# Van der Waerden's Theorem and Roth

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# Roadmap

1. Definitions and examples
2. Van der Waerden's theorem
3. The model proof: $W(2,3) \leq 325$
4. The full proof: focused progressions and induction

## Guiding idea and first definitions

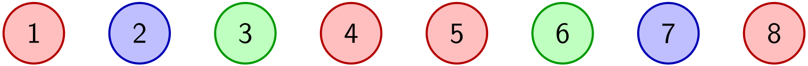
- Ramsey theory studies why large enough structures must contain order.
- Here, the ordered pattern is an arithmetic progression:

$$a,\ a+d,\ a+2d,\dots,a+(k-1)d, \qquad d\neq 0.$$

- An $r$-coloring of a set $S$ is a function

$$C : S \to \{1,2,\dots,r\}.$$

- A $k$-term AP is **monochromatic** if every term has the same color.




Each integer receives exactly one color.

# Blocks and block-colors

- A block of length $L$ is a set of $L$ consecutive integers:

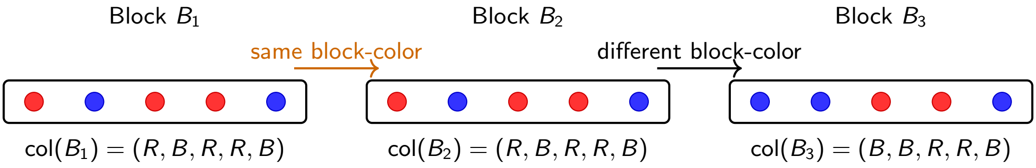
$$B = \{x+1, x+2, \dots, x+L\}.$$

- The block-color records the entire color pattern inside the block:

$$\mathrm{col}(B) = (C(x+1), C(x+2), \dots, C(x+L)).$$

- Two blocks have the same block-color exactly when corresponding entries have the same colors.

# Example: block-colors




Same block-color means the internal pattern agrees position by position.

# Focused progressions

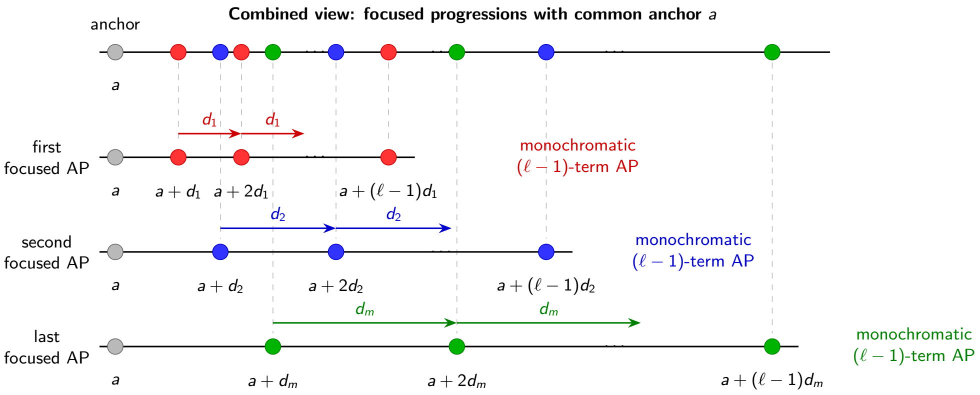
- A focused progression is an almost-complete AP with a shared missing first point.
- At anchor $a$, each focused $(\ell - 1)$-term AP has the form

$$a + d,\ a + 2d, \ldots, a + (\ell - 1)d.$$

- Adding the anchor completes

$$a,\ a + d,\ a + 2d, \ldots, a + (\ell - 1)d.$$

# Focused progressions share one missing anchor




Each focused progression is missing the same first point $a$.
Adding the grey anchor completes each row to an $\ell$-term arithmetic progression.

This is the object that the general proof keeps building.

