

Why Does Gaussian Quadrature Achieve Exactness up to Degree $2n - 1$?

Orthogonality, Legendre Polynomials, and Optimal Numerical Integration

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Abstract

This expository paper explains why the n -point Gauss–Legendre quadrature rule integrates every polynomial of degree at most $2n - 1$ exactly. We begin with the theory of polynomial spaces and numerical quadrature before introducing orthogonal polynomials and the Legendre polynomial family. We show that the roots of the Legendre polynomial naturally determine the optimal quadrature nodes and that orthogonality is the key property responsible for the exceptional degree of exactness. Using polynomial decomposition, we prove that Gauss–Legendre quadrature integrates every polynomial of degree at most $2n - 1$ exactly and establish that no quadrature rule using only n nodes can integrate every polynomial of degree $2n$. Throughout the paper, we emphasize the relationship between approximation theory, linear algebra, and numerical analysis while illustrating why Gaussian quadrature remains one of the most powerful techniques for numerical integration.

1 Introduction

Numerical integration is one of the fundamental problems in numerical analysis. Given a function defined on a closed interval, the objective is to approximate its definite integral using only a finite number of function evaluations. Such approximations arise naturally in scientific computing, engineering, computational physics, machine learning, and many other areas where exact antiderivatives are unavailable or computationally expensive to evaluate.

Classical quadrature methods, including the Trapezoidal Rule and Simpson’s Rule, approximate an integral by evaluating the function at equally spaced nodes. Although these methods are simple and effective for many applications, their degree of exactness is limited by the choice of equally spaced sampling points. Consequently, one may ask whether a different choice of nodes can produce a more accurate quadrature rule while using the same number of function evaluations.

Gaussian quadrature provides an affirmative answer to this question. Instead of fixing equally spaced nodes, the method selects both the nodes and the corresponding weights to maximize the degree of exactness. In the case of the Gauss–Legendre quadrature rule, the nodes are chosen to be the roots of the Legendre polynomial $P_n(x)$. This choice yields an n -point quadrature rule that integrates every polynomial of degree at most $2n - 1$ exactly, which is the highest degree theoretically attainable for any quadrature rule using only n nodes.

The purpose of this paper is to explain why this remarkable property holds. We first review the mathematical foundations of polynomial spaces, quadrature rules, and orthogonal polynomials. We then develop the theory of Legendre polynomials and demonstrate how their orthogonality determines the optimal quadrature nodes. Using polynomial decomposition and orthogonality, we prove the exactness theorem for Gauss–Legendre quadrature and establish its optimality.

among all n -point quadrature rules. Finally, we discuss the relationship between Gaussian quadrature and approximation theory, compare it with classical Newton–Cotes formulas, and present several applications and possible extensions.

The paper is organized as follows. Section 2 introduces the mathematical preliminaries required throughout the paper, including polynomial spaces, quadrature rules, orthogonal polynomials, and Legendre polynomials. Section 3 develops the construction of the Gauss–Legendre quadrature rule. In Section 4, we prove the exactness and optimality theorems. Section 5 discusses the relationship between Gaussian quadrature and numerical analysis, including the classical error formula and comparisons with Newton–Cotes methods. Finally, Section 6 presents future directions and concluding remarks.

2 Preliminaries

This section introduces the mathematical concepts used throughout the paper. We begin with polynomial spaces and quadrature rules before reviewing orthogonal polynomials and the Legendre polynomial family. These ideas provide the theoretical foundation for the construction of the Gauss–Legendre quadrature rule in the following section.

2.1 Polynomial Spaces

Throughout this paper, we consider real-valued polynomials defined on the interval $[-1, 1]$.

[Polynomial Space] Let $n \in \mathbb{N} \cup \{0\}$. The polynomial space of degree at most n is

$$\mathbb{P}_n = \{p(x) : \deg(p) \leq n\}.$$

The space $\mathbb{P}_n$ is a vector space of dimension $n + 1$ with basis

$$\{1, x, x^2, \dots, x^n\}.$$

Consequently, every polynomial $p \in \mathbb{P}_n$ has a unique representation

$$p(x) = a_0 + a_1x + \dots + a_nx^n,$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$. Since Gaussian quadrature is characterized by its ability to integrate polynomials exactly, the analysis of polynomial spaces forms the starting point of our discussion.

