ELLIPTIC INTEGRALS AND FUNCTIONS: AN OVERVIEW

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Abstract. In this paper, we will provide an overview of, and a few key results related to, elliptic integrals, as well as their closely related counterpart: elliptic functions. The work done in understanding and developing these constructs have demonstrated practical applicability in science, as well as rich mathematical results in complex analysis. Elliptic integrals, derived from the arc length of curves like the ellipse and lemniscate, are rational functions that involve the square root of a cubic or quartic polynomial. They cannot be solved using elementary functions. Their inverses, known as elliptic functions, have important types: the 12 Jacobi elliptic functions and the single Weierstrass elliptic function. The former are generalizations of the trigonometric functions in real space, but can be extended to complex space. The Weierstrass elliptic $\wp$-function, defined in the complex plane, has many surprising properties, connecting to elliptic curves.

1. Introduction

We start by examining the early work done in the 18th and early 19th centuries by mathematicians (including the Bernoullis, Euler, Fagnano, and Legendre). Their progress allowed for discovering, defining, and applying real-valued elliptic integrals in a variety of ways including generalizing inverse trigonometric functions and calculating arc lengths of curves (such as ellipses and lemniscates).

After discussing the early work on elliptic integrals in real space, we move on to the work done by Abel and Jacobi in the 19th century. They extended elliptic integrals to complex space and also defined elliptic functions, the inverses of elliptic integrals. In particular, they showed that elliptic functions have very rich properties including double periodicity.

We then discuss a few key results in complex analysis that built upon the seminal work done on elliptic integrals and functions. First, we look at Liouville's theorem, which helped show that elliptic functions that are holomorphic must be constant. Second, we look at Weierstrass' work on his namesake elliptic ($\wp$) function. We prove why it is doubly periodic and derive its Laurent expansion (which is easier to work with). We also derive the important differential equation of the Weierstrass elliptic function. This helps us show the third result in complex analysis in our paper: the Weierstrass $\wp$-function and its derivative parametrize elliptic curves over the real and complex numbers. Furthermore, the group structure of elliptic curves allows us to compute more results involving $\wp(z)$.

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2. Beginnings of Elliptic Integrals

Elliptic integrals rose from certain practical scenarios which resulted in integrals that could not be solved using elementary functions.

One such example is the perimeter of an ellipse. Given an ellipse $x^2/a^2 + b^2/y^2 = 1$, we have the parametric equations $x(\theta) = a\cos(\theta)$ and $y(\theta) = b\sin(\theta)$, where θ is the central angle. Hence, we can define the length of the entire ellipse as

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{\cos^2(\theta) + \left(\frac{b^2}{a^2}\right)\sin^2(\theta)}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2(\theta)}\, d\theta \\ &= a\int_0^{2\pi} \sqrt{1 - k^2\sin^2(\theta)}\, d\theta \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2(\theta)}\, d\theta \end{aligned}$$

where $k = \sqrt{1 - \frac{b^2}{a^2}}$ is the eccentricity of the ellipse. We can substitute $t = sin(\theta)$ and $x = sin(\varphi)$ to get an algebraic form of the integral:

$$L = a\int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt = a\int_0^1 \frac{1-k^2t^2}{\sqrt{(1-t^2)(1-k^2t^2)}} dt.$$

Similarly, the length of a lemniscate (a curve in the shape ∞ sign, defined by the polar curve $r^2 = a^2cos(2\theta)$ on $0 \le \theta \le 2\pi$) can be written in both trigonometric and algebraic forms as the following:

$$L_l = 4a\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{cos(2\theta)}} = 4a\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

Note that the upper bounds on the above integrals can be modified to calculate incomplete arc lengths of the curves.

