FROM VAN DER WAERDEN'S THEOREM TO ROTH'S THEOREM: COLORINGS, DENSITY, AND ARITHMETIC PROGRESSIONS

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ABSTRACT. This paper gives an expository introduction to arithmetic progressions in Ramsey theory and additive combinatorics. We begin with the language of colorings, using small examples to illustrate how simple coloring rules can force surprisingly rigid structure. The main result discussed is Van der Waerden's theorem, which states that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. We prove this theorem using a color-focusing argument, emphasizing the central ideas behind the proof rather than treating it as a purely technical construction.

After presenting Van der Waerden's theorem, we explain how the coloring perspective naturally leads to questions about density. Instead of asking whether a pattern must appear in one color class of every coloring, one can ask whether every sufficiently large subset of the integers must contain the same kind of arithmetic structure. This shift connects Ramsey theory to additive combinatorics and motivates density results such as Roth's theorem on three-term arithmetic progressions. The goal of this paper is to make these connections accessible, showing how elementary questions about colorings lead to deeper themes involving structure, randomness, and unavoidable patterns in the integers.

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1. Introduction

Ramsey theory is based on the principle that disorder cannot persist indefinitely: sufficiently large mathematical structures always exist to enforce highly organized patterns. An important result of Ramsey Theory is Van der Waerden's theorem, which states that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. Although the statement is elementary and requires little background to understand, it demonstrates the surprising extent to which order is enforced by simple combinatorial constraints.

Van der Waerden's theorem is particularly well suited to expository treatment for a variety of reasons. It occupies a central position in Ramsey theory and its proof utilizes many fundamental techniques that recur throughout these areas. The language of colorings provides a powerful combinatorial technique for partitioning mathematical structures and investigating the patterns and bounds that must inevitably arise, while remaining accessible to readers with a typical high school mathematical background. Furthermore, the theorem serves as a natural gateway to later developments in additive combinatorics, such as Roth's theorem and Szemerédi's theorem, illustrating how this seemingly elementary question gave rise to a rich body of mathematical research.

2. Preliminaries

In this section, we introduce the basic definitions and notation that will be used throughout the paper. The main objects are arithmetic progressions, colorings, blocks, and focused progressions.

2.1. **Definitions.**

Definition 2.1 (Arithmetic progression)**.** An *arithmetic progression* is a sequence of integers in which the difference between consecutive terms is constant. Precisely, it is a sequence of the form

$$a,\ a+d,\ a+2d,\ \dots,$$

where $a, d \in \mathbb{Z}$. The integer a is called the first term, and the integer d is called the common difference.

Throughout this paper, we consider only *non-trivial* arithmetic progressions, meaning arithmetic progressions with common difference $d \neq 0$.

Definition 2.2 (k-term arithmetic progression)**.** A *k-term arithmetic progression*, abbreviated as a *k-AP*, is a finite arithmetic progression consisting of exactly k terms:

$$a,\ a+d,\ a+2d,\ \dots,\ a+(k-1)d,$$

where $d \neq 0$.

For example,

$$3,\ 7,\ 11,\ 15$$

is a 4-term arithmetic progression with common difference 4.

Definition 2.3 (r-coloring)**.** Let S be a set and let r be a positive integer. An *r-coloring* of S is a function

$$C : S \to [r].$$

For each $i \in [r]$, the set

$$S_i = \{x \in S : C(x) = i\}$$

is called the i-th color class.

Thus an r-coloring is equivalently a decomposition of S into r pairwise disjoint, possibly empty, color classes whose union is S.

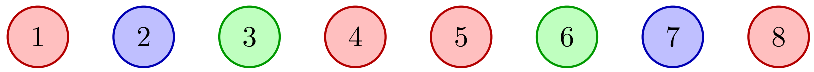

FIGURE 1. A 3-coloring of the interval [8].

For example, Figure 1 illustrates a 3-coloring of [8]. In this coloring,

$$C(1) = C(4) = C(5) = C(8) = 1, \qquad C(2) = C(7) = 2, \qquad C(3) = C(6) = 3.$$

Equivalently, the corresponding color classes are

$$S_1 = \{1, 4, 5, 8\}, \qquad S_2 = \{2, 7\}, \qquad S_3 = \{3, 6\}.$$

Definition 2.4 (Monochromatic arithmetic progression)**.** Let $C : S \to \{1, 2, \dots, r\}$ be an r-coloring. An arithmetic progression contained in S is said to be *monochromatic* if all of its terms have the same color.

Equivalently, a k-term arithmetic progression

$$a,\ a + d,\ a + 2d,\ \dots,\ a + (k - 1)d$$

is monochromatic if there exists some $i \in \{1, 2, \dots, r\}$ such that

$$C(a) = C(a + d) = C(a + 2d) = \cdots = C(a + (k - 1)d) = i.$$

Definition 2.5 (Block)**.** A *block of length* L is a set of L consecutive integers. Thus a block of length L has the form

$$B = \{x + 1, x + 2, \dots, x + L\}$$

for some integer x.

When an interval is divided into consecutive blocks of equal length L, we denote these blocks by

$$B_1, B_2, \dots.$$

For example, if $[N]$ is divided into blocks of length L, then

$$B_t = \{(t - 1)L + 1, (t - 1)L + 2, \dots, tL\}.$$

Remark 2.6. Some diagrams in this paper omit the numbers inside a block; instead representing them as dots.

Definition 2.7 (Block-color)**.** Let

$$B = \{x + 1, x + 2, \dots, x + L\}$$

be a block of length L, and suppose that C is a coloring defined on B. The *block-color* of B is the ordered tuple

$$\mathrm{col}(B) = \big(C(x + 1), C(x + 2), \dots, C(x + L)\big).$$

Thus, the block-color records the entire color pattern inside the block. Two blocks have the same block-color precisely when their corresponding entries have the same colors.

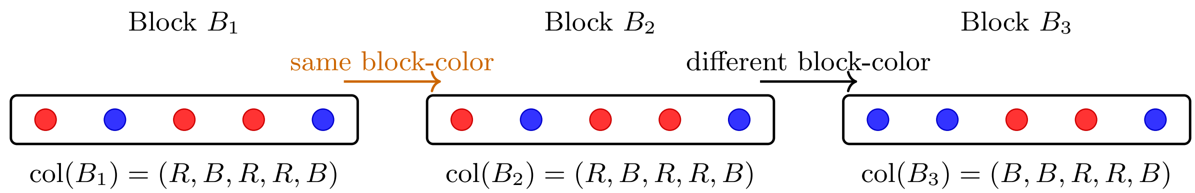

FIGURE 2. Two blocks are assigned the same block-color precisely when their first entries have the same color, their second entries have the same color, and so on for every position in the block.

Figure 2 shows the block-color viewpoint. The first two blocks have the same block-color because their internal color patterns agree position by position. The third block has a different block-color because its pattern is different.

Definition 2.8 (Focused progressions and anchor)**.** Let $\ell \geq 2$, and let a be an integer. A collection of $(\ell - 1)$-term arithmetic progressions is said to be *focused at the anchor a* if each progression has the form

$$a + d,\ a + 2d,\ \dots,\ a + (\ell - 1)d$$

for some positive integer d.

The point a is called the *anchor* because adding a to any one of these progressions produces the full ℓ-term arithmetic progression

$$a,\ a + d,\ a + 2d,\ \dots,\ a + (\ell - 1)d.$$

2.2. **Notation.** Throughout the paper, we use the following notation.

- Given a positive integer N, we write
$$[N] = \{1, 2, \dots, N\}.$$
- $W(r, \ell)$ denotes the Van der Waerden number, defined in theorem 3.1.
- The letter r denotes the number of colors in a coloring.
- The letter ℓ denotes the length of the arithmetic progression we are trying to force.
- Unless stated otherwise,
$$C : [N] \to \{1, 2, \dots, r\}$$
denotes an r-coloring of the interval $[N]$.
- If $x \in [N]$, then $C(x)$ denotes the color assigned to x.
- We use $B_1, B_2, \dots$ to denote consecutive blocks of equal length.
- The symbol d denotes the common difference of an arithmetic progression. In arguments involving several progressions, we may write $d_1, d_2, \dots, d_m$ for their common differences.
- The symbol D is used to describe the difference between two blocks.

3. Van der Waerden's Theorem

Having introduced arithmetic progressions and colorings of integers, we now turn to the main result of this paper. Rather than presenting the theorem immediately, it is helpful to first introduce a motivating problem.

3.1. **Motivating Problem.** Suppose each positive integer is colored either red or blue. At first glance, it may seem possible a careful coloring might avoid monochromatic arithmetic progressions. Indeed, for small intervals one can deliberately choose colors so that no three integers forming an arithmetic progression have the same color.

