

CONCENTRATION OF THE CHROMATIC NUMBER

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1 Introduction

Deciding whether the chromatic number $\chi(G)$ is less than k is an NP-complete problem for $k \geq 3$, it takes nondeterministic polynomial to solve but is easy to verify. However, for a random graph $G(n, p)$, $\chi(G)$ is highly predictable.

Main result

The result established by the Shamir-Spencer theorem [3] is that for any fixed $p \in (0, 1)$ and any $t > 0$,

$$\mathbb{P}[|\chi(G(n, p)) - \mathbb{E}[\chi(G(n, p))]| \geq t\sqrt{n}] \leq 2e^{-\frac{t^2}{2}}.$$

The chromatic number lies within an interval of width $O(\sqrt{n})$ around its mean.

In order to prove this result, we first define several concepts, including Hoeffding's lemma, the Azuma-Hoeffding inequality, McDiarmid's inequality, the vertex exposure martingale, and then the concentration of $\chi(G)$. Finally, we discuss Alon's improvement [4] to $O(\frac{\sqrt{n}}{\log(n)})$ and extensions to other random graph models.

2 Background

2.1 The Erdős-Rényi Model

The Erdős-Rényi Random Graph [1] definition: Start with n vertices labeled $1, 2, \dots, n$. For each pair of vertices $\{i, j\}$, flip a coin that is heads with probability $p \in [0, 1]$, independently of all other coins. Include the edge $\{i, j\}$ if and only if the coin is heads. The result is $G(n, p)$, the Erdős-Rényi model of random graphs.

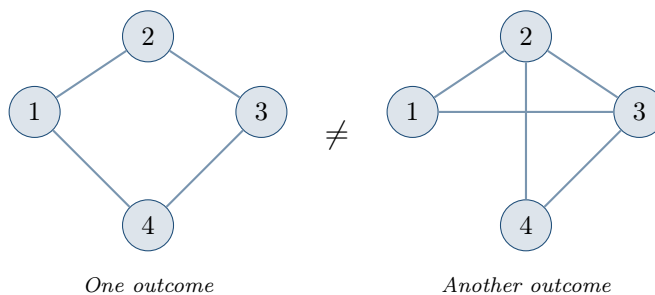


Figure 1: Two different outcomes of $G(4)$, every run is random.

Let N be the actual number of edges in $G(n, p)$. $G(n, p)$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ possible edges. Each is present independently with probability p , so the expected number of edges is

$$E[N] = p \cdot \binom{n}{2} = p \cdot \frac{n(n-1)}{2}.$$

For instance, for $n = 100$ and $p = 0.5$, $E[N] = 0.5 \cdot \frac{100 \cdot 99}{2} = 2475$.

The actual count varies but concentrates tightly by the law of large numbers.

To find $E[\chi(G(n, p))]$ for fixed $p \in [0, 1]$, with probability tending to 1 as $n \rightarrow \infty$:

$\chi(G(n, p))$ is approximately $\frac{n}{2 \cdot \log_b(n)}$, where $b = \frac{1}{1-p}$.

n	p	$b = \frac{1}{1-p}$	$\log_b(n)$	$E(\chi(G(n, p))) \approx \frac{n}{2 \cdot \log_b(n)}$
100	0.5	2	6.6	8
1000	0.5	2	10	50
10000	0.5	2	13.3	376
1000	0.9	10	3	167
1000	0.1	1.11	68	7

Table 1: Chromatic number for various n and p , higher p gives denser graphs and higher χ

2.2 Graph Coloring

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges that connect vertices. An edge $\{u, v\}$ means u and v are directly connected, or adjacent. This can also be written as $E \subseteq \{\{u, v\} : u, v \in V\} = [V]^2$.

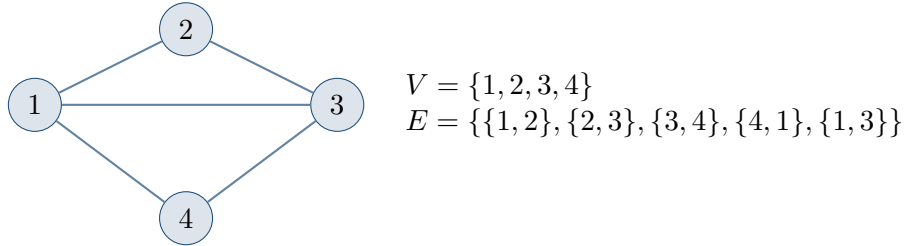


Figure 2: A graph with 4 vertices and 5 edges. The line between two vertices is an edge.