3. Definition of Elliptic Integrals

Notice that the integrals above in algebraic form are rational functions involving the square root of a quartic polynomial, and that these integrals cannot be solved using closed-form expressions with elementary functions. It is from here that elliptic integrals arose. They are defined as functions that satisfy the following:

$$f\left(x\right)=\int_{c}^{x}R\left(t,\sqrt{P\left(t\right)}\right)dt=\int_{c}^{x}\frac{A\left(t\right)+B\left(t\right)\sqrt{P\left(t\right)}}{C\left(t\right)+D\left(t\right)\sqrt{P\left(t\right)}}dt$$

where c is a constant, R is a rational function, A, B, C, and D polynomials in t (where B and D cannot be nonzero simultaneously) and P is a polynomial of degree 3 or 4. These cannot be expressed in terms of elementary functions, which was proven by Liouville in 1833, long after the discovery of elliptic integrals.

Note that elliptic integrals are also a generalization of the inverse trigonometric functions. We only list three for brevity's sake:

$$\begin{aligned} sin^{-1}(x) &= \int_0^x \frac{1-t}{\sqrt{(1-t^2)(1-t)^2}}dt \\ tan^{-1}(x) &= \int_0^x \frac{1}{\sqrt{(1+t^2)^2}}dt \\ sec^{-1}(x) &= \int_0^x \frac{1}{\sqrt{t^2(t^2-1)}}dt \end{aligned}$$

In the late 18th to early 19th centuries, Adrien-Marie Legendre defined three kinds of elliptic integrals and proved that all elliptic integrals are a linear combination of them ([3]). He primarily worked with them in trigonometric forms. In 1829, Carl Gustav Jacob Jacobi rewrote them ([2]) in algebraic forms by substituting $t = sin(\theta), x = sin(\varphi)$ to match our definition from earlier.

The three different kinds are shown below. Note the following:

- Incomplete elliptic integrals are with respect to the bounds $0 \leq \varphi \leq \frac{\pi}{2}$, $0 < x < 1$
- Complete elliptic integrals fix the bounds at $\varphi = \frac{\pi}{2}, x = 1$
  - $K(k), E(k)$ are the complete elliptic integrals of the $1^{st}, 2^{nd}$ kind, respectively
- φ is the "elliptic amplitude", $0 < k < 1$ is the "elliptic modulus"

Kind	Jacobi's Algebraic Form	Legendre's trigonometric Form
1^{st}	$F(x\|k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$	$F\left(\varphi\|k\right) = \int_0^\varphi \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}d\theta$
2^{nd}	$E(x\|k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt$	$E\left(\varphi\|k\right) = \int_0^\varphi \sqrt{1-k^2\sin^2\left(\theta\right)}d\theta$
3^{rd}	$\Pi\left(x\,\|\,n,k\right) = \int_0^x \frac{dt}{(1-nt^2)\sqrt{(1-k^2t^2)(1-t^2)}}$	$\Pi\left(\varphi\,\|\,n,k\right) = \int_0^\varphi \frac{1}{1-n\sin^2(\theta)}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$

Table 1. The incomplete elliptic integrals.

4. Properties of Elliptic Integrals

Below are approximations of $K(k)$ and $E(k)$ that can be derived through an application of Taylor series:

$$\begin{aligned}
K(k) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{2n!!}\right)^2 k^{2n} \\
&= \frac{\pi}{2}\left(1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n} + \dots\right) \\
E(k) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \frac{k^{2n}}{1-2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{1-2n} \\
&= \frac{\pi}{2}\left(1-\left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \cdots - \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{2n-1} - \dots\right)
\end{aligned}$$

They help to prove an elegant result known as *Legendre's Relation*:

Theorem 4.1.
$$K(k)\cdot E(k') + E(k)\cdot K(k') - K(k)\cdot K(k') = \frac{\pi}{2},$$
where $k' = \sqrt{1-k^2}$, and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively.

