Elliptic Integrals and Functions: An Overview

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Some Related Integrals

What is common among all of these integrals?

1. All inverse trigonometric functions, e.g.

$$sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \frac{1-t}{\sqrt{(1-t^2)(1-t)^2}} dt$$

2. Perimeter of an ellipse (eccentricity k, semi-major length axis a):

$$a \int_0^1 \frac{1-k^2t^2}{\sqrt{(1-t^2)(1-k^2t^2)}} dt$$

3. Perimeter of a lemniscate (where d is the focal distance):

$$4d \int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

Elliptic Integrals

Definition of Elliptic Integrals

Elliptic integrals are defined as:

$$f(x) = \int_c^x R\left(t, \sqrt{P(t)}\right) dt = \int_c^x \frac{A(t) + B(t)\sqrt{P(t)}}{C(t) + D(t)\sqrt{P(t)}} dt$$

where c is a constant, R is a rational function, A, B, C, and D are polynomials in t (where B and D cannot be nonzero simultaneously) and P is a polynomial of degree 3 or 4.

Elliptic integrals *cannot be expressed using elementary functions* (polynomials, rational functions, logarithms, trig functions, etc.).

Kinds of Elliptic Integrals

Kind	Jacobi's Algebraic Form	Legendre's Trigonometric Form
1^{st}	$\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$	$\int_0^\varphi \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}d\theta$
2^{nd}	$\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt$	$\int_0^\varphi \sqrt{1-k^2\sin^2(\theta)}d\theta$
3^{rd}	$\int_0^x \frac{dt}{(1-nt^2)\sqrt{(1-k^2t^2)(1-t^2)}}$	$\int_0^\varphi \frac{1}{1-n\sin^2(\theta)} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$

Table 1: $0 < k < 1$ is the *elliptic modulus*, $0 \leq \varphi \leq \frac{\pi}{2}$ is the *elliptic amplitude*.

Complete elliptic integrals fix the bounds at $\varphi = \frac{\pi}{2}, x = 1$.

Key Facts

1. All elliptic integrals are linear combinations of the three kinds.
2. *Legendre's Relation*, with $K(k)$ complete 1^{st} kind, $E(k)$ 2^{nd} kind, $k' = \sqrt{1-k^2}$:

$$K(k) \cdot E(k') + E(k) \cdot K(k') - K(k) \cdot K(k') = \frac{\pi}{2}.$$

3. Useful approximations e.g.

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \right)^2 k^{2n}$$

4. Addition theorems for $K(k), E(k)$

Applications of Elliptic Integrals

Motion of a Pendulum

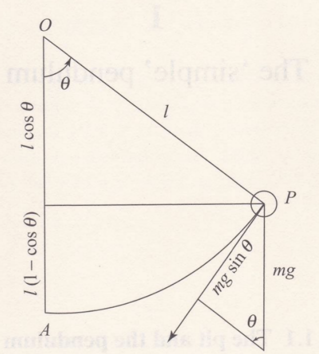

Figure 1: Motion diagram of pendulum. θ is the angle at any point during the swing.

Using the equation for conservation of energy of a pendulum, the period is:

$$T = 4\sqrt{\frac{l}{g}}\,K(k),$$

where l is the length of the pendulum, g is the gravitational constant, and $k = \sin(\frac{\alpha}{2})$, where α is the initial angle of the swing. $K(k)$ is the *complete elliptic integral of the* 1^{st} *kind.*

Orbital Motion

Advance of perihelion per revolution:

$$\epsilon = \frac{2K}{\eta} - 2\pi,$$

where K is the complete elliptic integral of the 1^{st} kind, $\eta = \sqrt{1-e^2}$ (e is eccentricity)

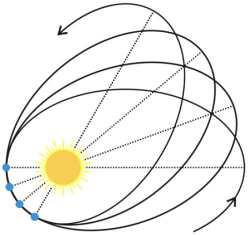

Figure 2: Every year, the perihelion of an orbit moves for planets, represented by ϵ.

Elliptic Functions

Jacobi Elliptic Functions in $\mathbb{R}$

Define $am(u, k) = \varphi$ as the inverse of the *incomplete elliptic integral of the* 1^{st} *kind*, i.e.

$$u = F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}}.$$

Jacobi elliptic functions are functions on φ, e.g.

- $sn(u, k) = sin(am(u, k)) = sin(\varphi)$
- $cn(u, k) = cos(am(u, k)) = cos(\varphi)$
- $dn(u, k) = \sqrt{1 - k^2 sin^2(\varphi)}$

The other 9 functions are some fractional combination of these three.

