The Outer Automorphism of S_6
Completeness of Groups in Abstract Algebra

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A group is a set with an operation that satisfies:

- Closure: $\forall\ a, b \in G, a \cdot b \in G$
- Identity: $\exists\ e \in G$ such that $\forall\ g \in G, e \cdot g = g \cdot e = g$.
- Inverses: $\forall\ g \in G\ \exists\ g^{-1} \in G$ such that $g \cdot g^{-1} = e$.
- Associativity: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Automorphism

An *automorphism* of a group G is a map $\phi : G \to G$ that is bijective and homomorphic, $\phi(xy) = \phi(x)\phi(y)$. We denote the automorphism group of G as $\mathrm{Aut}(G)$

The *symmetric group* S_n is defined as the set of bijections from $[n]$ to $[n]$. This is simply the permutations of the set $[n]$ which means that S_n has the order $n!$.

$$S_3 = \{(1)(2)(3),\ (1\,2),\ (1\,3),\ (2\,3),\ (1\,2\,3),\ (1\,3\,2)\}.$$

$$(1\,2)\cdot\{1,2,3\} = \{\,(1\,2)(1), (1\,2)(2), (1\,2)(3)\,\} = \{2,1,3\} = \{1,2,3\}.$$

An inner automorphism is an automorphism which can be expressed through conjugation of a fixed element of G. An automorphism ϕ is then inner if for some $g \in G$ then

$$\phi_g : G \to G$$

$$\phi_g(h) := g^{-1}hg$$

An outer automorphism is simply an automorphism that is not inner. We denote the outer automorphism group by

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

Lemma
Let $\sigma \in S_n$ and let $\tau \in S_n$. Then $\sigma\tau\sigma^{-1}$ has the same cycle type as τ. In particular, conjugation preserves cycle structure.

Proof. Let $\tau = (a_1\, a_2\, \ldots\, a_k)$ be a cycle of length k in S_n, then $\tau(a_i) = a_{i+1}$. Consider the conjugate

$$\sigma\tau\sigma^{-1}.$$

Then we will compute its action of the elements of the form $\sigma(a_i)$. For $1 \leq i \leq k$, we have

$$\sigma\tau\sigma^{-1}(\sigma(a_i)) = \sigma\tau(a_i) = \sigma(a_{i+1}).$$

Since we have a cycle take the indices modula k, so $a_{k+1} = a_1$. Thus $\sigma\tau\sigma^{-1}$ sends

$$\sigma(a_1) \mapsto \sigma(a_2) \mapsto \cdots \mapsto \sigma(a_k) \mapsto \sigma(a_1),$$

so

$$\sigma\tau\sigma^{-1} = (\sigma(a_1)\,\sigma(a_2)\,\ldots\,\sigma(a_k)),$$

which is again a cycle of length k ■

This is an essential part of proving that S_6 has an outer automorphism as we will see that we can construct an automorphism that sends one cycle type to another.

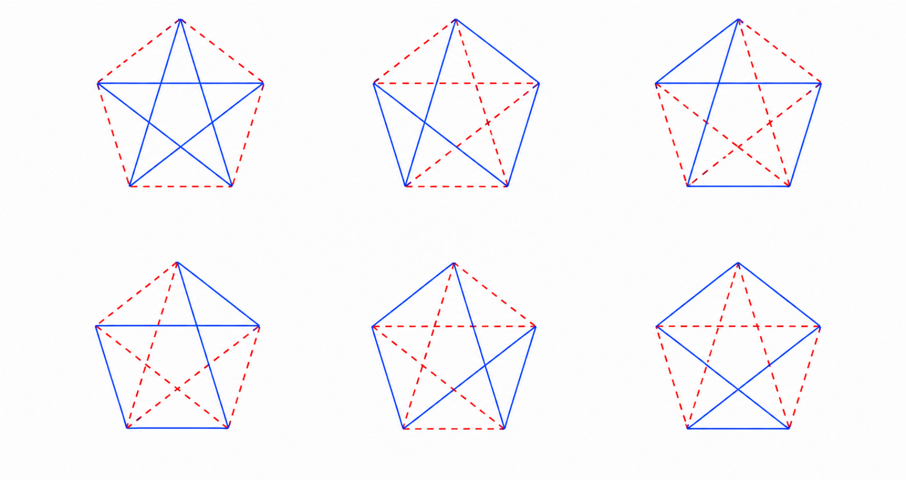
We say a group G is complete if the center of G is trivial and $Aut(G) = Inn(G)$.

Theorem

S_n is complete $\forall\ n \neq 2, 6$.

S_2 is not complete as its center is the group itself. On the other hand, S_6 is incomplete because of the existence of an outer automorphism.

The Six Mystic Pentagons



We can define the group homomorphism

$$\phi : S_5 \mapsto S_6$$

where for a permutation $\sigma \in S_5$, $\phi(\sigma)$ records the rearrangement of the six mystic pentagons.

There are exactly 6 left cosets $S_6/\phi(S_5)$. We see that S_6 acts on $S_6/\phi(S_5)$, so S_6 acts on a set of size 6, but not in the normal way that S_6 usually does. This gives the homomorphism $f : S_6 \mapsto S_6$. We can prove that f is bijective and computationally we notice that f sends

$$(1\,2) \mapsto (P_1\,P_4)(P_2\,P_3)(P_5\,P_6).$$

This symmetry is due to the fact that there are 15 transpositions and 15 triple transpositions.

Alternative Constructions

- Labeled Triangles
- Labeled Icosahedra
- The Segre Cubic
- The Igusa Quartic
- Six Points on Projective Space

S_n is complete $\forall\ n \neq 2, 6$.

There are three main parts that need to be proven in this theorem:

1. Prove that S_n has a trivial center for $n \geq 3$
2. Prove that an automorphism that preserves transpositions is formed by conjugation
3. Prove that every automorphism on S_n sends transpositions to transpositions for $n \neq 6$

The outer automorphism of S_6 is not used much in mathematics, however complete groups in general are very useful in many fields:

- Galois Theory
- Algebraic Topology
- Representation Theory
- Symmetry Groups of Topological Objects

Complete groups provide rigidity in the structure of the fundamental group which makes working with manifolds and covering spaces much simpler.