For example, the coloring

$$R,\ R,\ B,\ B,\ R,\ R,\ B,\ B$$

of the integers 1 through 8 contains no monochromatic 3-term arithmetic progression. Since such examples exist, it is natural to ask whether this strategy can be continued indefinitely.

This leads to the following fundamental question.

> *Is it always possible to color the positive integers using finitely many colors while avoiding monochromatic arithmetic progressions of a given length?*

Interestingly, the answer is no. Although we may avoid these patterns for a while, eventually any coloring is forced to contain one if it follows certain constraints. This phenomenon is precisely what Van der Waerden's theorem establishes.

3.2. **Statement of the Theorem.**

Definition 3.1 (Van der Waerden number)**.** For positive integers r and ℓ, let $W(r, \ell)$ denote the smallest positive integer N such that every r-coloring of $[N]$ contains a monochromatic arithmetic progression of length ℓ.

Remark 3.2. If $N \geq W(r, \ell)$, then every r-coloring of $[N]$ also contains a monochromatic ℓ-term arithmetic progression, since its restriction to the first $W(r, \ell)$ integers has this property.

Theorem 3.3 (Van der Waerden)**.** *Van der Waerden's theorem is commonly stated in the following two forms.*

Finite formulation. *For every pair of positive integers r and ℓ, there exists a positive integer $W(r, \ell)$ such that every r-coloring of*

$$\{1, 2, \ldots, W(r, \ell)\}$$

contains a monochromatic arithmetic progression of length ℓ.

Infinite formulation. *Every finite coloring of the positive integers contains monochromatic arithmetic progressions of arbitrarily long length.*

3.3. **Preliminary Cases.** Before considering nontrivial examples, it is helpful to examine the simplest Van der Waerden numbers.

If $\ell = 1$, then every single integer is already an arithmetic progression of length one. Hence every coloring of any nonempty set contains a monochromatic progression of length one, so

$$W(r, 1) = 1$$

for every positive integer r.

Similarly, if there is only one color available, every arithmetic progression is automatically monochromatic. Therefore the smallest set that always contains an arithmetic progression of length k is simply

$$\{1, 2, \ldots, k\},$$

giving

$$W(1, \ell) = \ell.$$

Now, let us consider $W(r, 2)$. We first claim that

$$W(r, 2) \leq r + 1.$$

Indeed, if we have $r + 1$ integers, then, by Pigeonhole Principle, at least two integers will share the same color, resulting in a two term Arithmetic Progression. We next claim that

$$W(r, 2) > r.$$

Indeed, if we color each of the first r integers one of the r colors, then we have found a construction where no monochromatic 2-term AP exists. Thus we have proven that

$$W(r, 2) = r + 1.$$

These cases are straightforward. The first genuinely interesting Van der Waerden numbers arise when both the number of colours and the desired progression length are at least two.

3.4. **Examples.** The smallest nontrivial Van der Waerden number is

$$W(2, 3) = 9.$$

This means that every coloring of the integers

$$\{1, 2, \ldots, 9\}$$

using two colors necessarily contains a monochromatic 3-term arithmetic progression.

The number 9 is the smallest positive integer with this property. Indeed, the coloring

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$$

of $\{1, \ldots, 8\}$ contains no monochromatic 3-term arithmetic progression, demonstrating that eight integers do not suffice. We encourage the reader to verify this by considering the possible colorings or by attempting a case analysis.

As another example,

$$W(2, 4) = 35,$$

meaning that every red–blue coloring of $\{1, \ldots, 35\}$ contains a monochromatic arithmetic progression of length four.

Although only a handful of Van der Waerden numbers are currently known exactly (as we will understand in later sections), these examples already illustrate how rapidly they grow, as the number of possible colorings that must be considered increases exponentially with the size of the interval.

4. Proof of Existence

We now present one of the classical proofs of Van der Waerden's theorem. To build an intuition for these proofs, we will first use smaller examples before proving the general case. This proof is widely known as the color focusing argument.

4.1. **Existence of $W(2, 3)$.** We begin with the special case of three-term arithmetic progressions. We show that there exists an integer N such that every 2-coloring of $[N]$ contains a monochromatic 3-term arithmetic progression.

A good intuition for this case is to imagine a situation where we have no available color to assign N that avoids a 3-term arithmetic progression. This phenomenon is illustrated by the case $W(2,3) = 9$. For example, the coloring

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$$

of $\{1, 2, \dots, 8\}$ contains no monochromatic 3-term arithmetic progression. However, the ninth integer has no valid color choice: if 9 is colored red, then

$$1,\ 5,\ 9$$

is a monochromatic 3-term arithmetic progression, while if 9 is colored blue, then

$$3,\ 6,\ 9$$

is a monochromatic 3-term arithmetic progression. Thus, the ninth position has no available color that avoids creating a monochromatic 3-term arithmetic progression.

This motivates us to seek a proof showing that, regardless of how the integers are 2-colored, such a configuration must eventually arise. Instead of proving the sharp bound $W(2,3) = 9$, we give a less efficient argument proving

$$W(2,3) \le 325.$$

Although this bound is far from optimal, the method illustrates the block-coloring ideas used later in the general proof.

To begin, we establish a simple combinatorial observation concerning blocks of five consecutive integers.

Lemma 4.1. *Let $C : B \to \{\text{red}, \text{blue}\}$ be a 2-coloring of a block B of five consecutive integers. Then there exist integers a and $d > 0$ such that*

$$a,\ a+d,\ a+2d \in B$$

and

$$C(a) = C(a+d).$$

Proof. Write the five elements of B as

$$x,\ x+1,\ x+2,\ x+3,\ x+4.$$

Consider the following 3-term arithmetic progressions inside B:

$$x,\ x+1,\ x+2$$

and

$$x,\ x+2,\ x+4.$$

If $C(x) = C(x+1)$, then the first progression satisfies the lemma with $a = x$ and $d = 1$.

If $C(x) \ne C(x+1)$, then since there are only two colors, it remains to consider $C(x+2)$. If $C(x+1) = C(x+2)$, then the progression

$$x+1,\ x+2,\ x+3$$

satisfies the lemma with $a = x+1$ and $d = 1$.

Otherwise, $C(x+1) \neq C(x+2)$. Since there are only two colors and $C(x) \neq C(x+1)$, we must have

$$C(x) = C(x+2).$$

Therefore the progression

$$x,\ x+2,\ x+4$$

satisfies the lemma with $a = x$ and $d = 2$.

Thus, in all cases, there exist a and $d > 0$ such that

$$a,\ a+d,\ a+2d \in B$$

and

$$C(a) = C(a+d).$$

□

We now prove that $W(2,3) \leq 325$. Suppose, for the sake of contradiction, that there exists a 2-coloring

$$C : [325] \to \{\text{red}, \text{blue}\}$$

with no monochromatic 3-term arithmetic progression.

Divide [325] into 65 consecutive blocks of length 5:

$$B_1 = \{1,2,3,4,5\}, \quad B_2 = \{6,7,8,9,10\}, \quad \ldots, \quad B_{65} = \{321,322,323,324,325\}.$$

Each block has five entries, and each entry has one of two colors. Hence there are only

$$2^5 = 32$$

possible color patterns for a block. We now view C as coloring the blocks themselves: two blocks have the same block-color if their five entries have exactly the same color pattern.

Consider only the first 33 blocks,

$$B_1, B_2, \ldots, B_{33}.$$

Since there are 33 blocks but only 32 possible block-color patterns, the Pigeonhole Principle implies that two of these blocks, say B_i and B_j with $i < j$, have the same color pattern.

Let D be the distance between corresponding elements of B_i and B_j. Since the blocks have the same pattern, corresponding elements have the same color.

By the lemma, inside B_i there exist integers a and $d > 0$ such that

$$a,\ a+d,\ a+2d \in B_i$$

and

$$C(a) = C(a+d).$$

Without loss of generality, assume this common color is red:

$$C(a) = C(a+d) = \text{red}.$$

Because B_i and B_j have the same color pattern, we also have

$$C(a+D) = C(a+D+d) = \text{red}.$$

If $C(a+2d)$ were red, then

$$a,\ a+d,\ a+2d$$

would already be a monochromatic red 3-term arithmetic progression. Thus

$$C(a+2d) = \text{blue}.$$

Similarly, since the corresponding point in B_j has the same color,

$$C(a+D+2d) = \text{blue}.$$

Now consider the point

$$a+2D+2d.$$

This point lies in the block obtained by translating B_j forward by the same block distance $j-i$. The index of this block is

$$j+(j-i) = 2j-i.$$

Since

$$1 \le i < j \le 33,$$

we have

$$2j - i \le 2 \cdot 33 - 1 = 65.$$

Therefore this translated block is among

$$B_1, B_2, \ldots, B_{65},$$

and hence

$$a+2D+2d \in [325].$$

There are two cases.