Proof. First we can find the derivative of $E(k), K(k)$:
$$K'(k) = \frac{E(k)-(k')^2K(k)}{k(k')^2}, E'(k) = \frac{E(k)-K(k)}{k}.$$
This means that
$$K'(k') = \frac{E(k')-(k)^2K(k')}{(k')k^2}, E'(k') = \frac{E(k')-K(k')}{k'}.$$
Now we will take the derivative of the left hand side of the relation. Note that $\frac{dk'}{dk} = -\frac{k}{k'}$:

$$\begin{aligned}
&\frac{d}{dk}\left[K(k)\cdot E(k') + E(k)\cdot K(k') - K(k)\cdot K(k')\right] \\
&= K'(k)\cdot E(k') + K(k)\cdot E'(k')\cdot\left(-\frac{k}{k'}\right) + E'(k)\cdot K(k') \\
&+ E(k)\cdot K'(k')\cdot\left(-\frac{k}{k'}\right) - K'(k)\cdot K(k') - K(k)\cdot K'(k')\left(-\frac{k}{k'}\right) \\
&= \left(\frac{E(k)-(k')^2K(k)}{k(k')^2}\right)\cdot E(k') - \left(\frac{k\,K(k)}{(k')^2}\right)(E(k')-K(k')) + \left(\frac{E(k)-K(k)}{k}\right)\cdot K(k') \\
&- \frac{E(k)\cdot(E(k')-k^2K(k'))}{k(k')^2} - \left(\frac{E(k)-(k')^2K(k)}{k(k')^2}\right)K(k') + K(k)\cdot\left(\frac{E(k')-k^2K(k')}{k(k')^2}\right) \\
&= 0.
\end{aligned}$$

This means that the expression must be a constant. To find out what this constant is, we find its values as k approaches 0:

$$\lim_{k\to 0,\, k'\to 1}\left[K(k)\cdot E(k') + E(k)\cdot K(k') - K(k)\cdot K(k')\right].$$

Note that

$$K(0) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(0)^2 sin^2(\theta))}} = \frac{\pi}{2},$$

$$E(0) = \int_0^{\frac{\pi}{2}} \sqrt{1-(0)^2\sin^2(\theta)}\,d\theta = \frac{\pi}{2},$$

$$E(1) = \int_0^{\frac{\pi}{2}} \sqrt{1-(1)^2\sin^2(\theta)} = \int_0^{\frac{\pi}{2}} \cos(\theta)d\theta = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$$

Using this results, our limit above becomes

$$\left(\frac{\pi}{2}\right)(1) + \lim_{k\to 0,\, k'\to 1}\left[K(k')\right]\left[E(k) - K(k)\right].$$

As $k \to 0$, higher powers of k $(k^4, k^6$, etc.$)$ become negligible. Hence, using the power series for $E(k), K(k)$ above, we can say that

$$K(k) \approx \frac{\pi}{2}\left(1+\frac{1}{4}k^2\right), E(k) \approx \frac{\pi}{2}\left(1-\frac{1}{4}k^2\right).$$

Furthermore, (recall $k' = \sqrt{1-k^2}$) $K(k) \sim \ln(\frac{4}{k'})$ as $k \to 1$, which means that $K(k') \sim \ln(\frac{4}{k})$ as $k' \to 1, k \to 0$. Plugging in these results into our limit yields

$$\frac{\pi}{2} + \lim_{k\to 0,\, k'\to 1}\left[\ln\left(\frac{4}{k}\right)\right]\left[\frac{\pi}{2}\left(1-\frac{1}{4}k^2\right) - \frac{\pi}{2}\left(1+\frac{1}{4}k^2\right)\right]$$
$$= \frac{\pi}{2} + \frac{\pi}{4}\lim_{k\to 0,\, k'\to 1}\left[\frac{\ln\left(\frac{4}{k}\right)}{k^{-2}}\right].$$

Using L'Hôpital's rule, we can show that the limit goes to 0, which means that for $k = 0$, the relation is $\pi/2$, and since the derivative is 0, the relation is $\pi/2$ for all $0 < k < 1$.