Definition 1. A proper k -coloring of graph G is an assignment of one of k colors to each vertex in G such that no two adjacent vertices share the same color.

Definition 2. The chromatic number $\chi(G)$ is the smallest k for which a proper k -coloring exists.

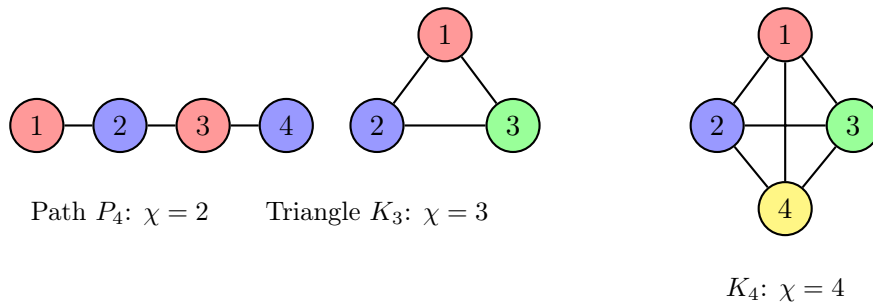


Figure 3: Examples of proper colorings. Adjacent vertices must receive different colors.

Theorem 1. $\chi(G) \geq \omega(G)$ where $\omega(G)$ is the clique number, or the size of the largest complete subgraph.

Informal proof: A clique is a set of vertices that are all connected to each other. If G contains a clique of size k , then those k vertices are all directly connected, and therefore all need different colors. Hence $\chi(G) \geq k$.

Theorem 2. $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree, or the largest number of neighbors any vertex has.

Informal proof: Number the vertices (in any order) and color them one by one, always picking the smallest number not already used by an adjacent vertex. When vertex v is colored, at most $\deg(v) \leq \Delta(G)$ adjacent vertices are already colored, so a color from $\{1, \dots, \Delta(G) + 1\}$ is always available.

3 Martingales and Azuma's Inequality

3.1 Martingales

We want to show that $\chi(G(n, p))$ is close to the expected value $E[\chi(G(n, p))]$.

The strategy is to build a random graph one vertex at a time and track the best prediction for χ at each step. Then, we want to show that the predictions are concentrated, so the final value $\chi(G)$ cannot be far from the initial guess $E(\chi(G))$.

To compute X_2 , for instance, for a specific graph, the problem becomes averaging $\chi(G)$ over all the possible ways the remaining coin flips could go. So if after 2 vertices, edge $\{v_1, v_2\}$ is present, we would then consider all possible graphs on vertices $\{v_1, v_2, v_3, v_4\}$ that are consistent with that one edge, weight them by their probabilities, and average their chromatic numbers. That average is X_2 , but this would be an inefficient computation in practice.

The martingale approach is to prove χ concentrates without computing the intermediate estimates.

Definition 3. A martingale is a sequence of random variables $(X_0, X_1, X_2, \dots, X_n)$ where the best prediction of the next value is the current value.

A biased coin would not be a martingale. For instance, if the coin is weighted 0.6 chance of heads and 0.4 chance of tails, this is not a martingale: $\mathbb{E}[X_{i+1} \mid X_i = 10] = 0.6(11) + 0.4(9) = 10.2 \neq 10$.

Define a filtration $(F_0 \subseteq F_1 \subseteq F_2 \dots \subseteq F_n)$ as the amount of information known by the current coin flip. So F_0 , there are no vertices placed and no edges revealed. At F_1 , vertex v_1 exists. At F_2 , vertex v_2 exists, and the coin has flipped to determine whether edge $\{v_1, v_2\}$ exists. At F_3 , vertex v_3 has been placed and coins have been flipped to determine whether edges $\{v_1, v_3\}$ and $\{v_2, v_3\}$ exist. By F_i , the entire subgraph with $\{v_1, v_2, \dots, v_i\}$ is known (all edges among the i vertices). F_n is the full graph, with all n vertices.