If

$$C(a+2D+2d) = \text{blue},$$

then

$$a+2d, \quad a+D+2d, \quad a+2D+2d$$

is a monochromatic blue 3-term arithmetic progression.

If instead

$$C(a+2D+2d) = \text{red},$$

then

$$a, \quad a+(D+d), \quad a+2(D+d)$$

is a monochromatic red 3-term arithmetic progression, since these three terms are

$$a, \quad a+D+d, \quad a+2D+2d.$$

In either case, we obtain a monochromatic 3-term arithmetic progression, contradicting our assumption. Therefore every 2-coloring of [325] contains a monochromatic 3-term arithmetic progression, and hence

$$W(2,3) \leq 325.$$

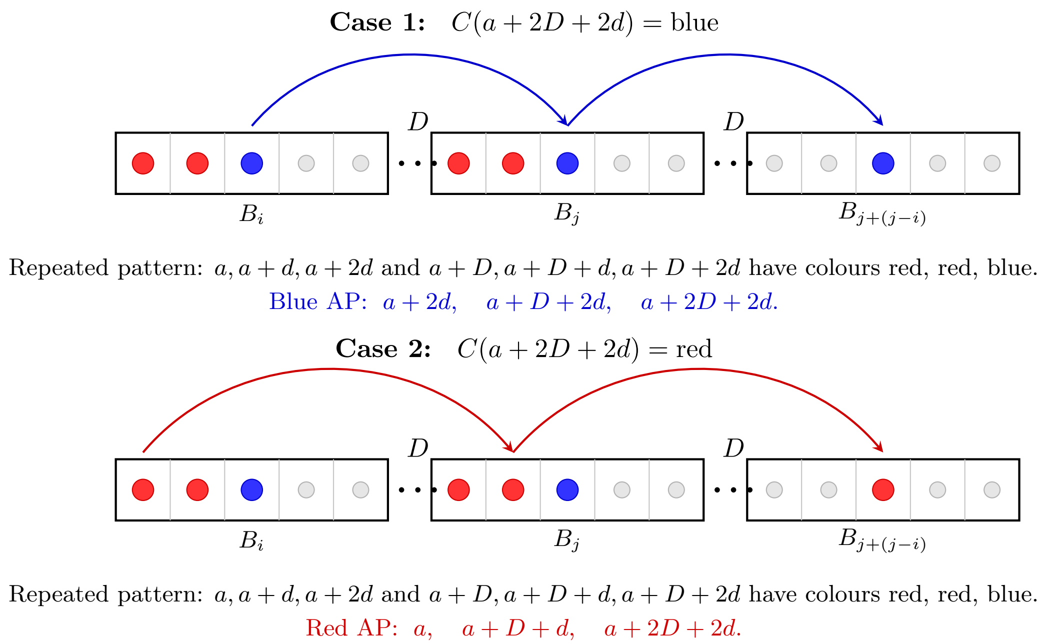

FIGURE 3. The two possible colors of $a+2D+2d$ force either a blue 3-term progression with common difference D or a red 3-term progression with common difference $D+d$.

4.2. **Existence of $W(2,4)$.** We now use the previous ideas to prove the existence of $W(2,4)$. In other words, we show that there exists an integer N such that every red–blue coloring of

$$\{1, 2, \ldots, N\}$$

contains a monochromatic arithmetic progression of length 4.

In the previous section, we proved the existence of a number M such that every red–blue coloring of an interval of length M contains a monochromatic 3-term arithmetic progression. For consistency with the previous proof, we take

$$M = 325.$$

The exact value of M is not important for the argument below. The only property we use is this:

every red–blue coloring of an interval of length M contains a monochromatic 3-term AP.

Lemma 4.2 (Three-term case for finitely many colors)**.** *For every positive integer q, there exists an integer $W(q,3)$ such that every q-coloring of*

$$\{1, 2, \ldots, W(q,3)\}$$

contains a monochromatic 3-term arithmetic progression.

We will defer the proof of this lemma to when we show the full proof.

The key idea is to combine two levels of structure; the color of each number within the blocks and the color of the entire block.

First, inside a sufficiently large block, the 3-term case forces a monochromatic 3-term arithmetic progression of numbers. Second, by treating entire blocks as single objects, the general 3-term theorem forces three identical blocks (same number colorings) to occur in arithmetic progression.

Let

$$L = 2M.$$

We divide our interval into consecutive blocks of length L. Since each block has $2M$ positions and each position is either red or blue, there are

$$2^{2M}$$

possible block-colors.

Let

$$q = 2^{2M}.$$

By the lemma, there exists an integer

$$T = W(q, 3)$$

such that every q-coloring of $\{1, 2, \ldots, T\}$ contains a monochromatic 3-term arithmetic progression.

We now choose N large enough to contain $2T$ blocks of length L. For instance, take

$$N = 2TL.$$

Write these blocks as

$$B_1, B_2, \ldots, B_{2T},$$

where

$$B_t = \{(t-1)L + 1, (t-1)L + 2, \ldots, tL\}.$$

Now suppose, for the sake of contradiction, that we have a red–blue coloring

$$C : [N] \to \{\text{red}, \text{blue}\}$$

with no monochromatic 4-term arithmetic progression.

We first look inside a single block. Consider any block B_t. Its first half has length M. Therefore, by the definition of M, the first half of B_t contains a monochromatic 3-term arithmetic progression. Hence there exist integers a and $d > 0$ such that

$$a, \quad a + d, \quad a + 2d$$

lie inside the first half of B_t, and all have the same color.

Without loss of generality, assume that this color is red:

$$C(a) = C(a + d) = C(a + 2d) = \text{red}.$$

The point $a + 3d$ also lies inside B_t. To see this, suppose that $x + 1$ is the first element of B_t, so that

$$B_t = \{x + 1, x + 2, \ldots, x + 2M\}.$$

Since

$$a, \quad a + d, \quad a + 2d$$

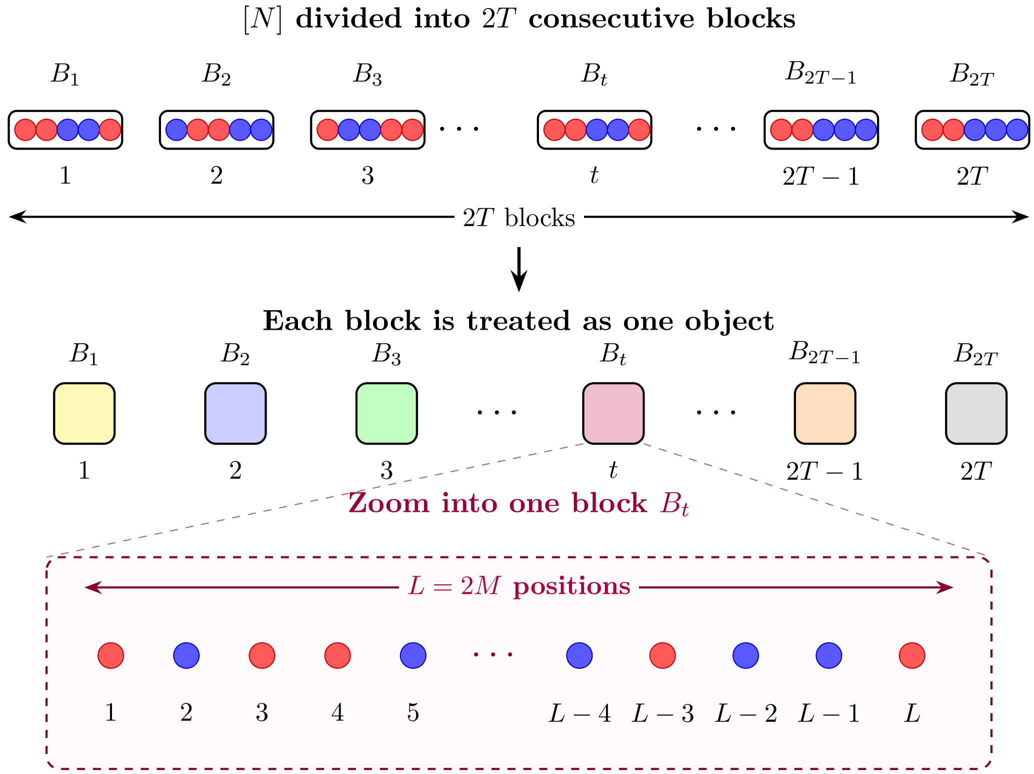

FIGURE 4. A two-level view of the construction: the interval is divided into $2T$ blocks, each of length $L = 2M$, and every block is colored by its complete internal red–blue pattern.

all lie in the first half of B_t, we have

$$a + 2d \leq x + M.$$

Moreover, because both a and $a + 2d$ lie in an interval of length M,

$$2d \leq M - 1,$$

and therefore

$$d \leq \frac{M-1}{2}.$$

It follows that

$$a + 3d = (a + 2d) + d \leq x + M + \frac{M-1}{2} < x + 2M + 1.$$

Since $a + 3d$ is an integer, this implies

$$a + 3d \leq x + 2M.$$

Thus $a + 3d \in B_t$.

