Theorem

We now pass to higher dimensions and the axiomatic theory of intrinsic volumes, following Hadwiger [8], Klain–Rota [11], and Chen [9].

6.1 Axioms for valuations

Definition 6.1 (Val^n). *A* valuation *on $\mathcal{K}^n$ is a function $v : \mathcal{K}^n \to \mathbb{R}$ such that $v(\emptyset) = 0$ and for $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$,*

$$v(K) + v(L) = v(K \cap L) + v(K \cup L).$$

Axiom 6.2 (V1: Normalization). $v(\emptyset) = 0$.

Axiom 6.3 (V2: Additivity). *$v(K) + v(L) = v(K \cap L) + v(K \cup L)$ whenever $K \cup L$ is convex.*

Axiom 6.4 (V3: Motion invariance). *$v(gK) = v(K)$ for all $g \in \text{SE}(n)$.*

Axiom 6.5 (V4: Continuity). *v is continuous with respect to the Hausdorff metric on $\mathcal{K}^n$.*

Definition 6.6 (Intrinsic volumes). *The* intrinsic volumes *$V_j(K)$ (also denoted $W_j(K)$) are defined via the Steiner formula: for $K \in \mathcal{K}^n$ and $r > 0$,*

$$\text{vol}_n(K + rB^n) = \sum_{j=0}^{n} V_j(K)\, \omega_{n-j}\, r^{n-j},$$

where B^n is the unit ball, ω_k is the volume of the k-unit ball, and $K + rB^n = \{x : \text{dist}(x, K) \leq r\}$.

Theorem 6.7 (Steiner formula)**.** *The coefficients $V_j(K)$ exist, are valuations, and satisfy $V_n(K) = \mathrm{vol}_n(K)$, $V_0(K) = \chi(K)$ (Euler characteristic).*

Proof. For polytopes, $\mathrm{vol}_n(K + rB^n)$ is a polynomial in r by triangulation; the general case follows by approximation in Hausdorff metric and continuity of volume. Additivity of each $V_j(\cdot)$ follows from the polynomial structure of Minkowski sums. □

Theorem 6.8 (Hadwiger, 1957)**.** *Every continuous, $\mathrm{SE}(n)$-invariant valuation v on $\mathcal{K}^n$ has a unique representation*

$$v(K) = \sum_{j=0}^{n} c_j \, V_j(K)$$

for constants $c_j \in \mathbb{R}$.

Chen's simplified proof. **Step 1 (Homogeneity decomposition).** Define $v_j(K) = \lim_{t\to 0^+} t^{-j} v(tK)$ when the limit exists. Each v_j is a homogeneous valuation of degree j.

Step 2 (Projection reduction). For $K \subset \mathbb{R}^n$ and a hyperplane Π, the projection $\pi_\Pi(K)$ satisfies a kinematic identity linking $v(K)$ to $v(\pi_\Pi(K))$ and fiber integrals. Induction on n reduces to $\mathbb{R}^1$, where the only continuous translation-invariant valuations are multiples of length and χ.

Step 3 (Spherical harmonics alternative). Klain [10] shows v_j is a multiple of $V_j(\cdot)$ using spherical harmonics on $\mathrm{SO}(n)$. See Section A.2 for details.

Step 4. Summing degrees $j = 0, \ldots, n$ gives the representation. □

6.2 Principal kinematic formula

Theorem 6.9 (Principal kinematic formula)**.** *For $K, M \in \mathcal{K}^n$ and $0 \le j \le n$,*

$$\int_{\mathrm{SE}(n)} V_j(K \cap gM) \, \mathrm{d}\mu(g) = \sum_{k+m=n+j} c_{j,k,m} \, V_k(K) \, V_m(M)$$

for explicit combinatorial constants $c_{j,k,m}$ depending only on n and j. In particular for $j = n$ (volume),

$$\int_{\mathrm{SE}(n)} \mathrm{vol}_n(K \cap gM) \, \mathrm{d}\mu(g) = \sum_{k=0}^{n} c_k \, V_k(K) \, V_{n-k}(M).$$

Proof. Fix M. The functional $K \mapsto \int V_j(K \cap gM) \, \mathrm{d}\mu(g)$ is a continuous, $\mathrm{SE}(n)$-invariant valuation in K (invariance from bi-invariance of Haar measure). By Axiom 6.8, it is a linear combination of $V_k(K)$. The coefficients are determined by evaluating on test bodies (e.g., $K = tB^n$, $M = sB^n$) using homogeneity degrees. See Schneider–Weil [14], Ch. 9. □

Remark 6.10. The planar Santaló formula (Axiom 5.1) is the case $n = 2$, $j = 2$ (area) of the principal kinematic formula, with $K = \Omega_1$, $M = \Omega_2$.

7 Applications

7.1 Tomography and the Radon transform

Computed tomography reconstructs a function $f : \mathbb{R}^n \to \mathbb{R}$ from its integrals over hyperplanes. The *Radon transform* [13, 12] is

$$(Rf)(\xi) = \int_\xi f(x) \, \mathrm{d}\mathcal{H}^{n-1}(x), \qquad \xi \in \Xi_n,$$

where Ξ_n is the space of hyperplanes in $\mathbb{R}^n$ with kinematic measure.