- $W(k, \ell)$ is the smallest $N$ such that every $k$-coloring of $[N]$ contains a monochromatic $\ell$-term AP.
- Van der Waerden's theorem says $W(k, \ell)$ exists for all positive integers $k, \ell$.
- Easy cases:
$$W(1, \ell) = \ell, \qquad W(r, 2) = r + 1.$$
- First nontrivial example:
$$W(2, 3) = 9.$$
- The coloring *RRBBRRBB* of [8] avoids monochromatic 3-term APs, but the ninth point is forced.

# Proof strategy for $W(2,3) \leq 325$

- We prove a non-sharp bound: $W(2,3) \leq 325$.
- Assume a red-blue coloring of [325] has no monochromatic 3-term AP.
- Split [325] into 65 blocks of length 5.
- A length-5 block has $2^5 = 32$ possible color patterns.
- Therefore, two among the first 33 blocks have identical block-color.

## Two-term monochromatic lemma

**Lemma**

In every red-blue coloring of five consecutive integers, there is a 3-term AP

$$a,\ a+d,\ a+2d$$

whose first two terms have the same color.

1. If $C(x) = C(x+1)$, use $x, x+1, x+2$.
2. Otherwise, if $C(x+1) = C(x+2)$, use $x+1, x+2, x+3$.
3. Otherwise, with only two colors, $C(x) = C(x+2)$, so use $x, x+2, x+4$.

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# Using the identical blocks

- Let $B_i$ and $B_j$ be identical blocks among the first 33, with $i < j$.
- Let $D$ be the distance between corresponding positions.
- The lemma gives points $a, a+d, a+2d \in B_i$ with

$$C(a) = C(a+d).$$

- Say this color is red. Since the blocks match,

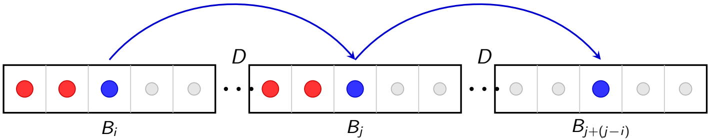
$$C(a+D) = C(a+D+d) = \text{red}.$$

- Since no red 3-term AP is allowed, both third points are forced blue:

$$C(a+2d) = C(a+D+2d) = \text{blue}.$$

# The final point has no safe color

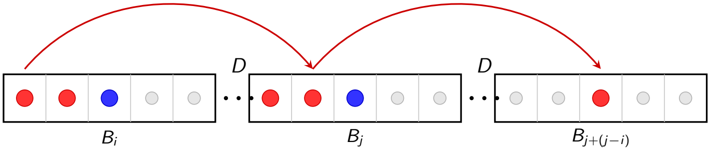
**Case 1:** $C(a + 2D + 2d) = \text{blue}$




Repeated pattern: $a, a+d, a+2d$ and $a+D, a+D+d, a+D+2d$ have colours red, red, blue.

Blue AP: $a+2d, \quad a+D+2d, \quad a+2D+2d.$

**Case 2:** $C(a + 2D + 2d) = \text{red}$




Repeated pattern: $a, a+d, a+2d$ and $a+D, a+D+d, a+D+2d$ have colours red, red, blue.

Red AP: $a, \quad a+D+d, \quad a+2D+2d.$

Blue gives a blue AP; red gives a red AP.

- The point $a+2D+2d$ still lies in $[325]$.
- If it is blue, then
$$a+2d, \quad a+D+2d, \quad a+2D+2d$$
  is a blue 3-term AP.
- If it is red, then
$$a, \quad a+D+d, \quad a+2D+2d$$
  is a red 3-term AP.
- Either way, we get a contradiction. Hence $W(2,3) \leq 325$.

# From the model case to the full proof

- The full proof proceeds by induction on the desired AP length $\ell$.
- Assume we can force $(\ell - 1)$-term APs using finitely many colors.
- To force an $\ell$-term AP, we build many focused $(\ell - 1)$-term APs at one anchor.
- Once every possible color is represented by a focused progression, the anchor's color completes one of them.

