Fair Division: Cake Cutting, Sperner's Lemma, and Rental Harmony

An expository presentation

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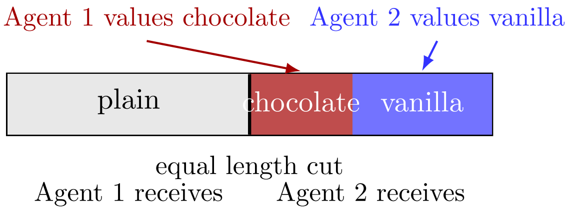
The central question

Fair division problem

How can we divide one resource among several agents so that no one feels cheated?



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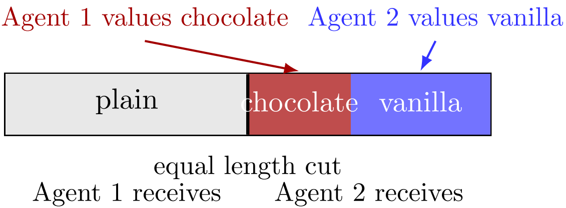

Key message

Equal physical size need not mean fair value.

Roadmap of the talk

1. Model fairness using valuations.
2. Compare proportionality and envy-freeness.
3. See constructive procedures: divide and choose, last diminisher, Selfridge-Conway.
4. Use Sperner's lemma to prove envy-free divisions exist.
5. Apply the same idea to rental harmony.

fairness → algorithms → simplexes → existence

Agents, shares, and valuations

Basic setup

- Agents: $N = \{1, 2, \ldots, n\}$.
- Resource: X.
- Allocation: $A = (A_1, A_2, \ldots, A_n)$, where agent i receives A_i.
- Valuation: $v_i(S)$ is the value agent i assigns to share S.

The same piece S can have different values:

$$v_i(S) \neq v_j(S).$$

Two fairness notions

Proportionality

If $v_i(X) = 1$, then agent i should get at least

$$v_i(A_i) \geq \frac{1}{n}.$$

Envy-freeness

Agent i should not prefer anyone else's share:

$$v_i(A_i) \geq v_i(A_j) \quad \text{for all } i, j.$$

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Important relation

For complete pairwise-disjoint allocations with additive normalized valuations,

$$\text{envy-free} \quad \Longrightarrow \quad \text{proportional}.$$

The converse is false.

Why proportionality is not enough

	P_1	P_2	P_3
Agent 1	2/5	1/2	1/10
Agent 2	3/10	2/5	3/10
Agent 3	3/10	3/10	2/5

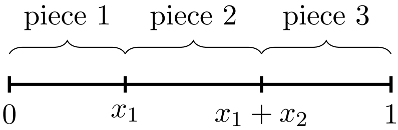
Assign P_i to agent i. Each agent gets at least 1/3, so the allocation is proportional.

But agent 1 envies agent 2

$$v_1(P_2) = \frac{1}{2} > \frac{2}{5} = v_1(P_1).$$

The cake-cutting model

- Model the cake as $C = [0, 1]$.
- Two cuts at x_1 and $x_1 + x_2$ create three consecutive pieces.
- Valuations are nonnegative, additive, normalized, and nonatomic.



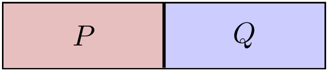
Nonatomicity means we can cut a piece to any desired value between 0 and its total value.

Divide and choose

1. Agent 1 cuts the cake into two pieces they value equally.
2. Agent 2 chooses their preferred piece.
3. Agent 1 gets the remaining piece.

Guarantee

The result is envy-free for two agents.



Agent 1: $v_1(P) = v_1(Q)$
Agent 2 chooses one piece

Last diminisher: proportionality for many agents

Idea

One proposed piece is repeatedly trimmed until no remaining agent thinks it is too large.

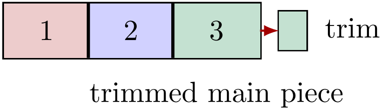
1. Agent 1 cuts a piece worth exactly $1/n$ to them.
2. Each later agent trims it only if they think it is worth more than $1/n$.
3. Each removed trimming is returned to the remainder.
4. The last person to trim receives the final piece; if nobody trims, Agent 1 receives it.
5. The other $n-1$ agents repeat on the remainder.

What it proves

Last diminisher guarantees proportionality, not necessarily envy-freeness.

Selfridge-Conway: why trimming appears

1. Agent A cuts three equal-value pieces.
2. Agent B trims a unique favorite until it ties with their second favorite.
3. Agent C chooses a main piece first.
4. If C did not choose the trimmed piece, then B must take it; otherwise B chooses next.
5. Agent A receives the remaining main piece.