If $C(a + 3d) = \text{red}$, then

$$a, \quad a + d, \quad a + 2d, \quad a + 3d$$

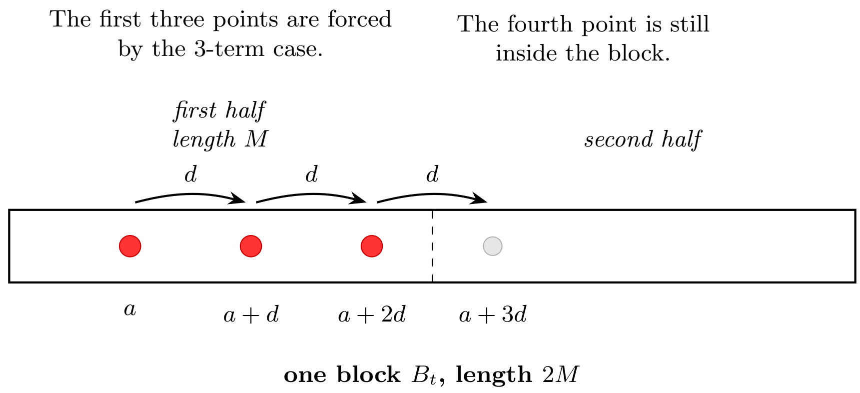

FIGURE 5. A monochromatic 3-term progression in the first half of B_t extends to a fourth point that remains inside the block.

would already be a monochromatic red 4-term arithmetic progression. This is impossible by our contradiction assumption. Therefore

$$C(a+3d) = \text{blue}.$$

Thus, inside any block, we can find a configuration of the form

$$\text{red}, \quad \text{red}, \quad \text{red}, \quad \text{blue}.$$

Now we pass to the block-coloring viewpoint. Each block B_t receives one of the $q = 2^{2M}$ possible block-colors. In other words, we define a new coloring of the block indices

$$\{1, 2, \ldots, 2T\}$$

by assigning to t the full red–blue pattern inside B_t.

Now apply the 3-term lemma with q colors to the first T blocks:

$$B_1, B_2, \ldots, B_T.$$

Since $T = W(q, 3)$, there exist indices

$$i, \quad i+s, \quad i+2s$$

forming a 3-term arithmetic progression such that the blocks

$$B_i, \quad B_{i+s}, \quad B_{i+2s}$$

all have the same block-color.

Because these three blocks were found among the first T blocks, we have

$$i + 2s \leq T.$$

Therefore

$$i + 3s \leq 2T,$$

so the next block B_{i+3s} also exists in our interval. This is why we chose $2T$ blocks instead of only T blocks.

The block indices i, $i+s$, $i+2s$ form a 3-term arithmetic progression.

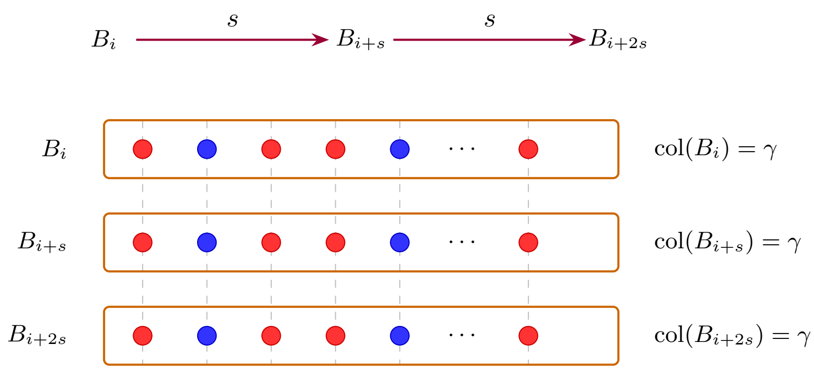

FIGURE 6. The blocks B_i, B_{i+s}, and B_{i+2s} have identical internal color patterns because their block-colors coincide.

Let

$$D = sL.$$

This is the distance between corresponding positions in consecutive blocks B_i, B_{i+s}, B_{i+2s}.

Inside B_i, we have a configuration

$$a, \quad a+d, \quad a+2d, \quad a+3d$$

with colors

$$\text{red}, \quad \text{red}, \quad \text{red}, \quad \text{blue}.$$

Since the blocks B_i, B_{i+s}, B_{i+2s} have the same block-color, the same pattern occurs in the corresponding positions of the next two blocks.

Thus

$$C(a) = C(a+d) = C(a+2d) = \text{red},$$

$$C(a+D) = C(a+D+d) = C(a+D+2d) = \text{red},$$

and

$$C(a+2D) = C(a+2D+d) = C(a+2D+2d) = \text{red}.$$

Similarly,

$$C(a+3d) = C(a+D+3d) = C(a+2D+3d) = \text{blue}.$$

Now consider the point

$$a+3D+3d.$$

This is the corresponding position in the next block B_{i+3s}. Its color is either blue or red.
Case 1. Suppose

$$C(a+3D+3d) = \text{blue}.$$

Then the four points

$$a+3d, \quad a+D+3d, \quad a+2D+3d, \quad a+3D+3d$$

form a 4-term arithmetic progression with common difference D. They are all blue, so this is a monochromatic blue 4-term arithmetic progression.
Case 2. Suppose instead

$$C(a+3D+3d) = \text{red}.$$

Then the four points

$$a, \quad a+(D+d), \quad a+2(D+d), \quad a+3(D+d)$$

form a 4-term arithmetic progression with common difference $D+d$. Written out, these terms are

$$a, \quad a+D+d, \quad a+2D+2d, \quad a+3D+3d.$$

They are all red, so this is a monochromatic red 4-term arithmetic progression.

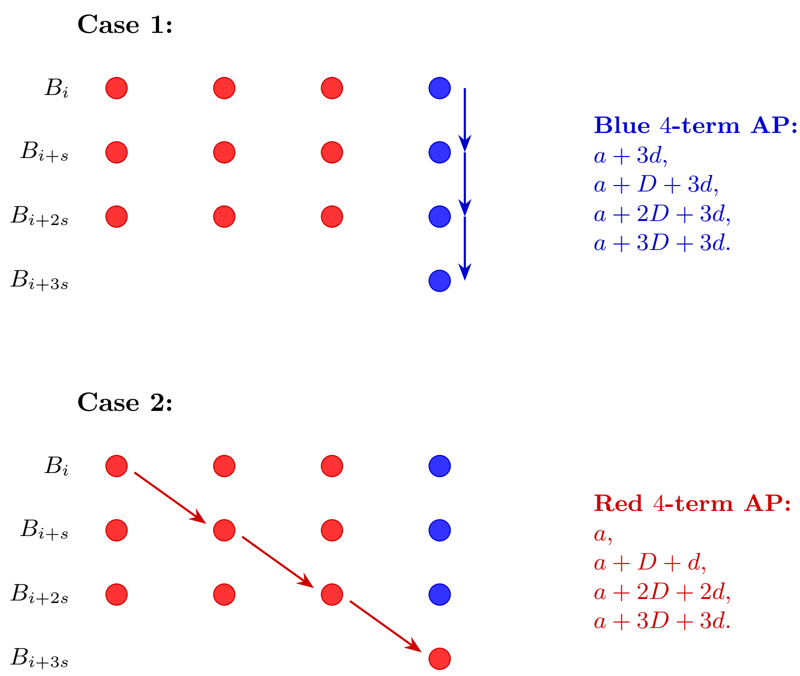

FIGURE 7. Depending on the color of $a+3D+3d$, the construction produces either a blue 4-term progression with difference D or a red 4-term progression with difference $D+d$.

In both cases, we obtain a monochromatic 4-term arithmetic progression. This contradicts our assumption that the coloring contained no monochromatic 4-term arithmetic progression.

Therefore every sufficiently long red–blue coloring contains a monochromatic 4-term arithmetic progression. Hence

$$W(2,4)$$

exists.

Remark 4.3. This argument proves the existence of $W(2,4)$, not the sharp value. The exact value is

$$W(2,4) = 35,$$

but the block-coloring argument above gives a much larger upper bound.

4.3. **The general case.** We now prove the full existence statement of Van der Waerden's theorem. Recall that in our notation,

$$W(r, \ell)$$

denotes the smallest integer N, if it exists, such that every r-coloring of

$$\{1, 2, \ldots, N\}$$

contains a monochromatic arithmetic progression of length ℓ.