# Color-focusing lemma: formal statement

Fix $r$ and $\ell \geq 3$, and assume $W(q, \ell - 1)$ exists for every positive integer $q$.
For every $m \geq 1$, there is an integer $U_m = U_m(r, \ell)$ such that every $r$-coloring of $[U_m]$ satisfies one of:

1. **Statement I:** there is a monochromatic $\ell$-term AP.
2. **Statement II:** there are an anchor $a$ and positive integers $d_1, \ldots, d_m$ such that

$$a + d_t,\ a + 2d_t, \ldots,\ a + (\ell - 1)d_t$$

   is monochromatic for each $t$, and these $m$ focused progressions have pairwise different colors.

Moreover, if Statement I does not occur, then $C(a)$ is different from the color of every focused progression.

## Lemma proof: base case $m = 1$

Take

$$U_1 = 2W(r, \ell - 1).$$

Given an $r$-coloring of $[U_1]$, look only at the second half. It has length $W(r, \ell - 1)$, so it contains a monochromatic $(\ell - 1)$-term AP:

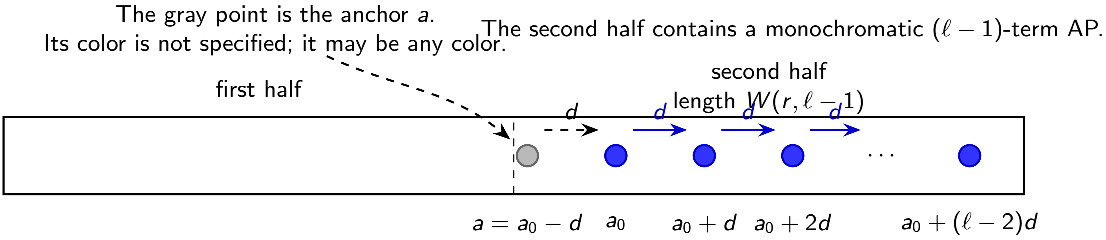
$$a_0,\ a_0 + d,\ a_0 + 2d, \ldots,\ a_0 + (\ell - 2)d.$$

Since this lies in the second half, $a = a_0 - d$ is still inside $[U_1]$. Therefore the same progression becomes

$$a + d,\ a + 2d, \ldots,\ a + (\ell - 1)d,$$

a focused progression at anchor $a$.

# Base focused progression




The anchor is one step before $a_0$. It does not have to lie at the midpoint; we only need $a = a_0 - d$ to remain inside the whole interval.

One focused progression is just an $(\ell-1)$-term AP missing its first point.

There are two cases for the anchor color.

- If $C(a) = C(a+d)$, then
$$a,\ a+d,\ a+2d, \ldots,\ a+(\ell-1)d$$
is a monochromatic $\ell$-term AP. This is Statement I.
- If $C(a) \neq C(a+d)$, then
$$a+d,\ a+2d, \ldots,\ a+(\ell-1)d$$
is one monochromatic focused $(\ell-1)$-term AP at $a$. This is Statement II for $m = 1$.

# Lemma proof: induction step setup

Assume $U_m$ has been constructed. We prove that $U_{m+1}$ exists.
Set

$$H = W(r^{U_m}, \ell - 1), \qquad U_{m+1} = 2U_m H.$$

Now take any $r$-coloring

$$C : [U_{m+1}] \to \{1, 2, \ldots, r\}.$$

Divide the second half into $H$ consecutive blocks

$$B_1, B_2, \ldots, B_H,$$

each of length $U_m$. The first half is reserved so we can later move one block-step backward.