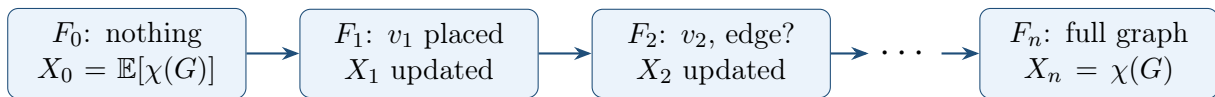


Figure 4: The Doob martingale: a sequence of improving estimates from $\mathbb{E}[\chi(G)]$ to the exact $\chi(G)$.

Definition: A sequence $(X_0, X_1, X_2, \dots, X_n)$ is a martingale with respect to a filtration $(F_0 \subseteq F_1 \subseteq F_2 \dots \subseteq F_n)$ if:

1. X_i is determined by the information in F_i .
2. $E[X_{i+1} | F_i] = X_i$

For step 2, on average, X_{i+1} is the same as X_i . Revealing one more vertex may move the actual prediction up or down, but the expected amount of movement of the prediction is 0 since it's a martingale.

The Doob Martingale. Definition: Let Z be any random variable. In this case, $Z = \chi(G)$. Let (F_i) be a filtration encoding information revealed step by step. Then, the sequence $(X_0, X_1, \dots, X_n)$ is always a martingale. It starts at $X_0 = E[Z]$ and ends at $X_n = Z$.

3.2 The moment generating function trick

The variable X used in this section denotes a single step of movement, $D_i = X_i X_{i-1}$. The moment generating function trick is applied to each individual step, not to the whole sequence. We want to show that the probability that X is greater than t is small to prove that the chromatic number concentrates. Without knowing the exact distribution of X , we can use a workaround by transforming X into e^{sX} for $s > 0, s \in \mathbb{R}$. Then, since e^{sX} is an increasing function, $X \geq t$ means $e^{sX} \geq e^{st}$.

Next, we bring in Markov's inequality. Markov's inequality states that for any non-negative quantity W , $P[W \geq a] \leq \frac{E[W]}{a}$.

Then, applying this to $e^{sX} \geq e^{st}$, we get that $P[e^{sX} \geq e^{st}] \leq \frac{E[e^{sX}]}{e^{st}}$. We choose s to make the bound as tight as possible.

3.3 Hoeffding's Lemma

Hoeffding's lemma [6] gives a bound on the moment generating function of a bounded, mean-zero random variable. In our application, X is a single step $D_i = X_i X_{i-1}$. The martingale property gives $E[D_i | F_{i-1}] = 0$, and X is also bounded by $[-1, 1]$ since adding one vertex changes χ by at most one. Hoeffding's lemma gives the upper bound on $E[e^{sX}]$, $E[e^{sX}] \leq e^{(\frac{s^2}{2})}$.

From the previous section we have $P[X \geq t] \frac{E[e^{sX}]}{e^{st}}$. By Hoeffding's lemma, we have the explicit value to plug in for $E[e^{sX}]$, so it is possible to get an actual number bounding $P[X \geq t]$. Once we can bound the moment generating function of each individual step, we can bound the total drift of the martingale over all n steps.

3.4 Azuma-Hoeffding Inequality

The Azuma-Hoeffding inequality [5] extends Hoeffding's lemma from a single bounded variable to a martingale, bounding how far the final value can drift from its starting value. **Theorem 3.** Let $(X_0, X_1, \dots, X_n)$ be a martingale with $|X_i - X_{i-1}| \leq 1$. Then for all $t > 0$:

$$\mathbb{P}[|X_n - X_0| \geq t] \leq 2e^{-\frac{t^2}{2n}}.$$

Here $X_n = \chi(G)$ and $X_0 = \mathbb{E}[\chi(G)]$, so $|X_n - X_0|$ is exactly how far $\chi(G)$ is from its expected value.