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5. Applications of Elliptic Integrals

In a pre-calculator 18^{th} and 19^{th} century, elliptic integrals proved invaluable for representing and computing values in physics. Below are two examples:

5.1. **Period of a Pendulum.** Below is the equation for conservation of energy for a pendulum:

$$\frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 - mgl\,cos(\theta) = -mgl\,cos(\alpha),$$

where t is the variable time, θ is the variable for the angle, l is the length of the pendulum, g is the gravitational constant, and α is the angle of the initial swing. Rearranging, we get:

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos(\theta) - \cos(\alpha)) = \frac{4g}{l}(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}).$$

Integrating and using the substitutions $k = \sin(\frac{\alpha}{2})$ $\sin(\frac{\theta}{2}) = k\sin(\phi)$ yields the period:

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}} = 4\sqrt{\frac{l}{g}}K(k).$$

5.2. **Orbital Motion.** Advance of perihelion per revolution:

$$\epsilon = \frac{2K}{\eta} - 2\pi,$$

where K is the complete elliptic integral of the 1^{st} kind, $\eta = \sqrt{1-e^2}$ (e is eccentricity). More information about these examples can be found in ([6]).

6. Jacobi Elliptic Functions in Real Space

There are a group 12 functions known as the *Jacobi elliptic functions* which are defined based on the inversion of the incomplete elliptic integral of the first kind. Given

$$u = F(\varphi|k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2 sin^2(\theta)}},$$

the 12 Jacobi elliptic functions are defined on $\varphi = am(u,k) = F^{-1}(u,k)$, where $am(u,k)$ is also called the *Jacobi amplitude function*. The 12 functions are as follows:

- $sn(u,k) = sin(\varphi) = sin(am(u,k))$
- $cn(u,k) = cos(\varphi) = cos(am(u,k))$
- $dn(u,k) = \sqrt{1-k^2sin^2(\varphi)} = \sqrt{1-k^2sin^2(am(u,k))}$
- $sc(u,k) = tan(\varphi) = tan(am(u,k))$
- $sd(u,k) = sn(u,k)/dn(u,k)$
- $cd(u,k) = cn(u,k)/dn(u,k)$

The other 6 elliptic functions are defined by taking the reciprocal of one of the functions above and switching the letters. For example, $ns(u,k) = (sn(u,k))^{-1} = csc(\varphi)$. Note that the remaining 4 "functions" $cc(u,k), ss(u,k), nn(u,k)$ and $dd(u,k)$, which are the other combinations of the letters c, n, d, s, are equated to 1 for simplicity.

A geometric interpretation of these elliptic functions on a unit ellipse (with semi-minor axis 1) is closely related to the 6 unit circle trigonometric functions, as shown below. Using central angle φ, four similar right triangles are drawn, where nd is the radial distance of the ellipse for this particular φ.

Note that dn, ds, dc, (and dd) are excluded from the diagram because they are purely algebraic definitions. However, dn is used to calculate angular velocity.