2.2 Quadrature Rules

The objective of numerical integration is to approximate a definite integral by evaluating the integrand at finitely many points.

[Quadrature Rule] An n -point quadrature rule is an approximation of the form

$$Q_n(f) = \sum_{i=1}^n w_i f(x_i),$$

where

- $x_1, \dots, x_n$ are the quadrature nodes, and
- $w_1, \dots, w_n$ are the corresponding quadrature weights.

The approximation is intended to estimate

$$I(f) = \int_{-1}^1 f(x) dx.$$

Different quadrature rules arise from different choices of nodes and weights.

[Trapezoidal Rule] The Trapezoidal Rule approximates

$$\int_{-1}^1 f(x) dx$$

using the endpoints of the interval,

$$Q_T(f) = f(-1) + f(1).$$

Although simple, the rule is exact only for polynomials of degree at most one.

[Simpson's Rule] Simpson's Rule employs three equally spaced nodes

$$-1, 0, 1,$$

and approximates the integral by

$$Q_S(f) = \frac{1}{3} [f(-1) + 4f(0) + f(1)].$$

This rule is exact for all cubic polynomials.

These examples illustrate that the choice of nodes strongly influences the accuracy of a quadrature rule.

2.3 Degree of Exactness

One of the most important measures of the accuracy of a quadrature rule is its degree of exactness.

[Degree of Exactness] A quadrature rule has degree of exactness d if

$$Q_n(p) = I(p)$$

for every polynomial

$$p \in \mathbb{P}_d,$$

while there exists at least one polynomial of degree $d + 1$ for which

$$Q_n(p) \neq I(p).$$

A quadrature rule with a larger degree of exactness generally produces more accurate approximations for sufficiently smooth functions.

The central problem considered in this paper is determining how an n -point quadrature rule can achieve the highest possible degree of exactness.

2.4 Orthogonal Polynomials

Orthogonal polynomials provide the mathematical framework underlying Gaussian quadrature.

[Inner Product] For continuous functions on $[-1, 1]$, define

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

Two functions are said to be orthogonal whenever

$$\langle f, g \rangle = 0.$$

Orthogonality allows polynomial spaces to be decomposed into mutually independent components, making many approximation problems significantly simpler.

[Orthogonal Polynomial Sequence] A sequence of polynomials

$$\{P_n\}_{n=0}^{\infty}$$

is called orthogonal if

$$\int_{-1}^1 P_m(x)P_n(x) dx = 0, \quad m \neq n.$$

Among the many orthogonal polynomial families, the Legendre polynomials are the most important for Gaussian quadrature because their roots determine the optimal quadrature nodes.

2.5 Legendre Polynomials

The Legendre polynomials form an orthogonal basis for polynomial approximation on the interval $[-1, 1]$.

[Legendre Polynomial] The Legendre polynomial of degree n is defined by Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

The first four Legendre polynomials are

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

The polynomial $P_n(x)$ has degree exactly n .

Proof. Since $(x^2 - 1)^n$ has degree $2n$, differentiating it n times leaves a polynomial of degree n . The leading coefficient remains nonzero, so

$$\deg(P_n) = n.$$

□

[Orthogonality of Legendre Polynomials] For integers $m \neq n$,

$$\int_{-1}^1 P_m(x)P_n(x) dx = 0.$$

Proof. The result follows from Rodrigues' formula together with repeated integration by parts. Since $(x^2 - 1)^n$ vanishes at both endpoints with multiplicity n , every boundary term is zero. After repeatedly transferring derivatives to the lower-degree polynomial, the remaining integral vanishes identically. $\square$

If $q(x) \in \mathbb{P}_{n-1}$, then

$$\int_{-1}^1 P_n(x)q(x) dx = 0.$$

This property will play a fundamental role in the proof of the Gaussian quadrature exactness theorem.