Proof of the Azuma-Hoeffding Inequality:

Write $X_n - X_0 = D_1 + D_2 + \dots + D_n$ where $D_i = X_i - X_{i-1}$ are the martingale steps. First, apply the moment-generating function trick: $\mathbb{P}[X_n - X_0 \geq t] \leq e^{-st} \cdot \mathbb{E}[e^{s(X_n - X_0)}]$. Then, Use $e^{a+b} = e^a \cdot e^b$ to factor:

$$\mathbb{E}[e^{s(D_1 + \dots + D_n)}] = \mathbb{E}[e^{sD_1} \dots e^{sD_n}].$$

Applying Hoeffding's lemma to each D_i separately (each has $\mathbb{E}[D_i | F_{i-1}] = 0$ and $D_i \in [-1, 1]$):

$$\mathbb{E}[e^{sD_i}] \leq e^{s^2/2}.$$

Multiplying over all n steps: $\mathbb{E}[e^{s(X_n - X_0)}] \leq e^{ns^2/2}$. Plug back in: $\mathbb{P}[X_n - X_0 \geq t] \leq e^{-st} \cdot e^{ns^2/2}$. Setting $s = t/n$ minimizes the right-hand side, giving $e^{-t^2/(2n)}$. The two-sided bound $\mathbb{P}[|X_n - X_0| \geq t] \leq 2e^{-t^2/(2n)}$ gives the factor of 2. $\square$

3.5 McDiarmid's Inequality

McDiarmid's inequality [7] restates the same idea directly in terms of a function of independent inputs, which is often more convenient to apply than working with the martingale sequence itself.

Theorem 4. Let $\chi(G(X_1, \dots, X_n))$ satisfy: changing any single input X_i to any other value in $(X_1, \dots, X_n)$ changes χ by at most 1 since adding one vertex can change χ by at most 1. Then for any $t > 0$:

$$\mathbb{P}[\chi(G(X_1, \dots, X_n)) - \mathbb{E}[\chi] \geq t] \leq e^{-\frac{2t^2}{n}}.$$

Here $X_1, \dots, X_n$ are the coin flip outcomes that build $G(n, p)$: each X_i saves the edges revealed when vertex v_i is added, so $\chi(G(X_1, \dots, X_n))$ just means the chromatic number of the graph determined by those coin flips.

How McDiarmid's follows from Azuma

Define the Doob martingale $M_k = \mathbb{E}[\chi(G) \mid X_1, \dots, X_k]$. The bounded differences property ensures $|M_k - M_{k-1}| \leq 1$. Apply Azuma. $\square$

4 Edge-Exposure and Vertex-Exposure Martingales

4.1 Edge-Exposure Martingale

There are two ways to reveal $G(n, p)$ step by step: edge by edge, or vertex by vertex.

Fix an arbitrary ordering of all $\binom{n}{2} = \frac{n(n-1)}{2}$ potential edges. Reveal them one at a time, and let F_i = the information from the first i edges. Let $X_i = \mathbb{E}[\chi(G) \mid F_i]$.

Revealing one more edge changes χ by at most 1, since adding an edge can require at most one new color, and removing an edge can free at most one color. So $|X_i - X_{i-1}| \leq 1$ for all i .

Applying Azuma with $\frac{n(n-1)}{2}$ steps:

$$\mathbb{P}[|\chi(G) - \mathbb{E}[\chi(G)]| \geq t] \leq 2e^{-\frac{t^2}{n(n-1)}}.$$

Since $\chi(G) \leq n$ always, this bound only guarantees concentration within $O(n)$ of the mean, which we already knew. So we must use vertex exposure instead.

Approach	Number of steps	Window from Azuma
Edge exposure	$\frac{n(n-1)}{2}$	$O(n)$
Vertex exposure	n	$O(\sqrt{n})$

Table 2: The improvement comes from reducing the number of steps from $O(\frac{n(n-1)}{2})$ to n .

4.2 Vertex-Exposure Martingale

Reveal the graph vertex by vertex. At step i , we add vertex v_i and reveal all $i - 1$ edges between v_i and the previous vertices $v_1, \dots, v_{i-1}$. Let $G_i = G[\{v_1, \dots, v_i\}]$ (the induced subgraph on the first i vertices). Define:

$$Y_i = \mathbb{E}[\chi(G) \mid G_i].$$

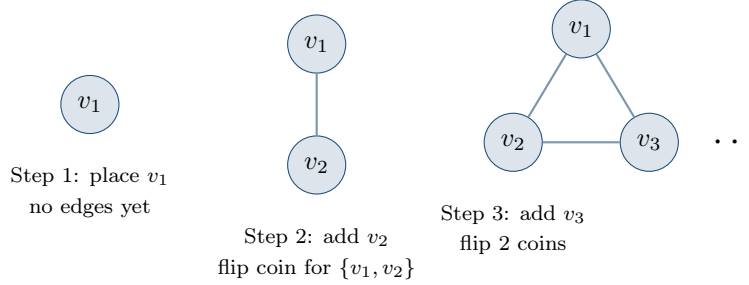


Figure 5: Vertex exposure: the graph is revealed one vertex at a time, with all its edges to prior vertices.