The goal is to prove that for every pair of positive integers r and ℓ, the number

$$W(r, \ell)$$

exists.

The proof below is an abstraction of the ideas already seen in the cases $W(2, 3)$ and $W(2, 4)$. The central idea is called *color focusing.* Instead of immediately trying to force a full ℓ-term arithmetic progression, we first force many shorter $(\ell - 1)$-term progressions which all point back toward a common missing first term. Eventually, since there are only finitely many colors, this creates a contradiction unless a full monochromatic ℓ-term progression already exists.

The color-focusing lemma. We now prove the main lemma used in the full proof.

Lemma 4.4 (Color-focusing lemma). *Fix positive integers r and ℓ, with $\ell \geq 3$. Assume that for every positive integer q, the number*

$$W(q, \ell - 1)$$

exists.

For fixed r and ℓ, let $U_m = U_m(r, \ell)$ denote a sufficiently large positive integer depending on m, r, and ℓ. The number U_m is chosen so that every r-coloring of $[U_m]$ either already contains a monochromatic ℓ-term arithmetic progression, or contains m monochromatic $(\ell - 1)$-term arithmetic progressions focused at the same anchor.

More precisely, for every positive integer m, there exists a positive integer U_m such that every r-coloring

$$C : [U_m] \to \{1, 2, \ldots, r\}$$

satisfies at least one of the following two conclusions.

Statement I. *There is a monochromatic ℓ-term arithmetic progression.*

Statement II. *There exist an anchor a and positive integers*

$$d_1, d_2, \ldots, d_m$$

such that each of the $(\ell - 1)$-term arithmetic progressions

$$a + d_t, \quad a + 2d_t, \quad \ldots, \quad a + (\ell - 1)d_t$$

is monochromatic, and these m progressions have pairwise different colors. Moreover, if Statement I does not occur, then the color of the anchor a is different from the color of each of these m progressions.

Proof. We prove the lemma by induction on m. The number m counts how many focused $(\ell - 1)$-term progressions we can force.

Base case: $m = 1$. We claim that it is enough to take

$$U_1 = 2W(r, \ell - 1).$$

Let

$$C : [U_1] \to \{1, 2, \dots, r\}$$

be any r-coloring.

Look at the second half of $[U_1]$. This second half has length

$$W(r, \ell - 1).$$

By the induction hypothesis on the progression length, the second half contains a monochromatic $(\ell - 1)$-term arithmetic progression. Hence there exist integers a_0 and $d > 0$ such that

$$a_0, \quad a_0 + d, \quad a_0 + 2d, \quad \dots, \quad a_0 + (\ell - 2)d$$

are all the same color.

Since this progression lies in the second half of $[U_1]$, the point

$$a_0 - d$$

still lies in $[U_1]$. Define

$$a = a_0 - d.$$

Then the monochromatic $(\ell - 1)$-term progression can be rewritten as

$$a + d, \quad a + 2d, \quad \dots, \quad a + (\ell - 1)d.$$

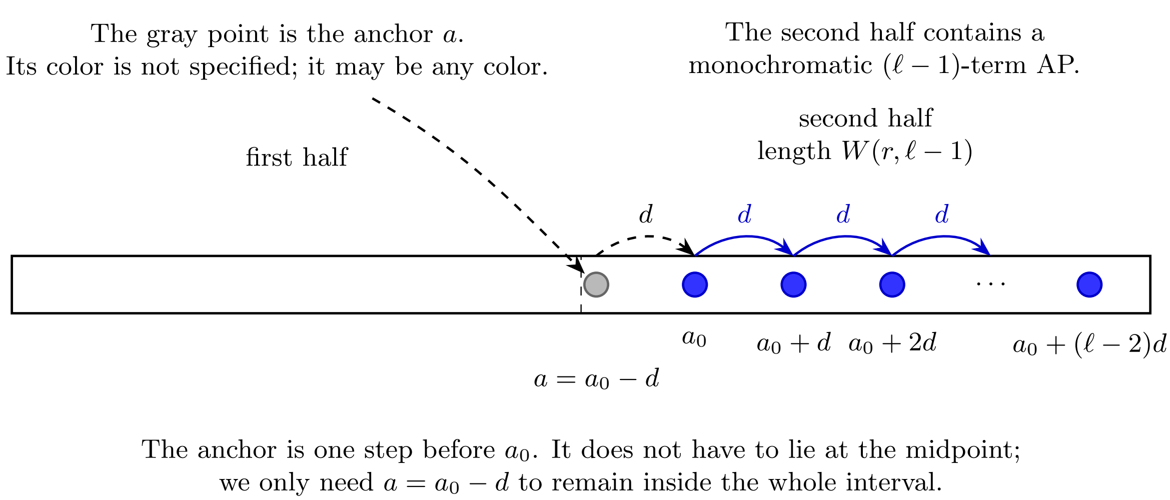

FIGURE 8. Visualization of the monochromatic progression and its anchor point.

There are now two possibilities.

If

$$C(a) = C(a + d),$$

then

$$a, \quad a + d, \quad a + 2d, \quad \dots, \quad a + (\ell - 1)d$$

is a monochromatic ℓ-term arithmetic progression. This is Statement I.

If instead

$$C(a) \neq C(a+d),$$

then the progression

$$a+d, \quad a+2d, \quad \dots, \quad a+(\ell-1)d$$

is a monochromatic $(\ell-1)$-term arithmetic progression focused at a. This gives Statement II with $m=1$.

Thus the base case is proved.

Induction step. Assume that U_m exists. We will prove that U_{m+1} exists.

Let

$$H = W(r^{U_m}, \ell-1).$$

We will take

$$U_{m+1} = 2U_m H.$$

Now let

$$C : [U_{m+1}] \to \{1, 2, \dots, r\}$$

be any r-coloring.

We divide the second half of $[U_{m+1}]$ into H consecutive blocks, each of length U_m:

$$B_1, \quad B_2, \quad \dots, \quad B_H.$$

The first half is kept unused for the moment. Its purpose is to make sure that, later, we can move one block-step backwards and still remain inside the interval.

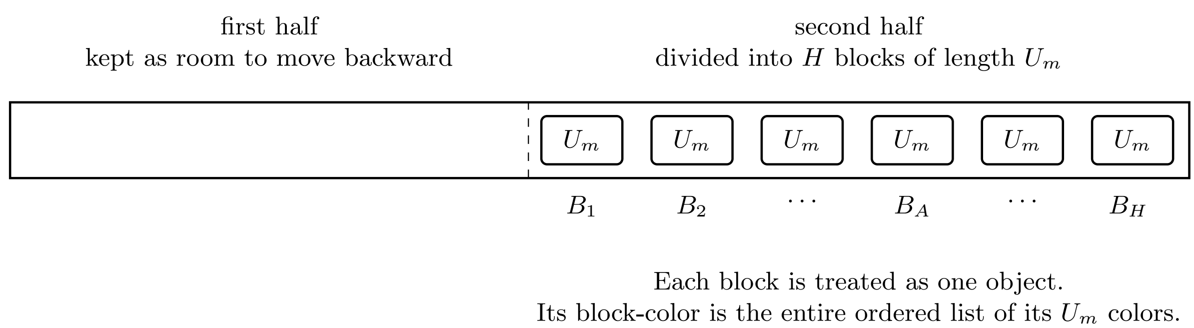

FIGURE 9. Division of the second half into uniform blocks of length U_m.

Each block has U_m entries, and each entry has one of r colors. Therefore there are at most

$$r^{U_m}$$

possible block-colors. Thus the coloring C induces an

$$r^{U_m}\text{-coloring}$$

of the block indices

$$\{1, 2, \dots, H\}.$$

Since

$$H = W(r^{U_m}, \ell - 1),$$

the block-coloring contains a monochromatic $(\ell-1)$-term arithmetic progression of blocks. Therefore there exist block indices

$$A, \quad A+E, \quad A+2E, \quad \dots, \quad A+(\ell-2)E$$

such that

$$B_A, \quad B_{A+E}, \quad B_{A+2E}, \quad \dots, \quad B_{A+(\ell-2)E}$$

all have the same block-color.

Let

$$D = EU_m.$$

This is the distance between corresponding positions in consecutive selected blocks. Since $E \leq H - 1$, we have

$$D = EU_m < HU_m,$$

which is the length of the first half of $[U_{m+1}]$. Since a_0 lies in the second half, it follows that $a_0 - D \geq 1$.

