Why Does Gaussian Quadrature Achieve Exactness up to Degree $2n - 1$?

Orthogonality, Legendre Polynomials, and Optimal Numerical Integration

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Motivation: Numerical Integration

Many integrals are difficult or impossible to solve exactly.

$$I(f) = \int_a^b f(x)\,dx$$

Numerical integration approximates this value using finitely many function evaluations.

Research Question

Why does an n-point Gaussian Quadrature rule integrate every polynomial up to degree $2n - 1$ exactly?

Classical Numerical Integration

Method	Approximation Type	Exact Degree
Trapezoidal Rule	Linear	1
Simpson's Rule	Quadratic	3
Gaussian Quadrature	Optimized nodes	$2n - 1$

Main Idea

Classical methods usually use equally spaced points. Gaussian Quadrature chooses better points.

Gaussian Quadrature

Gaussian Quadrature approximates

$$\int_{-1}^{1} f(x)\,dx$$

using

$$Q_n(f) = \sum_{i=1}^{n} w_i f(x_i).$$

- x_i: nodes, or sample points
- w_i: weights

Key Difference

The nodes are not equally spaced. They are chosen optimally.

The Surprising Result

An n-point Gaussian Quadrature rule is exact for every polynomial of degree at most

$$2n - 1.$$

Number of points n	Highest exact degree
1	1
2	3
3	5
4	7
5	9

Inner Product and Orthogonality

For functions on $[-1, 1]$, define

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Two functions are orthogonal if

$$\langle f, g \rangle = 0.$$

Why this matters

Orthogonality allows certain polynomial terms to cancel completely.

Legendre Polynomials

The first few Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

They satisfy

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad (m \neq n).$$

Important

Gaussian Quadrature uses the roots of $P_n(x)$ as its nodes.

Main Proof Idea

Take any polynomial $p(x)$ with

$$\deg p \leq 2n - 1.$$

Using polynomial division,

$$p(x) = P_n(x)q(x) + r(x),$$

where

$$\deg q \leq n - 1, \qquad \deg r \leq n - 1.$$

Key Idea

The term $P_n(x)q(x)$ disappears twice.

First Cancellation: Orthogonality

Integrate the decomposition:

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} P_n(x)q(x)\,dx + \int_{-1}^{1} r(x)\,dx.$$

Because P_n is orthogonal to every polynomial of degree less than n,

$$\int_{-1}^{1} P_n(x)q(x)\,dx = 0.$$

Therefore,

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} r(x)\,dx.$$

Second Cancellation: Roots

Gaussian nodes are roots of $P_n(x)$, so

$$P_n(x_i) = 0.$$

Substitute x_i into

$$p(x) = P_n(x)q(x) + r(x).$$

Then

$$p(x_i) = P_n(x_i)q(x_i) + r(x_i) = r(x_i).$$

Meaning

The quadrature rule only sees the lower-degree remainder $r(x)$.

Why the Rule Is Exact

From the previous slides,

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} r(x)\,dx$$

and

$$Q_n(p) = Q_n(r).$$

Since $r(x)$ has degree at most $n-1$, the rule integrates r exactly.

$$Q_n(r) = \int_{-1}^{1} r(x)\,dx.$$

Therefore,

$$Q_n(p) = \int_{-1}^{1} p(x)\,dx.$$

Why Exactly $2n - 1$?

An n-point Gaussian rule has

$$n \text{ nodes} + n \text{ weights} = 2n$$

adjustable values.
These can satisfy exactness conditions for

$$1, x, x^2, \ldots, x^{2n-1}.$$

That is $2n$ polynomial conditions.

Conclusion

The highest possible exact degree is $2n - 1$.

Why Not Degree $2n$?

Let

$$\omega_n(x) = \prod_{i=1}^{n} (x - x_i).$$

Now define

$$g(x) = \omega_n(x)^2.$$

Then $g(x)$ has degree $2n$, and

$$g(x_i) = 0$$

for every node.
Therefore,

$$Q_n(g) = \sum_{i=1}^{n} w_i g(x_i) = 0.$$

Why Degree $2n$ Fails

Even though

$$Q_n(g) = 0,$$

we have

$$g(x) = \omega_n(x)^2 \geq 0, \qquad g(x) \not\equiv 0.$$

So

$$\int_a^b g(x)\,dx > 0.$$

Conclusion

No n-point quadrature rule can integrate every polynomial of degree $2n$ exactly.

Conclusion

Gaussian Quadrature achieves degree $2n - 1$ exactness because:

1. Legendre polynomials are orthogonal.
2. Their roots are used as quadrature nodes.
3. The error-producing term disappears twice.

Final Answer

Gaussian Quadrature is optimal because it reaches the maximum possible degree of exactness for any rule using n nodes.

Thank You

Questions?