Theorem 5. Bounded Differences Lemma for Vertex Exposure: $|Y_i - Y_{i-1}| \leq 1$ for all $i = 1, \dots, n$.

Theorem 6. For *any* graph H and any vertex v , $|\chi(H) - \chi(H - v)| \leq 1$, where $H - v$ denotes H with v isolated.

Proof

1. $\chi(H) \leq \chi(H - v) + 1$: Take an optimal coloring of $H - v$ using $\chi(H - v)$ colors. When v is restored, it may conflict with all $\chi(H - v)$ colors among its neighbors, requiring at most one additional color.
2. $\chi(H - v) \leq \chi(H)$: Any proper coloring of H restricts to a proper coloring of $H - v$ by ignoring v .

The step from Y_{i-1} to Y_i reveals the edges from v_i to $\{v_1, \dots, v_{i-1}\}$. For any two outcomes of these edges, the resulting full graphs differ only in edges adjacent to v_i . By the bound above, their chromatic numbers differ by at most 1, so the conditional expectations Y_i differ by at most 1. $\square$

5 Chromatic Number Concentration

5.1 Main Theorem (Shamir-Spencer)

Theorem 7. Shamir-Spencer [3] For fixed $p \in (0, 1)$ and any $t > 0$:

$$\mathbb{P}[|\chi(G(n, p)) - \mathbb{E}[\chi(G(n, p))]| \geq t\sqrt{n}] \leq 2e^{-t^2/2}.$$

Proof:

1. Let $(Y_i)_{i=0}^n$ be the vertex-exposure Doob martingale, so $Y_0 = \mathbb{E}[\chi(G)]$ and $Y_n = \chi(G)$.
2. By the Bounded Differences Lemma: $|Y_i - Y_{i-1}| \leq 1$ with $c_i = 1$ for all i .
3. Apply Azuma–Hoeffding with $\sum c_i^2 = n$:

$$\mathbb{P}[|Y_n - Y_0| \geq t] \leq 2e^{-t^2/(2n)}.$$

4. Substitute $t \mapsto t\sqrt{n}$: $\mathbb{P}[|\chi(G) - \mathbb{E}[\chi(G)]| \geq t\sqrt{n}] \leq 2e^{-t^2n/(2n)} = 2e^{-t^2/2}$. □

5.2 Alon’s Improvement

The Shamir–Spencer bound of $O(\sqrt{n})$ is not the tightest possible. In the 1990s, Alon improved it to $O(\sqrt{n}/\log n)$ for fixed $p \in (0, 1)$ [4]

Theorem 8. Alon’s improvement: For fixed $p \in (0, 1)$, $\chi(G(n, p))$ is concentrated in an interval of width $O(\sqrt{n}/\log n)$.

The Shamir–Spencer proof uses only the fact that adding one vertex changes χ by at most 1. This is true, but for a *typical* $G(n, p)$, adding a new vertex almost never requires a new color. The new vertex can almost always be accommodated by an existing color class, because a random graph has large independent sets available. Only a small fraction of vertices, roughly $O(\sqrt{n}/\log n)$ of them, actually force χ to increase.

Alon’s argument adapts a coloring argument of Bollobás [2]. Using the structure of colorings of $G(n, p)$, one can show that the number of vertices that force a new color is $O(n/\log n)$, and plugging into Azuma gives a bound of $O(\sqrt{n/\log n}) = O(\sqrt{n}/\sqrt{\log n})$. Later, the bound was decreased to $O(\sqrt{n}/\log n)$.

However, determining when a new color is needed requires knowing the structure of colorings of $G(n, p)$, which goes well beyond the Lipschitz bound used in Shamir–Spencer. The martingale approach treats every vertex equally and therefore doesn’t see this structure.

Further, it is conjectured that $\chi(G(n, p))$ concentrates in a window of constant width $O(1)$ for fixed p , but this remains open; see [10] and [11] for further progress toward this conjecture.

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