2.6 Transition to Gaussian Quadrature

The Legendre polynomials possess two remarkable properties. First, they form an orthogonal basis for polynomial approximation on $[-1, 1]$. Second, each polynomial $P_n(x)$ has exactly n distinct roots in the interval $(-1, 1)$. These roots will be used as the quadrature nodes of the Gauss–Legendre rule.

In the next section, we show how these nodes, together with appropriately chosen weights, produce an n -point quadrature rule that achieves the maximum possible degree of exactness. The orthogonality established in this chapter will provide the key ingredient in the proof of the main theorem.

3 Gaussian Quadrature

3.1 Construction of the Gauss–Legendre Rule

Having introduced polynomial spaces and Legendre polynomials, we now construct the Gauss–Legendre quadrature rule. Unlike classical Newton–Cotes formulas, which use equally spaced nodes, Gaussian quadrature selects both the nodes and the weights to maximize the degree of exactness.

[Gauss–Legendre Quadrature] The n -point Gauss–Legendre quadrature rule approximates

$$\int_{-1}^1 f(x) dx$$

by

$$Q_n(f) = \sum_{i=1}^n w_i f(x_i),$$

where $x_1, \dots, x_n$ are the roots of the Legendre polynomial $P_n(x)$, and $w_1, \dots, w_n$ are chosen so that the rule is exact for polynomials of the highest possible degree.

3.2 Quadrature Nodes

The location of the quadrature nodes determines the accuracy of the approximation. The Gauss–Legendre rule chooses these nodes as the zeros of the Legendre polynomial.

The Legendre polynomial $P_n(x)$ has exactly n distinct real roots in the interval

$$(-1, 1).$$

Proof. This is a classical property of Legendre polynomials and follows from Sturm–Liouville theory. Consequently, these roots provide n distinct quadrature nodes on the interval $[-1, 1]$. $\square$

3.3 Quadrature Weights

Once the quadrature nodes have been selected, the corresponding weights are uniquely determined.

For every set of distinct nodes

$$x_1, x_2, \dots, x_n,$$

there exists a unique set of weights

$$w_1, w_2, \dots, w_n$$

such that the quadrature rule is exact for every polynomial in

$$\mathbb{P}_{n-1}.$$

Proof. The result follows by integrating the Lagrange interpolation polynomial associated with the chosen nodes. Since interpolation is unique, the corresponding weights are uniquely determined. $\square$

For the Gauss–Legendre rule, the weights may be expressed as

$$w_i = \frac{2}{(1 - x_i^2) (P'_n(x_i))^2}, \quad i = 1, \dots, n.$$

This formula guarantees positive weights and plays an important role in the numerical stability of the quadrature rule.

3.4 Why Legendre Polynomials?

The defining feature of Gaussian quadrature is the choice of nodes. Rather than using equally spaced points, the Gauss–Legendre rule uses the roots of $P_n(x)$. This choice ensures that every polynomial of degree less than n is orthogonal to $P_n(x)$, allowing higher-degree polynomial terms to vanish when integrated.

The orthogonality established in Section 2 is therefore not merely a theoretical property; it is precisely the reason why Gaussian quadrature achieves a significantly higher degree of exactness than classical quadrature rules.

3.5 Polynomial Decomposition

The following decomposition is the key step toward the main theorem.

Let

$$p(x) \in \mathbb{P}_{2n-1}.$$

Then there exist unique polynomials

$$q(x), r(x) \in \mathbb{P}_{n-1}$$

such that

$$p(x) = P_n(x)q(x) + r(x).$$

Proof. Since $\deg(P_n) = n$, the polynomial division algorithm gives unique polynomials $q(x)$ and $r(x)$ satisfying

$$p(x) = P_n(x)q(x) + r(x),$$

where

$$\deg(r) < n.$$

Hence,

$$r(x) \in \mathbb{P}_{n-1}.$$

□

The decomposition above separates every polynomial of degree at most $2n - 1$ into two parts. Since the quadrature nodes are precisely the roots of $P_n(x)$, the first term vanishes when evaluated by the quadrature rule, while the second term is integrated exactly because its degree is less than n . Together with the orthogonality of Legendre polynomials, this observation forms the basis of the main exactness theorem proved in the next section.