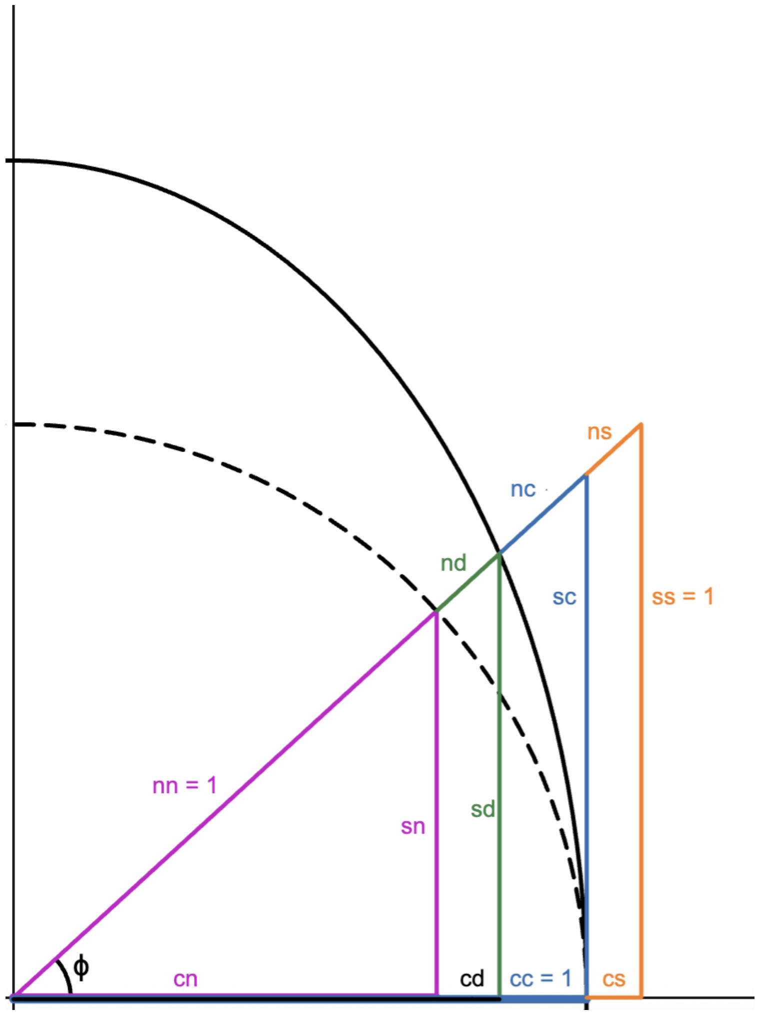


Figure 1. The diagram above includes all the elliptic functions with geometric interpretations in relation to the central angles, ϕ.

7. Elliptic Functions and Integrals in Complex Space

So far, we have only discussed elliptic integrals and functions in real space. However, Jacobi and Niels Henrik Abel defined them in complex space as well. Before we delve into that, recall some basics of complex analysis:

- A complex-valued function $f(z)$ is *analytic* or *holomorphic* over a region U if any for any point $p \in U$, f can be represented using a convergent power series with a positive radius of convergence.

- A function $f(z)$ has a *pole* at point p if $(z-p)^n f(z)$ is holomorphic for an $n \in \mathbb{N}$. The smallest value of n for which this holds is the *order* of the pole at point p.

- A complex-valued function $f(z)$ is called *meromorphic* if it is analytic except at isolated poles.

Below is the definition of complex-valued elliptic functions:

Definition 7.1. A complex-valued meromorphic function $f(z)$ is an *elliptic function* if it is doubly periodic, or more precisely, if two $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$ (i.e. one is not a scalar multiple of the other) exist such that

$$f(z) = f(z + \omega_1) = f(z + \omega_2).$$

The lattice $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ that defines the elliptic function results in a *fundamental parallelogram* in the complex plane, which represents the entire domain. It can be defined as:

$$R_\Lambda = \{a\omega_1 + b\omega_2 : 0 \leq a, b < 1\}.$$

The double periodicity allows us to "glue" one edge to another (in the direction of one of the periods, resulting in a tube. Then, using the second period, we can connecting the sides of the tube, meaning that $\mathbb{C}/\Lambda$ is a *torus* generated by Λ.