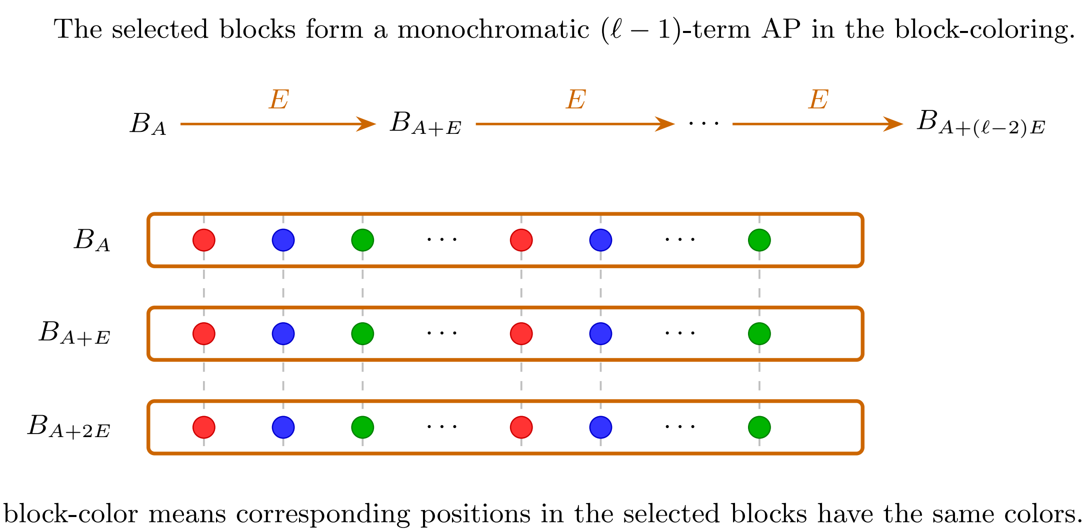

FIGURE 10. Structural alignment and color consistency across corresponding blocks within the arithmetic progression.

Now look inside the first selected block B_A. Since B_A has length U_m, the induction hypothesis on m applies to this block.

If Statement I occurs inside B_A, then we already have a monochromatic ℓ-term arithmetic progression, and we are done.

So suppose Statement I does not occur inside B_A. Then Statement II occurs inside B_A. Therefore there exist an anchor $a_0 \in B_A$ and positive integers

$$d_1, d_2, \dots, d_m$$

such that each progression

$$a_0 + d_t, \quad a_0 + 2d_t, \quad \dots, \quad a_0 + (\ell-1)d_t$$

is monochromatic, and these m progressions have pairwise different colors. Also, since there is no monochromatic ℓ-term progression, the color of a_0 is different from the color of each of these progressions.

The picture inside B_A looks like this.

Inside the block B_A, the induction hypothesis gives m focused spikes

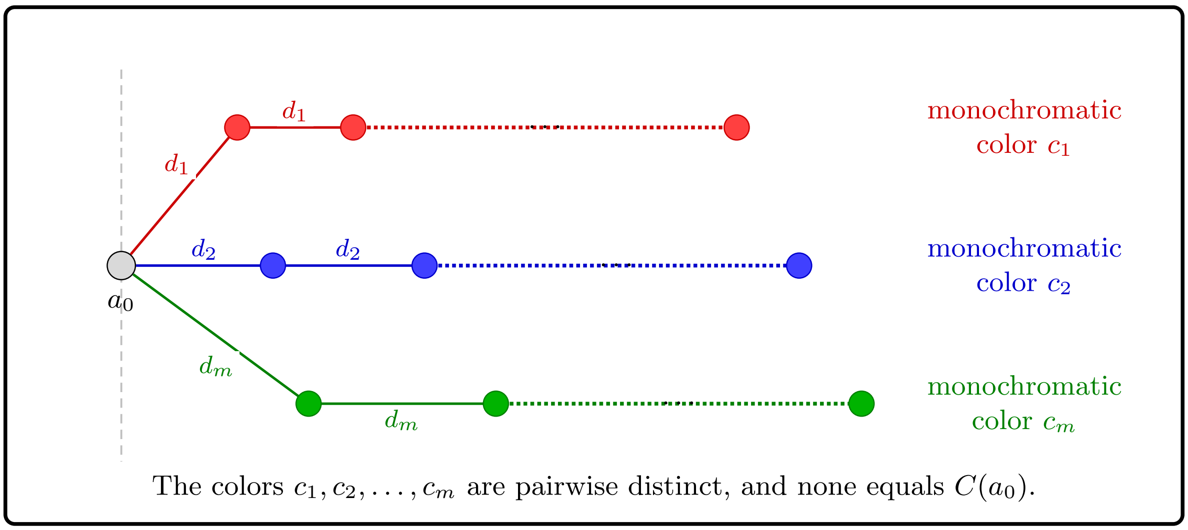

FIGURE 11. The induction hypothesis produces m monochromatic $(\ell-1)$-term progressions focused at a_0, with pairwise distinct colors.

Because the selected blocks

$$B_A, \quad B_{A+E}, \quad \ldots, \quad B_{A+(\ell-2)E}$$

all have the same block-color, the same configuration repeats in corresponding positions of all these blocks.

Now define

$$a = a_0 - D.$$

This point lies in $[U_{m+1}]$. The reason is that all selected blocks lie in the second half of $[U_{m+1}]$, and D is at most the length of that second half. Hence moving back by D still stays inside the full interval.

We now show that the anchor a gives $m+1$ focused progressions.

First, each old focused progression produces a new one. For each $t \in \{1, 2, \ldots, m\}$, consider the common difference

$$D + d_t.$$

Then

$$a + (D + d_t), \quad a + 2(D + d_t), \quad \ldots, \quad a + (\ell - 1)(D + d_t)$$

is monochromatic.

Indeed, substituting $a = a_0 - D$, the terms become

$$a_0 + d_t, \quad a_0 + D + 2d_t, \quad a_0 + 2D + 3d_t, \quad \ldots, \quad a_0 + (\ell - 2)D + (\ell - 1)d_t.$$

These are corresponding copies of the original focused progression in the selected blocks, so they all have the same color.

Second, we get one new focused progression using the common difference D:

$$a + D, \quad a + 2D, \quad \dots, \quad a + (\ell - 1)D.$$

Since $a = a_0 - D$, this is

$$a_0, \quad a_0 + D, \quad \dots, \quad a_0 + (\ell - 2)D.$$

These are corresponding positions in blocks with the same block-color, so they are all the same color.

The first m new progressions retain the pairwise distinct colors of the original m focused progressions. The progression with common difference D has color $C(a_0)$, which is different from all of those colors by the induction hypothesis.

Finally, if the new anchor $a = a_0 - D$ had the same color as any one of these $m + 1$ progressions, adjoining a would complete that progression to a monochromatic ℓ-term arithmetic progression. Thus, unless Statement I already holds, the new anchor has a different color from every new progression.

This completes the induction on m, and hence proves the color-focusing lemma. □

Proof of Van der Waerden's theorem. We now use the color-focusing lemma to prove the theorem.

Theorem 4.5 (Van der Waerden's theorem)**.** *For every pair of positive integers r and ℓ, there exists a positive integer*

$$W(r, \ell)$$

such that every r-coloring of

$$\{1, 2, \dots, W(r, \ell)\}$$

contains a monochromatic arithmetic progression of length ℓ.

Proof. We prove the theorem by induction on ℓ, the desired length of the arithmetic progression.

Base cases. If $\ell = 1$, then every single point is already a monochromatic 1-term arithmetic progression, so

$$W(r, 1) = 1.$$

If $\ell = 2$, then the Pigeonhole Principle gives

$$W(r, 2) = r + 1.$$

Indeed, among $r + 1$ integers colored with r colors, two integers must have the same color, and any two distinct integers form a 2-term arithmetic progression.

Thus the theorem holds for $\ell = 1$ and $\ell = 2$.

Induction step. Now fix $\ell \geq 3$. Assume that for every positive integer q, the number

$$W(q, \ell - 1)$$

exists. We will prove that for every positive integer r, the number

$$W(r, \ell)$$

exists.

Fix r. Apply the color-focusing lemma with

$$m = r.$$

The lemma gives an integer U_r such that every r-coloring of

$$[U_r]$$

satisfies either Statement I or Statement II.

If Statement I holds, then we have a monochromatic ℓ-term arithmetic progression, and we are done.

Suppose instead that Statement II holds. Then there is an anchor a, together with r focused monochromatic $(\ell - 1)$-term arithmetic progressions:

$$\begin{array}{cccc} a + d_1, & a + 2d_1, & \ldots, & a + (\ell - 1)d_1, \\ a + d_2, & a + 2d_2, & \ldots, & a + (\ell - 1)d_2, \\ & & \vdots & \\ a + d_r, & a + 2d_r, & \ldots, & a + (\ell - 1)d_r. \end{array}$$

These r progressions have pairwise different colors. Moreover, if no monochromatic ℓ-term progression exists, then the color of a must be different from the color of each progression. Otherwise, if $C(a)$ matched the color of one of the progressions, then adding a to that progression would produce a monochromatic ℓ-term arithmetic progression.