4 Exactness of Gaussian Quadrature

In this section, we prove the principal result of the paper. The key idea is that every polynomial of degree at most $2n - 1$ can be decomposed into a multiple of the Legendre polynomial $P_n(x)$ together with a polynomial of lower degree. The orthogonality of Legendre polynomials and the choice of quadrature nodes then imply the exactness of the Gauss–Legendre quadrature rule.

4.1 Exactness Theorem

We first establish the main result.

[Exactness of Gauss–Legendre Quadrature] The n -point Gauss–Legendre quadrature rule integrates every polynomial of degree at most $2n - 1$ exactly. That is,

$$Q_n(p) = \int_{-1}^1 p(x) dx,$$

for every polynomial

$$p(x) \in \mathbb{P}_{2n-1}.$$

Proof. Let

$$p(x) \in \mathbb{P}_{2n-1}.$$

By Proposition 3.4, there exist unique polynomials

$$q(x), r(x) \in \mathbb{P}_{n-1}$$

such that

$$p(x) = P_n(x)q(x) + r(x).$$

Since the quadrature nodes are precisely the roots of $P_n(x)$,

$$P_n(x_i) = 0, \quad i = 1, \dots, n.$$

Therefore,

$$Q_n(P_n q) = 0,$$

and hence

$$Q_n(p) = Q_n(r).$$

Because

$$\deg(r) \leq n - 1,$$

the quadrature rule is exact on $\mathbb{P}_{n-1}$. Thus,

$$Q_n(r) = \int_{-1}^1 r(x) dx.$$

On the other hand, by Theorem 2.11, the Legendre polynomial $P_n(x)$ is orthogonal to every polynomial of degree less than n . Since

$$q(x) \in \mathbb{P}_{n-1},$$

it follows that

$$\int_{-1}^1 P_n(x)q(x) dx = 0.$$

Hence,

$$\int_{-1}^1 p(x) dx = \int_{-1}^1 r(x) dx.$$

Combining these equalities gives

$$Q_n(p) = \int_{-1}^1 p(x) dx.$$

Therefore, the Gauss–Legendre quadrature rule is exact for every polynomial of degree at most $2n - 1$. $\square$

4.2 Optimality

The previous theorem shows that Gaussian quadrature achieves a remarkably high degree of exactness. We now prove that this result is optimal.

[Optimality] No quadrature rule using only n nodes can integrate every polynomial of degree $2n$ exactly.

Proof. Suppose an n -point quadrature rule were exact for every polynomial in

$$\mathbb{P}_{2n}.$$

Let

$$x_1, \dots, x_n$$

be the quadrature nodes and define

$$g(x) = \prod_{i=1}^n (x - x_i).$$

Then

$$g(x)$$

has degree n , and therefore

$$g(x)^2$$

has degree $2n$.

Since

$$g(x_i) = 0,$$

we have

$$Q_n(g^2) = 0.$$

However,

$$g(x)^2 \geq 0$$

for all

$$x \in [-1, 1],$$

and $g(x)^2$ is not identically zero. Consequently,

$$\int_{-1}^1 g(x)^2 dx > 0.$$

Thus,

$$Q_n(g^2) \neq \int_{-1}^1 g(x)^2 dx,$$

which contradicts the assumption that the quadrature rule is exact for every polynomial of degree $2n$.

Therefore, no n -point quadrature rule can have degree of exactness greater than $2n - 1$. $\square$

4.3 Discussion

Theorems 4.1 and 4.2 together characterize the Gauss–Legendre quadrature rule as an optimal numerical integration method. The choice of the roots of the Legendre polynomial as quadrature nodes is not arbitrary; rather, it is precisely this choice that allows the rule to attain the maximum possible degree of exactness for a fixed number of function evaluations. This optimality distinguishes Gaussian quadrature from classical Newton–Cotes formulas and explains its widespread use in scientific computing, engineering, and applied mathematics.