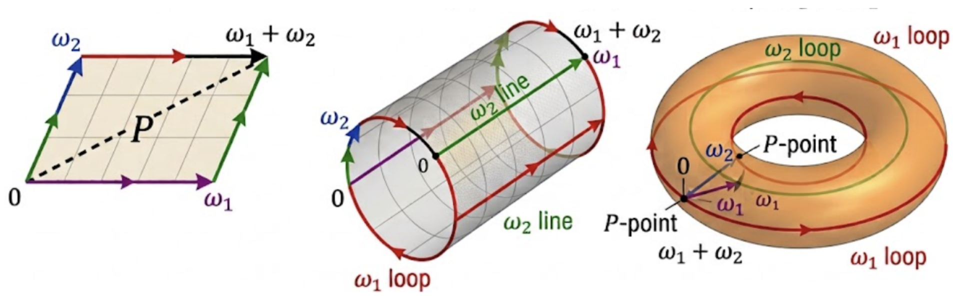


Figure 2. In the diagram above, we go from the fundamental parallelogram to a tube to a torus, as described.

If the 12 Jacobi functions earlier are truly elliptic functions, they must be doubly periodic in complex space. We can show that they satisfy this because they have one real and one imaginary period. To prove this, we first take the integral (where $n \in \mathbb{N}$)

$$u = \int_{n\pi - \frac{\pi}{2}}^{n\pi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}}.$$

Substituting $\phi = n\pi - \theta$ yields

$$\int_{n\pi-\frac{\pi}{2}}^{n\pi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} = \int_{\frac{\pi}{2}}^{0} \frac{-d\phi}{\sqrt{1 - k^2 sin^2(n\pi - \phi)}} = \int_{0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 sin^2(\phi)}} = K,$$

where K is the complete elliptic integral of the first kind. Note that the penultimate simplification is valid because $sin(2k\pi - \phi) = -sin(\phi)$ and $sin((2k+1)\pi - \phi) = sin(\phi)$. Similarly, we can show that

$$u' = \int_{n\pi}^{n\pi + \frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} = \int_{0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 sin^2(\phi)}} = K.$$

For convenience, substituting $\Delta\theta = \sqrt{1 - k^2 sin^2(\theta)}$ yields

$$\int_0^{\frac{n\pi}{2}} \frac{d\theta}{\Delta\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\Delta\theta} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{\Delta\theta} + ... + \int_{\frac{(n-1)\pi}{2}}^{\frac{n\pi}{2}} \frac{d\theta}{\Delta\theta} = nK,$$

as each of those n pieces matches either u or u' from earlier. By the definition of $am(u)$, we have $am(nK) = \frac{n\pi}{2}$. Furthermore, since $am(K) = \frac{\pi}{2}$, we have

$$am(nK) = n \cdot am(K).$$

Lemma 7.2. *$am(-u) = -am(u)$ (i.e. $am(u)$ is odd). We define*

$$u = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} \implies am(u) = \varphi.$$

Because the integrand is even ((sin(θ) is odd, so $sin^2(\theta)$ is even), we have:

$$u' = \int_{-\varphi}^{0} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} = -\int_0^{-\varphi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} = u.$$

Hence, because $u = u'$, we can conclude that:

$$-\varphi = am(-u') = am(-u) = -am(u).$$

For any $\alpha = n\pi \pm \beta : 0 \leq \beta \leq \frac{\pi}{2}$, we have:

$$\int_0^{n\pi+\beta} \frac{d\theta}{\Delta\theta} = \int_0^{n\pi} \frac{d\theta}{\Delta\theta} + \int_{n\pi}^{n\pi+\beta} \frac{d\theta}{\Delta\theta} = 2nK + u,$$

where $u = \int_{n\pi}^{n\pi+\beta} \frac{d\theta}{\Delta\theta} = \int_0^{\beta} \frac{d\phi}{\Delta\phi}$ with the substitution $\theta = n\pi + \phi$. By definition, we have

$$\alpha = n\pi + \beta = am(2nK + u) = n\pi + am(u).$$

Because $am(u)$ is odd,

$$am(2nK - u) = n\pi + am(-u) = n\pi - am(u) = n\pi - \beta,$$

or more concisely (and using our prior results),

$$\alpha = n\pi \pm \beta = 2n \cdot am(k) \pm am(u) = am(2nK \pm u).$$

Now, using the definition(s) of the Jacobi elliptic functions, we have:

$$\begin{aligned} sn(u \pm 2nK) &= sin(am(u \pm 2nK)) = sin(am(u) \pm 2n \cdot am(K)) \\ &= sin(am(u) \pm n\pi) = -sin(am(u)) \\ sn(u \pm 4nK) &= sin(am(u \pm 4nK)) = sin(am(u) \pm 4n \cdot am(K)) \\ &= sin(am(u) \pm 2n\pi) = sin(am(u)) \end{aligned}$$