Geometric Interpretation of Jacobi Elliptic Functions

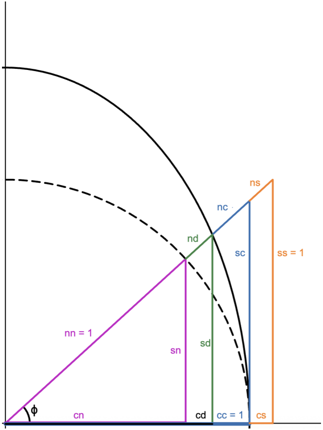

Figure 3: An extension of the trig functions to the unit ellipse.

Note the following:

1. 4 similar right triangles using the ray at angle ϕ to the x-axis
2. Most Jacobi functions are side lengths on these triangles
3. dn, ds, dc based on angular velocity

Elliptic Functions in $\mathbb{C}$

A complex-valued function $f(z)$ is **meromorphic** if it is analytic except at isolated poles.

An meromorphic function is elliptic if it is doubly periodic in $\mathbb{C}$, i.e.

$$f(z) = f(z + \omega_1) = f(z + \omega_2)$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

All 12 Jacobi elliptic functions have one real ($2K$ or $4K$) and one imaginary ($2iK'$ or $4iK'$) period. Note that $K = K(k), K' = K(k') = K(\sqrt{1-k^2})$.

Fundamental Parallelogram to Torus

The lattice $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ that comes from the periods ω_1, ω_2 of an elliptic function creates a *fundamental parallelogram* in the complex plane, which represents the entire domain. It can be defined as:

$$R_\Lambda = \{a\omega_1 + b\omega_2 : 0 \leq a, b < 1\}.$$

This can be transformed as follows:

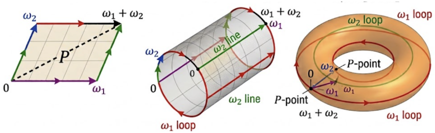

Figure 4: $\mathbb{C}/\Lambda$ is a *torus* generated by Λ.

Weierstrass Elliptic Function: Definition

For the lattice $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, where $\omega_1, \omega_2 \in \mathbb{C}$, the Weierstrass elliptic function associated to Λ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

$\wp(z)$ is even and absolutely convergent.

Results of $\wp(z)$

Laurent Expansion using Eisenstein series ($G_n = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-n}$):

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) z^{2n} G_{2n+2}.$$

Differential Equation ($g_2 = 60 G_4$, $g_3 = 140 G_6$ are *elliptic invariants*):

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\wp(z)) - g_3.$$

$\wp(z)$ is the inverse of the elliptic integral

$$u(z) = \int_z^{\infty} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}.$$

Elliptic Curves

Elliptic Curves in Weierstrass Form

All elliptic curves over the real or complex numbers can be written in either of the two *Weierstrass forms*:

$$y^2 = x^3 + ax + b,\ y^2 = 4x^3 - g_2 x - g_3$$

for constants a, b and elliptic invariants g_2, g_3. Note that this looks similar to the differential equation from earlier:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\wp(z)) - g_3.$$

Hence, $\wp(z)$ and its derivative *parameterize* elliptic curves.

Group Structure of Elliptic Curves

"Adding" $P + Q$ yields the reflection across the x-axis of the 3^{rd} point where the line through P, Q intersects the curve. The "point at infinity" is the identity element (0). Points on an elliptic curve have an Abelian (commutative) group structure over this operation.

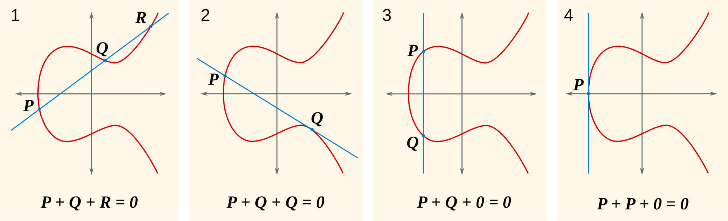

Figure 5: Cases $2, 3, 4$ define the operation in cases where the line goes through 1 or 2 points on the curve.

Weierstrass Addition Formula

Because $\wp(z)$ parametrizes elliptic curves, we can use the group structure to show that

$$P = (\wp(z), \wp'(z)), Q = (\wp(\omega), \wp'(\omega)), R = (\wp(z+\omega), -\wp'(z+\omega))$$

are collinear, yielding:

$$\wp(z+\omega) = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(\omega)}{\wp(z) - \wp(\omega)}\right]^2 - \wp(z) - \wp(\omega).$$

Conclusion

Key Takeaways

1. Elliptic integrals $\neq$ elementary functions
   - Origin: Inverse trig functions, arc length
   - Applications: Orbits, Pendulum, etc
2. Elliptic functions (meromorphic, doubly periodic) are inverses of elliptic integrals, extended to $\mathbb{C}$.
   - Key examples: Jacobi, Weierstrass ($\wp$)
3. $\wp(z)$ and $\wp'(z)$ parametrize elliptic curves in $\mathbb{R}, \mathbb{C}$