Therefore Statement II would force

$$r + 1$$

different colors: one color for the anchor a, and r further distinct colors for the r focused progressions.

But this is impossible in an r-coloring.

Hence Statement II cannot occur. Therefore Statement I must occur, meaning every r-coloring of $[U_r]$ contains a monochromatic ℓ-term arithmetic progression. Thus

$$W(r, \ell) \leq U_r,$$

so $W(r, \ell)$ exists.

By induction on ℓ, the number $W(r, \ell)$ exists for every pair of positive integers r and ℓ. □

Remark 4.6. This proof also explains why the $W(2, 4)$ argument needed the general 3-term theorem for many colors. In the induction step above, the proof of length ℓ uses the theorem for length $\ell - 1$, but with a much larger number of colors, namely the number of possible block-colors.

5. Van der Waerden Numbers

The known exact values and computational lower bounds for small Van der Waerden numbers are summarized in table 1. The data are taken from the table on Wikipedia's "Van der Waerden number" page.

One computational method used to study Van der Waerden numbers is the use of *Boolean satisfiability solvers*, or SAT solvers. The basic idea is to translate the question of whether $W(r, \ell) > N$ into a large logical formula. For each integer $n \in [N]$ and each color $c \in \{1, 2, \ldots, r\}$, one introduces a Boolean variable which represents the statement that n has color c. The formula then includes clauses forcing each integer to receive exactly one color, and additional clauses forbidding any monochromatic ℓ-term arithmetic progression. Thus, if the SAT solver finds a satisfying assignment, this assignment

TABLE 1. Known exact values and lower bounds for Van der Waerden numbers $W(r,\ell)$. The parameter r is the number of colors, while ℓ is the length of the monochromatic arithmetic progression. An entry preceded by $>$ is a known strict lower bound; all other displayed entries are exact values.

ℓ	Number of colors r				
	2	3	4	5	6
3	9	27	76	> 170	> 225
4	35	293	$> 1{,}048$	$> 2{,}254$	$> 9{,}778$
5	178	$> 2{,}173$	$> 17{,}705$	$> 98{,}741$	$> 98{,}748$
6	1,132	$> 11{,}191$	$> 157{,}209$	$> 786{,}740$	$> 1{,}555{,}549$
7	$> 3{,}703$	$> 48{,}811$	$> 2{,}284{,}751$	$> 15{,}993{,}257$	$> 111{,}952{,}799$
8	$> 11{,}495$	$> 238{,}400$	$> 12{,}288{,}155$	$> 86{,}017{,}085$	$> 602{,}119{,}595$
9	$> 41{,}265$	$> 932{,}745$	$> 139{,}847{,}085$	$> 978{,}929{,}595$	$> 6{,}852{,}507{,}165$
10	$> 103{,}474$	$> 4{,}173{,}724$	$> 1{,}189{,}640{,}578$	$> 8{,}327{,}484{,}046$	$> 58{,}292{,}388{,}322$
11	$> 193{,}941$	$> 18{,}603{,}731$	$> 3{,}464{,}368{,}083$	$> 38{,}108{,}048{,}913$	$> 419{,}188{,}538{,}043$

corresponds to an explicit r-coloring of $[N]$ with no monochromatic ℓ-term progression, proving that

$$W(r,\ell) > N.$$

On the other hand, if the solver proves that the formula is unsatisfiable, then no such coloring exists, so every r-coloring of $[N]$ contains a monochromatic ℓ-term arithmetic progression, proving that

$$W(r,\ell) \leq N.$$

In principle, combining these two types of results can determine an exact value of $W(r,\ell)$. However, the number of variables and constraints grows very quickly as r, ℓ, and N increase, because every possible ℓ-term arithmetic progression inside $[N]$ must be considered. This rapid growth is one reason why so many Van der Waerden numbers remain unknown exactly, even though many lower bounds can be found computationally. Thus, the table should be read not as a failure of the theorem, but as evidence that the theorem is far easier to prove qualitatively than to compute quantitatively.

6. From Van der Waerden to Roth

Van der Waerden's theorem shows that if the positive integers are divided into finitely many color classes, then one of those color classes must contain arbitrarily long arithmetic progressions. In other words, no finite coloring can avoid monochromatic arithmetic progressions forever.

There is another natural way to think about the same phenomenon. Instead of coloring every integer, suppose we only choose a subset

$$A \subseteq \{1, 2, \ldots, N\}.$$

We may then ask whether A itself must contain arithmetic progressions. Of course, this cannot be true for every subset. For example, a very sparse set such as

$$\{1, 2, 4, 8, 16, 32, \ldots\}$$

does not behave like a typical large subset of the integers. It has very large gaps, so we should not expect it to contain many arithmetic progressions.

Thus, the correct question is not whether every subset contains arithmetic progressions, but whether every sufficiently dense subset contains them.

6.1. **Density.** The *density* of a subset $A \subseteq [N]$ is the proportion of numbers in $[N]$ which belong to A. That is, if

$$[N] = \{1, 2, \ldots, N\},$$

then the density of A inside $[N]$ is

$$\frac{|A|}{N}.$$

For example, if A contains 30 elements of $[100]$, then A has density

$$\frac{30}{100} = 0.3.$$

Roth's theorem concerns families of sets whose relative density is bounded below by a fixed constant $\delta > 0$, independently of N. Roth's theorem says that this condition alone is already enough to force a three-term arithmetic progression.

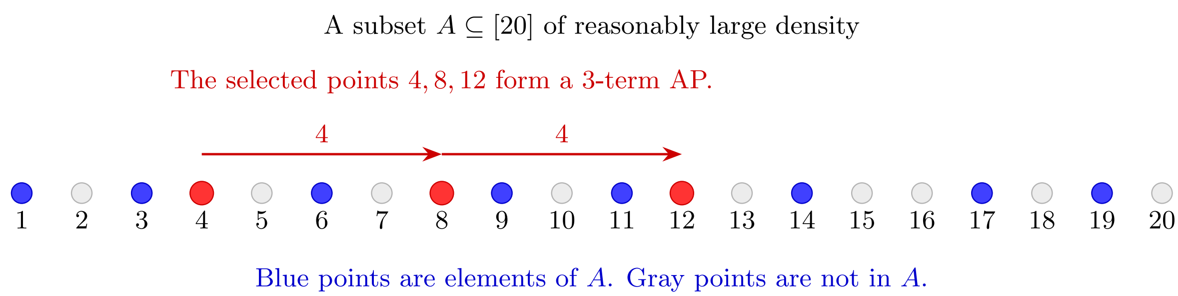

FIGURE 12. The set $A \subseteq [20]$ has relative density $11/20$; the highlighted elements $4, 8, 12$ form a 3-term arithmetic progression.

In the diagram above, the set A is represented by the blue points. The three red points

$$4, \quad 8, \quad 12$$

belong to A, and they form an arithmetic progression with common difference 4.

6.2. **Roth's Theorem.** Roth's theorem is one of the first major density versions of Van der Waerden's theorem.

Theorem 6.1 (Roth's theorem)**.** *For every real number $\delta > 0$, there exists an integer N_0 such that whenever $N \geq N_0$, every subset*

$$A \subseteq [N]$$

with

$$|A| \geq \delta N$$

contains a nontrivial three-term arithmetic progression.

In other words, if A occupies at least a fixed positive proportion of $[N]$, then once N is large enough, A must contain three numbers of the form

$$a, \quad a + d, \quad a + 2d,$$

where $d > 0$.

The important point is that the number δ is fixed. For example, Roth's theorem says that if a set always contains at least 10% of the numbers in $[N]$, then for all sufficiently large N, that set must contain a three-term arithmetic progression. The required value of N may be enormous, but it exists.

6.3. **How Roth differs from Van der Waerden.** Van der Waerden's theorem and Roth's theorem both force arithmetic progressions, but they do so in different ways.

Van der Waerden's theorem begins with a coloring

$$[N] = C_1 \cup C_2 \cup \cdots \cup C_r.$$

Since these r color classes cover all of $[N]$, at least one color class has size at least

$$\frac{N}{r}.$$

So one of the colors has density at least $1/r$.