Hence, $sn(u \pm 2K) = -sn(u)$ and $sn(u \pm 4K) = sn(u)$. Using a similar process, we can show that $cn(u \pm 2K) = -cn(u)$, $cn(u \pm 4K) = cn(u)$, $dn(u \pm 2K) = dn(u)$, etc. Hence, we have shown that $4K$ is a real period of all the Jacobi elliptic functions.

Before we start to look for an imaginary period, recall that $k' = \sqrt{1 - k^2}$ and $K' = K(k')$. If we now define ψ such that $sin(\phi) = i \cdot tan(\psi)$, it can be easily shown that $cos(\phi) = \frac{1}{cos(\psi)}$, $\Delta(\phi, k) = \frac{\Delta(\psi,k')}{cos(\psi)}$, $d\phi = i\frac{d\psi}{cos(\psi)}$, where like earlier, $\Delta(\phi, k) = \sqrt{1 - k^2 sin^2\phi}$. Hence it is clear that

$$\frac{d\phi}{\sqrt{1 - k^2 sin^2(\phi)}} = \frac{d\phi}{\Delta(\phi, k)} = \frac{i \cdot d\psi}{\Delta(\psi, k')} = \frac{i \cdot d\psi}{\sqrt{1 - (k')^2 sin^2(\psi)}},$$

so we have

$$\int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 sin^2(t)}} = i \int_0^{\psi} \frac{dt}{\sqrt{1 - (k')^2 sin^2(t)}} = iu,$$

which means that $\psi = am(u, k')$ and $\phi = am(iu, k)$. Plugging this in to our initial introduction of ψ yields $sin(iu, k) = i \cdot sc(u, k')$, where $sc(u, k') = tan(am(u, k')) = sn(u, k')/cn(u, k')$. Therefore, we have:

$$sn(i(u \pm 2K'), k) = i \cdot sc(u \pm 2K', k') = i\frac{-sn(u, k')}{-cn(u, k')} = i \cdot sc(u, k') = sn(iu, k).$$

Substituting u for iu yields $sn(u \pm 2iK', k) = sn(u, k)$, which shows that $2iK'$ is the imaginary period of $sn(u, k)$. Similarly, the imaginary periods for the remainder of the Jacobi elliptic functions can be shown.

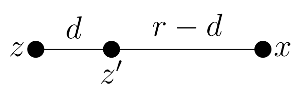
8. Liouville's Theorem

Before we visit the Weierstrass elliptic function, we will look at Liouville's Theorem, which is key to reasoning about it. Despite being credited to Liouville for presenting it in 1847, it was proven by Cauchy in 1844.

Theorem 8.1. *A* ***bounded entire function*** *(that is finite and holomorphic in the entirety of $\mathbb{C}$) must be constant.*

The proof below comes from [8, p.105], a useful compilation of late 19^{th} to early 20^{th} century analysis.