Theorem 7.1 (Schwartz). *$R : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\Xi_n)$ is a linear bijection on Schwartz functions.*

Theorem 7.2 (Support theorem). *If $f \in C_c(\mathbb{R}^n)$ and $Rf \equiv 0$ on all hyperplanes meeting* $\mathrm{supp}(f)$*, then $f \equiv 0$.*

These results are the analytic foundation of CT scanning: X-ray data are samples of Rf, and inversion recovers f.

7.2 Stereology and random sets

Stereology estimates geometric properties of a 3D specimen from 2D sections [16, 14]. The theoretical framework:

Definition 7.3 (Stationary random set). *A random closed set $Z \subset \mathbb{R}^d$ is* stationary *if $Z + x \stackrel{d}{=} Z$ for all translations x, and* isotropic *if also $\vartheta Z \stackrel{d}{=} Z$ for $\vartheta \in \mathrm{SO}(d)$.*

Theorem 7.4 (Crofton stereological estimator). *Let Z be a stationary random set with $\mathbb{E}V_j(Z \cap K) < \infty$ for a convex window K. If Π is a uniformly random k-flat through K, then*

$$\mathbb{E}\big[V_j(Z \cap \Pi \cap K)\big] = c_{j,k,d}\,\frac{V_{j+k}(Z \cap K)}{V_k(K)},$$

for a kinematic constant $c_{j,k,d}$ from the principal kinematic formula.

Proof. By Axiom 6.9 and Fubini, integrating $V_j(Z \cap \Pi)$ over the space of k-flats Π with invariant measure yields a multiple of $V_{j+k}(Z \cap K)$. Normalizing by $V_k(K)$ gives the estimator. □

Example 7.5 (Volume from area of sections). For $d = 3$, $j = 3$, $k = 2$: the mean area of a random plane section estimates surface area-related quantities; combined with Crofton's formula (Axiom 4.7), one obtains unbiased volume estimators from section areas (Delesse's principle).

7.3 Matheron's theory and Wijsman topology

Matheron [15] developed *random sets* as measurable maps $\Omega \to \mathcal{F}(\mathbb{R}^d)$ (closed sets). The *Wijsman topology* on closed sets is generated by $A \mapsto d(x, A)$ for each $x \in \mathbb{R}^d$; it is the topology of pointwise convergence of distance functions and provides a Polish space structure for stochastic geometry.

Theorem 7.6 (Miles formula). *For a stationary isotropic Boolean model with convex particles,*

$$\mathbb{E}\big[V_j(Z \cap K)\big] = \lambda\, \mathbb{E}\big[V_j(X)\big]\, \mathbb{E}\big[V_{d-j}(K \cap (-X))\big]$$

for intensity λ and typical particle X.

7.4 Firey's colliding cubes

Firey [19] showed that for two disjoint unit cubes in $\mathbb{R}^3$ in random collision, edge-to-edge contact has probability ≈ 0.54, exceeding corner-to-face contact ≈ 0.46. This uses kinematic measure on SE(3) and face/edge/contact manifold dimensions—a higher-dimensional integral geometry problem beyond the planar theory.

8 Abstract Extensions

We briefly indicate how integral geometry extends to sheaf theory, D-modules, and categorical formulations. Full proofs are beyond our scope; we state precise results with proof sketches.

8.1 Abstract Radon transform on homogeneous spaces

Following Helgason [12], let G be a Lie group, H, K closed subgroups with $H \subset K$, and $X = G/K$, $\Xi = G/H$. The double fibration

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & \Xi \end{tikzcd}$$

defines the Radon transform $Rf(\xi) = \int_{x \in \pi_2^{-1}(\xi) \cap \pi_1^{-1}(x)} f(x)\, \mathrm{d}m(x)$ and dual transform $\check{R}$.

Theorem 8.1 (Duality)**.** *With G-invariant measures on X and Ξ,*

$$\langle Rf, \varphi \rangle_\Xi = \langle f, \check{R}\varphi \rangle_X$$

for $f \in C_c(X)$, $\varphi \in C(\Xi)$.

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "R"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}(\Xi) \arrow[d, "\check{\ }"] \\ \mathcal{F}(X) \arrow[r, "\check{R}"'] & \mathcal{F}(\Xi) \end{tikzcd}$$

$$\langle Rf, \varphi \rangle_\Xi = \langle f, \check{R}\varphi \rangle_X$$

Figure 8: Adjointness of Radon and dual Radon transforms.

8.2 Sheaf-theoretic Radon transform

Sabo and D'Agnolo [18] define a *Radon transform functor* on derived categories of sheaves:

$$R : \mathbf{D}(X \times S^{n-1} \times \mathbb{R}) \longrightarrow \mathbf{D}(\mathbb{P}^n), \qquad R = \pi_{2,!}\, \pi_1^*,$$

using compactly supported sections along fibers. After microlocal localization (restricting to sheaves with prescribed singular support), R becomes an equivalence of categories.