# Block-coloring in the induction step

Each block has $U_m$ entries and each entry has one of $r$ colors, so there are at most

$$r^{U_m}$$

possible block-colors. Thus the original coloring induces an $r^{U_m}$-coloring of the block indices $[H]$. Because $H = W(r^{U_m}, \ell - 1)$, there are block indices

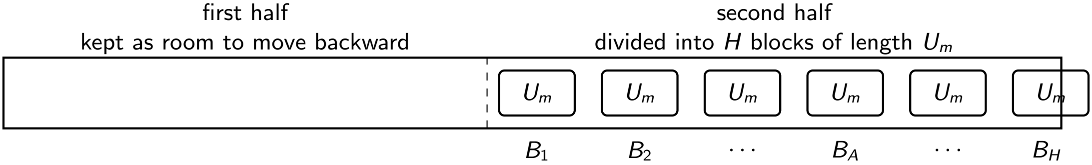
$$A,\ A + E,\ A + 2E, \dots,\ A + (\ell - 2)E$$

forming a monochromatic $(\ell - 1)$-term AP in the block-coloring. Equivalently,

$$B_A, B_{A+E}, \dots, B_{A+(\ell-2)E}$$

all have the same internal pattern. Put $D = EU_m$.

# Keeping room to move backward



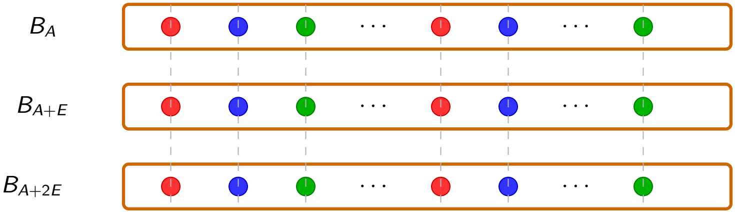

Each block is treated as one object.
Its block-color is the entire ordered list of its $U_m$ colors.

The construction reserves the first half so the new anchor stays inside the interval.

# Selected equal block-colors in AP

The selected blocks form a monochromatic $(\ell - 1)$-term AP in the block-coloring.

$$B_A \xrightarrow{\quad E \quad} B_{A+E} \xrightarrow{\quad E \quad} \cdots \xrightarrow{\quad E \quad} B_{A+(\ell-2)E}$$



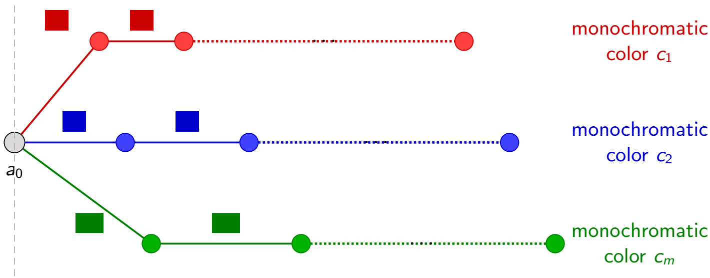

Same block-color means corresponding positions in the selected blocks have the same colors.

The selected blocks all have the same full internal color pattern.

Since $B_A$ has length $U_m$, apply the induction hypothesis defining $U_m$ inside this block.

- If Statement I occurs inside $B_A$, we already have a monochromatic $\ell$-term AP.
- Otherwise Statement II occurs: there are $m$ focused $(\ell - 1)$-term APs at some anchor $a_0 \in B_A$.

Inside the block $B_A$, the induction hypothesis gives $m$ focused spikes




The colors $c_1, c_2, \ldots, c_m$ are pairwise distinct, and none equals $C(a_0)$.

# Copying the old focused progressions

Because the selected blocks have the same block-color, the configuration inside $B_A$ repeats at corresponding positions in

$$B_A, B_{A+E}, \dots, B_{A+(\ell-2)E}.$$

Define the new anchor

$$a = a_0 - D.$$

For each old difference $d_t$, use the new difference

$$D + d_t.$$

Then the focused progression

$$a + (D + d_t),\ a + 2(D + d_t), \dots,\ a + (\ell - 1)(D + d_t)$$

is monochromatic, because its terms are corresponding copies of the old focused progression.