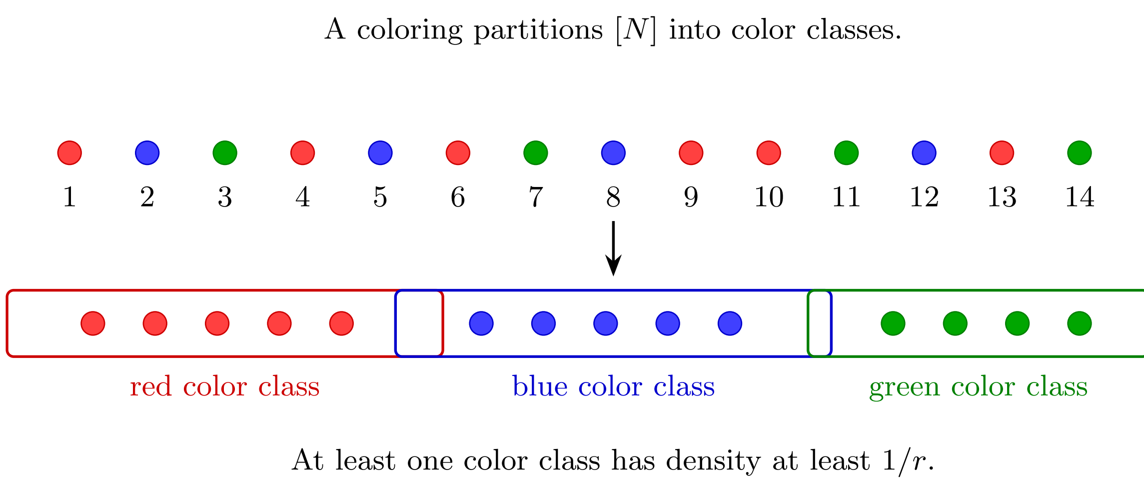

FIGURE 13. An r-coloring partitions $[N]$ into r color classes, at least one of which contains at least N/r elements.

This explains why Roth's theorem feels related to Van der Waerden's theorem. A color class in a finite coloring is simply a subset of $[N]$. If one color class is large enough, then density results can sometimes be used to find arithmetic progressions inside it.

However, Roth's theorem is not just a restatement of Van der Waerden's theorem. Van der Waerden's theorem guarantees monochromatic progressions in every finite coloring, while Roth's theorem studies a single set A and asks whether its density alone forces arithmetic structure.

The difference can be summarized as follows:

$$\text{Van der Waerden: finite coloring forces structure,}$$

while

$$\text{Roth: positive density forces structure.}$$

6.4. **Why density matters.** The density assumption in Roth's theorem is essential. If a set becomes too sparse, then it may avoid three-term arithmetic progressions for a long time, or even forever.

For example, consider the set

$$A = \{1, 2, 4, 8, 16\} \subseteq [20].$$

This set has density

$$\frac{5}{20} = \frac{1}{4}.$$

Although this is not extremely small for $N = 20$, the pattern suggests what can happen for sparse sets: the gaps grow quickly, so arithmetic progressions become harder to force.

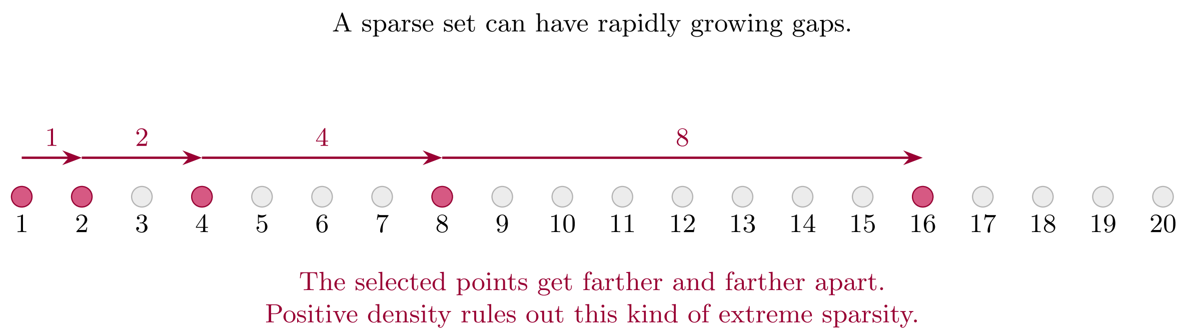

FIGURE 14. Example of an exponentially sparse subset demonstrating rapidly widening gaps between elements.

Roth's theorem says that once a set keeps a fixed positive density, this kind of avoidance becomes impossible. The set cannot spread out too thinly, so eventually three equally spaced points must appear.

6.5. **Examples.** A simple example of a dense set is the set of even numbers:

$$A = \{2, 4, 6, 8, \dots\}.$$

Inside $[N]$, this set has density approximately

$$\frac{1}{2}.$$

It contains many three-term arithmetic progressions, such as

$$2, \quad 4, \quad 6,$$

or

$$4, \quad 8, \quad 12.$$

Another example is the set of integers congruent to 1 modulo 3:

$$A = \{1, 4, 7, 10, 13, \dots\}.$$

This set has density approximately

$$\frac{1}{3},$$

and it also contains many three-term arithmetic progressions. For instance,

$$1, \quad 4, \quad 7$$

is a three-term arithmetic progression.

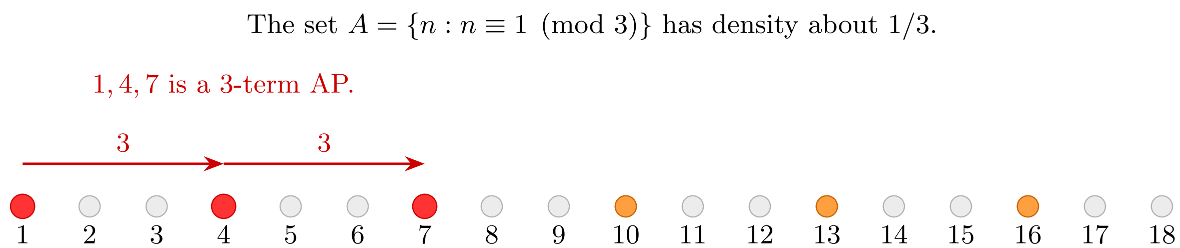

FIGURE 15. The residue class $1 \pmod 3$ has asymptotic density $1/3$ and contains the 3-term progression $1, 4, 7$.

These examples are simple because the sets are very structured. Roth's theorem is much stronger: it says that even if the set A is chosen in a complicated or irregular way, positive density alone eventually forces a three-term arithmetic progression.

6.6. **Connection to Szemerédi's theorem.** Roth's theorem deals with three-term arithmetic progressions. A much deeper result, Szemerédi's theorem, extends this to progressions of any fixed length.

Theorem 6.2 (Szemerédi's theorem)**.** *For every integer $\ell \geq 3$ and every real number $\delta > 0$, there exists an integer N_0 such that whenever $N \geq N_0$, every subset*

$$A \subseteq [N]$$

with

$$|A| \geq \delta N$$

contains an ℓ-term arithmetic progression.

Thus, the progression of ideas is:

$$\text{Van der Waerden} \quad \longrightarrow \quad \text{Roth} \quad \longrightarrow \quad \text{Szemerédi}.$$

Van der Waerden's theorem shows that finite colorings force monochromatic arithmetic progressions. Roth's theorem shows that positive-density subsets force three-term arithmetic progressions. Szemerédi's theorem completes this density perspective by showing that positive-density subsets force arithmetic progressions of every fixed length.

REFERENCES

[1] Veselin Jungic, *Introduction to Ramsey Theory.* Available at: `https://opentext.uleth.ca/Ramsey/book-1.html`. Accessed July 4, 2026.

[2] William Gasarch, *Van der Waerden's Theorem*. Available at: `https://www.cs.umd.edu/~gasarch/TOPICS/mathnotes/vdw3.pdf`. Accessed July 4, 2026.

[3] Richard G. Swan, *Van der Waerden's Theorem*. Available at: `http://www.math.uchicago.edu/~swan/expo/vdW.pdf`. Accessed July 4, 2026.

[4] A. Ya. Khinchin, *Three Pearls of Number Theory*. Dover Publications, New York.

[5] William Gasarch, Clyde Kruskal, and Andy Parrish, *Van der Waerden's Theorem: Variants and Applications*. Available at: `https://www.cs.umd.edu/~gasarch/TOPICS/vdw/GKPbook.pdf`. Accessed July 4, 2026.

[6] Ronald L. Graham and Bruce L. Rothschild, "A short proof of van der Waerden's theorem on arithmetic progressions," *Proceedings of the American Mathematical Society*, vol. 42, no. 2, pp. 385–386, 1974.

[7] *Proof of van der Waerden's Theorem*. Available at: `https://web.mat.upc.edu/juan.jose.rue/dmII-FU/Proof%20of%20VdW%20theorem.pdf`. Accessed July 4, 2026.

[8] Marijn Heule, *vdWaerden: SAT Encodings and Computational Data for Van der Waerden Numbers*. GitHub repository. Available at: `https://github.com/marijnheule/vdWaerden`. Accessed July 4, 2026.

[9] Wikipedia contributors, "Van der Waerden number," *Wikipedia, The Free Encyclopedia*. Available at: `https://en.wikipedia.org/wiki/Van_der_Waerden_number`. Accessed July 4, 2026.