Theorem 8.2 (Sabo–D'Agnolo)**.** *The sheaf Radon transform induces an equivalence $\mathbf{D}_\Lambda(X) \simeq \mathbf{D}_{R(\Lambda)}(\mathbb{P}^n)$ between microlocal sheaf categories associated to a Legendrian Λ.*

Sketch. The proof uses the microlocal theory of Kashiwara–Schapira: R is a quantized contact transformation preserving singular support. Invariance of Legendrian invariants under R mirrors classical reconstruction (R injective on appropriate function spaces). □

8.3 D-modules and range characterization

D'Agnolo and Schapira [17] show that range characterization for Radon-type transforms reduces to a general adjunction

$$f_! \circ f^* \dashv f_* \circ f^!$$

for a map f of manifolds, in the category of D-modules. The *range* of R is the image of a D-module pushforward; characterizing it is a microlocal problem about the singular support of Rf.

8.4 Categorical valuations

Klain and Rota [11] interpret Hadwiger's theorem categorically: continuous $\mathrm{SE}(n)$-equivariant functors from $\mathcal{K}^n$ (as a category with Minkowski sum as monoidal product) to $\mathbb{R}$ that satisfy a valuation axiom are classified by intrinsic volumes. In this view, Axiom 6.8 is a *representation theorem* for the additive structure on convex bodies.

8.5 Functional intrinsic volumes

Recent work extends intrinsic volumes to convex functions $u : \mathbb{R}^n \to \bar{\mathbb{R}}$ via mixed Monge–Ampère measures, yielding a *functional Hadwiger theorem*: every continuous, epi-translation and rotation invariant valuation on convex functions is a linear combination of functional intrinsic volumes. This connects integral geometry to variational analysis and optimal transport.

8.6 Blaschke–Petkantschin and matrix spaces

The Blaschke–Petkantschin formula disintegrates/decomposes Lebesgue measure on $\mathbb{R}^{n\times k}$ over the Grassmannian of k-planes:

$$\int_{\mathbb{R}^{n\times k}} f(X)\,\mathrm{d}X = \int_{\mathrm{Gr}(k,n)} \int_{\Pi} f(x_1, \dots, x_k)\,\mathrm{d}x_1 \cdots \mathrm{d}x_k\,\mathrm{d}\Pi.$$

Forrester's matrix polar decomposition generalizes this to symmetric matrix spaces, with applications to random matrix theory and multivariate statistics.

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A Supplementary Proofs

A.1 Jacobian for the flag parameterization

Lemma A.1. *For a C^1 curve $Z(s) = (x(s), y(s))$ parameterized by arc length and line coordinates $(\tilde{p}, \eta)$ with $\tilde{p} = x(s)\cos\eta + y(s)\sin\eta$, writing $\varphi(s)$ for the tangent angle,*

$$\mathrm{d}\tilde{p} \wedge \mathrm{d}\eta = |\cos(\varphi(s) - \eta)|\ \mathrm{d}s \wedge \mathrm{d}\eta.$$

Proof. Differentiate:

$$\mathrm{d}\tilde{p} = \dot{x}\cos\eta + \dot{y}\sin\eta \ \mathrm{d}s + (-x\sin\eta + y\cos\eta)\ \mathrm{d}\eta = \cos(\varphi - \eta)\ \mathrm{d}s + (\cdots)\ \mathrm{d}\eta.$$

Since $\mathrm{d}\eta \wedge \mathrm{d}\eta = 0$,

$$\mathrm{d}\tilde{p} \wedge \mathrm{d}\eta = \cos(\varphi - \eta)\ \mathrm{d}s \wedge \mathrm{d}\eta.$$

Using $\eta \in [0, \pi)$ for unoriented lines, we take $|\cos(\varphi - \eta)|$. □

A.2 Hadwiger theorem: spherical harmonics step

Lemma A.2 (Klain)**.** *If v is a continuous, $\mathrm{SO}(n)$-invariant valuation on $\mathcal{K}^n$ that is homogeneous of degree j, then $v = c\,V_j(\cdot)$ for some constant c.*

Sketch. The space of $\mathrm{SO}(n)$-invariant valuations homogeneous of degree j is acted on irreducibly by the representation of $\mathrm{SO}(n)$ on spherical harmonics of degree j on S^{n-1}. The intrinsic volume $V_j(\cdot)$ is the unique (up to scaling) such invariant. By Schur's lemma, any v in this space is a multiple of $V_j(\cdot)$. □

A.3 Curve intersection kinematic density

For curves C and Γ with arc length parameters s, t and tangent angles $\alpha(s), \beta(t)$, at intersection Z the angle $\psi = \varphi + \beta(t) - \alpha(s)$ relates motion parameters (a, b, φ) to (s, t, ψ). Computing the 3-form $\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}\varphi$ yields $-\sin\psi\ \mathrm{d}s \wedge \mathrm{d}t \wedge \mathrm{d}\psi$, as in Treiberg [2] and Santaló [1], Ch. 7.