Purpose of trimming

It removes the special advantage of the one piece that could create serious envy for B.

Selfridge-Conway: dividing the trimmings

If a trimming is created, let T be the agent among B, C who gets the trimmed main piece. Let D be the other one.

Trimming stage

1. Agent D divides the trimmings into three equal-value parts.
2. The choosing order is T, then A, then D.

main-piece allocation $+$ trimmings allocation $\longrightarrow$ envy-free bundles

From procedures to geometry

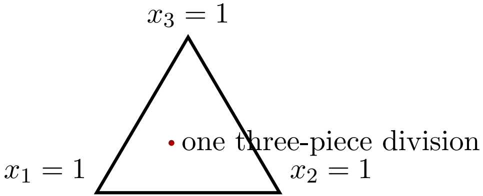
Instead of giving one explicit sequence of cuts, study all possible cut-sets at once.

A division into n consecutive pieces is a vector

$$(x_1, \ldots, x_n), \qquad x_i \geq 0, \qquad x_1 + \cdots + x_n = 1.$$

So all divisions live in the simplex

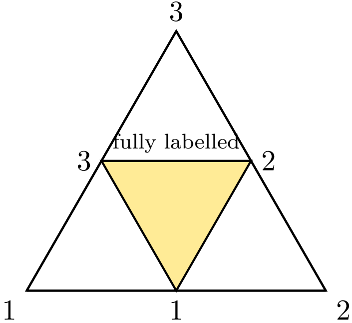
$$\Delta^{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i, \ \sum_{i=1}^{n} x_i = 1 \right\}.$$



Sperner's lemma

Sperner labelling

Facet j means the face opposite vertex j. A vertex on facet j is not allowed to receive label j.



Conclusion

Some elementary simplex is fully labelled.

Turning preferences into Sperner labels

1. Triangulate the cut-set simplex very finely.
2. Give every small simplex one vertex owned by each agent.
3. At each vertex z, ask its owner which piece they prefer.
4. Label z by the preferred piece number.

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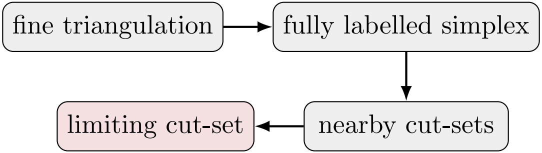
Why the boundary condition holds

If z lies on the facet $x_j = 0$, then piece j is empty. Since some other piece has positive length, hunger rules out choosing the empty piece j. So label j is forbidden there.

Existence of envy-free cake divisions



Proof idea

Fully labelled simplexes give nearby cut-sets with distinct agent-piece assignments. After passing to a subsequence with one fixed assignment, compactness gives a limiting cut-set. Closed preference sets preserve each agent's preference for their assigned piece.

Thus, each agent receives a distinct piece that they weakly prefer to every other piece.

Rental harmony

Roommate version

Set room prices so that each housemate receives one room and no housemate prefers another room at its assigned price.

Normalize the total rent to 1.

Rooms

$\boxed{R_1}$ $\boxed{R_2}$ $\boxed{R_3}$

Prices

$$p = (p_1, \ldots, p_n), \qquad \sum_{i=1}^{n} p_i = 1.$$

$$p \in \Delta^{n-1}.$$

Why rent division needs a twist

Cake cutting

Facet $x_j = 0$: piece j is empty.

Hunger says label j is impossible.

Rental harmony

Facet $p_j = 0$: room j is free.

If room j is the only free room, miserly tenants must choose room j.

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Dualization

This is the reverse of the cake-cutting boundary condition. Su's proof passes to a dual complex, where it becomes the Sperner condition again.

Existence is not the same as computation

$$\text{existence} \neq \text{constructibility} \neq \text{efficient computation.}$$

Robertson-Webb query model

Algorithms ask agents questions such as:

$$\mathrm{Eval}_i(a, b) = v_i([a, b])$$

$$\mathrm{Cut}_i(a, \alpha) = b \quad \text{where } v_i([a, b]) = \alpha.$$

Bounded envy-free protocols exist for all n, but known general bounds are far from practical.

Main takeaways

1. Fairness depends on subjective valuations, not physical size.
2. Envy-freeness is stronger than proportionality.
3. Simple procedures work beautifully for small cases.
4. Under hunger and closedness assumptions, Sperner's lemma gives a global existence proof from local preferences.
5. Rental harmony shows the same combinatorial idea beyond cake cutting.

Final message

A familiar question about sharing leads naturally to algorithms, combinatorics, geometry, and topology.

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