The exactness theorem demonstrates that the interaction between polynomial decomposition and orthogonality is the fundamental reason for the remarkable performance of Gaussian quadrature. The optimality theorem further shows that no alternative choice of n nodes can produce a higher degree of exactness. Together, these results establish the Gauss–Legendre quadrature rule as one of the most efficient and mathematically elegant methods for numerical

integration.

The next section examines the practical consequences of these theoretical results. In particular, we derive the classical error formula, compare Gaussian quadrature with Newton–Cotes methods, and discuss several applications in modern numerical analysis.

5 Connections to Numerical Analysis

The previous chapter established that the Gauss–Legendre quadrature rule is both exact for polynomials of degree at most $2n - 1$ and optimal among all n -point quadrature rules. In this section, we discuss the implications of these results by examining the error of Gaussian quadrature, comparing it with classical Newton–Cotes formulas, and highlighting its relationship to polynomial approximation and modern scientific computing.

5.1 Error of Gaussian Quadrature

Although the Gauss–Legendre quadrature rule is exact for all polynomials of degree at most $2n - 1$, most functions encountered in practice are not polynomials. Consequently, it is important to understand the approximation error for more general smooth functions.

Let

$$f \in C^{2n}([-1, 1]).$$

Then there exists some

$$\xi \in (-1, 1)$$

such that

$$\int_{-1}^1 f(x) dx - Q_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi).$$

The error depends on the $2n$ -th derivative of the integrand. Consequently, the smoother the function, the smaller the approximation error is expected to be. This explains why Gaussian quadrature performs exceptionally well for analytic and continuously differentiable functions.

5.2 Comparison with Newton–Cotes Rules

The principal difference between Gaussian quadrature and the classical Newton–Cotes formulas lies in the choice of quadrature nodes. Newton–Cotes formulas employ equally spaced nodes, whereas Gaussian quadrature selects the roots of the Legendre polynomial. This seemingly small modification leads to a substantial increase in the degree of exactness.

Method	Number of Nodes	Degree of Exactness
Trapezoidal Rule	2	1
Simpson’s Rule	3	3
Gauss–Legendre (2-point)	2	3
Gauss–Legendre (3-point)	3	5

Table 1: Comparison of the degree of exactness of several quadrature rules.

Table shows that Gaussian quadrature achieves a higher degree of exactness while using the same number of function evaluations. This increased efficiency is one of the primary reasons for its widespread use in numerical computation.

5.3 Polynomial Approximation

Gaussian quadrature is closely connected to polynomial approximation. Since sufficiently smooth functions can be approximated locally by polynomials, numerical integration can often be reduced to integrating these polynomial approximations.

The orthogonality of the Legendre polynomials suppresses the contribution of higher-degree polynomial components. As a result, the quadrature rule integrates all lower-degree components exactly while minimizing the influence of higher-order terms. This connection between approximation theory and orthogonal polynomials provides the mathematical foundation for the remarkable accuracy of Gaussian quadrature.

5.4 Applications

Because of its efficiency and accuracy, Gaussian quadrature has become a standard tool in many areas of computational mathematics.

Finite Element Analysis. Finite element methods require repeated numerical evaluation of element matrices and load vectors. Gaussian quadrature provides accurate integration while maintaining computational efficiency.

Computational Physics. Many physical models involve integrals that cannot be evaluated analytically. Gaussian quadrature is widely used in quantum mechanics, statistical mechanics, and numerical solutions of differential equations.

Engineering. Applications in structural mechanics, heat transfer, and fluid dynamics require reliable numerical integration. The high degree of exactness of Gaussian quadrature makes it a preferred method in many engineering software packages.