## The new focused progression

The same block-color also creates one completely new focused progression using difference $D$:

$$a + D,\ a + 2D, \ldots,\ a + (\ell - 1)D.$$

Since $a = a_0 - D$, this is

$$a_0,\ a_0 + D, \ldots,\ a_0 + (\ell - 2)D.$$

These are corresponding positions in the selected equal-pattern blocks, so they all have the same color.

Therefore, unless Statement I has already occurred, we have increased the number of focused progressions from $m$ to $m + 1$.

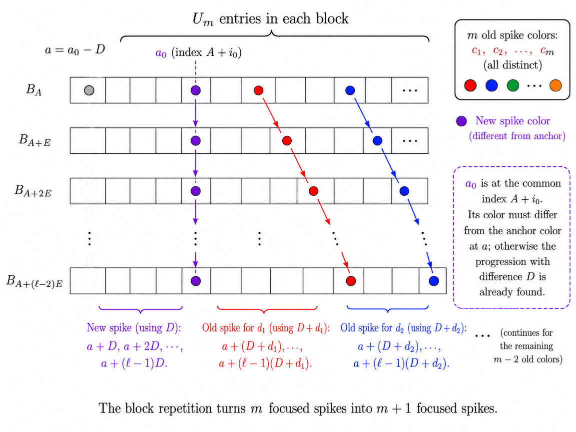



Figure: Forcing m+1 progressions

# Completing the induction on $\ell$

The theorem is proved by induction on $\ell$.

- For $\ell = 1$, take $W(r, 1) = 1$.
- For $\ell = 2$, take $W(r, 2) = r + 1$ by the Pigeonhole Principle.
- For $\ell \geq 3$, assume $W(q, \ell - 1)$ exists for every $q$.
- The color-focusing lemma gives $U_r$ for $m = r$.
- Statement II is impossible in an $r$-coloring: the anchor plus the $r$ focused progressions would require $r + 1$ distinct colors.
- Hence Statement I must occur, so every $r$-coloring of $[U_r]$ contains a monochromatic $\ell$-term AP. Therefore $W(r, \ell) \leq U_r$.

| $\ell \backslash r$ | 2 colors | 3 colors | 4 colors | 5 colors | 6 colors |
|---|---|---|---|---|---|
| 3 | 9 | 27 | 76 | $>$ 170 | $>$ 225 |
| 4 | 35 | 293 | $>$ 1,048 | $>$ 2,254 | $>$ 9,778 |
| 5 | 178 | $>$ 2,173 | $>$ 17,705 | $>$ 98,741 | $>$ 98,748 |
| 6 | 1,132 | $>$ 11,191 | $>$ 157,209 | $>$ 786,740 | $>$ 1,555,549 |
| 7 | $>$ 3,703 | $>$ 48,811 | $>$ 2,284,751 | $>$ 15,993,257 | $>$ 111,952,799 |
| 8 | $>$ 11,495 | $>$ 238,400 | $>$ 12,288,155 | $>$ 86,017,085 | $>$ 602,119,595 |
| 9 | $>$ 41,265 | $>$ 932,745 | $>$ 139,847,085 | $>$ 978,929,595 | $>$ 6,852,507,165 |
| 10 | $>$ 103,474 | $>$ 4,173,724 | $>$ 1,189,640,578 | $>$ 8,327,484,046 | $>$ 58,292,388,322 |
| 11 | $>$ 193,941 | $>$ 18,603,731 | $>$ 3,464,368,083 | $>$ 38,108,048,913 | $>$ 419,188,538,043 |

Table: Some known exact values and lower bounds for Van der Waerden numbers $W(r, \ell)$. Here $r$ denotes the number of colors and $\ell$ denotes the length of the monochromatic arithmetic progression. Entries without $>$ are exact values.