Scientific Computing. Modern numerical libraries, including MATLAB, SciPy, Mathematica, and Julia, implement Gaussian quadrature as a standard routine for evaluating definite integrals.

5.5 Discussion

The Gauss–Legendre quadrature rule illustrates how several branches of mathematics combine to produce an efficient numerical algorithm. Polynomial approximation provides the framework for representing smooth functions, orthogonal polynomials determine the optimal quadrature nodes, and the exactness theorem guarantees the highest possible degree of accuracy for a fixed number of function evaluations.

These results explain why Gaussian quadrature has remained one of the most important techniques in numerical analysis for over two centuries. Its combination of elegant mathematical theory and practical computational performance continues to make it indispensable in modern scientific computing.

6 Future Directions

Although Gauss–Legendre quadrature is one of the most accurate numerical integration methods, many important extensions have been developed. This section briefly discusses several directions for further study and concludes the paper.

6.1 Generalized Gaussian Quadrature

The Gauss–Legendre rule assumes a constant weight function on the interval $[-1, 1]$. More generally, Gaussian quadrature can be constructed for other orthogonal polynomial families associated with different weight functions.

[General Gaussian Quadrature] Let $w(x)$ be a positive weight function on an interval $[a, b]$. A Gaussian quadrature rule approximates

$$\int_a^b f(x)w(x) dx$$

using nodes chosen as the roots of the orthogonal polynomial associated with the weighted inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x) dx.$$

Different choices of the weight function $w(x)$ give rise to different Gaussian quadrature rules, including Gauss–Chebyshev, Gauss–Laguerre, and Gauss–Hermite quadrature, each designed for a particular class of integrals.

6.2 Adaptive Quadrature

Many practical problems involve functions with rapidly changing behavior, sharp peaks, or singularities. In such cases, applying a fixed quadrature rule over the entire interval may not be the most efficient approach.

[Adaptive Quadrature] Adaptive quadrature subdivides the integration interval into smaller subintervals and applies numerical integration separately on each one. Additional quadrature points are allocated to regions where the estimated approximation error is largest.

Adaptive Gaussian quadrature combines the high accuracy of Gaussian quadrature with local error estimation, making it well suited for evaluating challenging integrals encountered in practical applications.

6.3 Multidimensional Integration

Many scientific and engineering problems require evaluating integrals involving several variables.

[Tensor Product Quadrature] For multivariable functions, Gaussian quadrature can be extended by taking tensor products of one-dimensional quadrature rules.

Although this approach is highly effective for low-dimensional problems, the number of quadrature points increases rapidly as the dimension grows. This phenomenon is commonly known as the *curse of dimensionality* and motivates the development of more efficient high-dimensional integration techniques.

6.4 Applications in Modern Computing

Gaussian quadrature continues to play a central role in computational mathematics and scientific computing. Its combination of efficiency and accuracy makes it an indispensable tool in many disciplines.

Some important applications include:

- Finite element analysis,

- Computational physics,
- Uncertainty quantification,
- Numerical optimization,
- Machine learning,
- Numerical solutions of ordinary and partial differential equations.

As computational models continue to increase in complexity, efficient numerical integration remains a fundamental component of modern scientific computing.

6.5 Concluding Remarks

This paper investigated why the n -point Gauss–Legendre quadrature rule integrates every polynomial of degree at most $2n - 1$ exactly. Beginning with polynomial spaces and orthogonal polynomials, we developed the mathematical framework required to construct Gaussian quadrature and established both its exactness and optimality.

The results presented in this paper illustrate the close relationship between approximation theory, orthogonal polynomials, linear algebra, and numerical analysis. By selecting quadrature nodes as the roots of the Legendre polynomials, Gaussian quadrature achieves the highest possible degree of exactness for a fixed number of function evaluations.

Although Gaussian quadrature has been studied for more than two centuries, it remains one of the most elegant examples of how abstract mathematical theory can produce highly efficient computational algorithms. Its theoretical optimality, numerical stability, and wide range of applications ensure that it continues to be one of the fundamental techniques in modern numerical analysis.

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