Proof. First, let's call the bounded entire function $f(z)$ where $|f(z)| < K$ for some constant K. Let z, z' be two complex numbers contained within the boundary of a contour C defined as a circle with center z and radius $r \geq 2|z' - z| = 2d$ for the appropriate value d. We write the points on C as $x = z + re^{i\theta}$, yielding what we have below:



Hence, because $r > 2d$, it is clear that $r - d = |x - z'| > \frac{1}{2}r$. Recall that for any analytic function $f(z)$ in closed contour C, (see appendix for the proof)

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - a}dz,\ \forall a \in\ C.$$

We can use this result, as $f(z)$ is analytic, and points z, z' are in contour C.

$$\begin{aligned} |f(z') - f(z)| &= \left|\frac{1}{2\pi i}\oint_C \left(\frac{1}{x - z'} - \frac{1}{x - z}\right) f(x)dx\right| \\ &= \left|\frac{1}{2\pi}\oint_C \frac{z' - z}{(x - z')(x - z)} f(x)dx\right| \end{aligned}$$

Substituting $x = z + re^{i\theta}$, $|x - z'| > \frac{1}{2}r$, and $|f(z)| \leq K$ (from earlier), we get

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{z'-z}{(x-z')(x-z)}f(x)\cdot rie^{i\theta}d\theta\right| < \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{z-z'}{\frac{1}{2}r^2e^{i\theta}}K\cdot rie^{i\theta}d\theta\right|$$
$$< \left|\frac{Ki}{\pi r}\int_0^{2\pi}(z-z')d\theta\right|$$
$$< \left|\frac{K}{\pi r}(z-z')(2\pi)\right|$$
$$< 2K|z-z'|r^{-1}.$$

Keeping z, z' fixed (the two random points in the contour) and letting $r \to \infty$ (as r only had a lower bound) yields $|f(z') - f(z)| = 0$, which means that the entire bounded (analytic) function $f(z)$ is constant. ■

Note that using this theorem, we can derive the following result that was actually proven by Liouville himself. In brief, because of the double periodicity of elliptic functions, they must be bounded if they have no poles. This satisfies the criteria of Liouville's Theorem. Hence,

Theorem 8.2. *All holomorphic elliptic functions are constant.*

9. Weierstrass Elliptic Function ($\wp(z)$)

In the early 1860s, Karl Weierstrass presented his definition of his namesake elliptic $\wp$-function. Let Λ be a lattice in $\mathbb{C}$, the complex plane. In other words,

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \neq 0\},$$

where ω_1 and ω_2 are lineally independent complex numbers. The Weierstrass function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right].$$

Now we will start by proving some nice properties of $\wp(z)$:

Lemma 9.1. $\wp(-z) = \wp(z)$ *($\wp(z)$ is even).*

Proof.

$$\wp(-z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2}\right] = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2}\right]$$
$$= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] = \wp(z).$$

The last step is valid because our sum remains the same, as negative values are also present in the sum (as both positive and negative linear combinations of ω_1 and ω_2 are in the lattice, by definition).

■

Lemma 9.2. *$\wp(z)$ and $\wp'(z)$ are both doubly periodic.*

Proof. Differentiating with respect to z yields

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega\in\Lambda\setminus\{0\}} -\frac{2}{(z-\omega)^3} = \sum_{\omega\in\Lambda} -\frac{2}{(z-\omega)^3},$$

which is doubly periodic, as even at periods ω_1 and ω_2, we are effectively summing over the same lattice. This is because we have

$$\begin{aligned}\wp'(z) &= \sum_{\omega\in\Lambda} -\frac{2}{(z-\omega)^3} = \sum_{m,n\in\mathbb{Z}} -\frac{2}{(z-(m\omega_1+n\omega_2))^3}\\ \wp'(z+\omega_1) &= \sum_{\omega\in\Lambda} -\frac{2}{(z+\omega_1-\omega)^3} = \sum_{m,n\in\mathbb{Z}} -\frac{2}{(z-((m-1)\omega_1+n\omega_2))^3}\\ \wp'(z+\omega_2) &= \sum_{\omega\in\Lambda} -\frac{2}{(z+\omega_2-\omega)^3} = \sum_{m,n\in\mathbb{Z}} -\frac{2}{(z-(m\omega_1+(n-1)\omega_2))^3}\end{aligned}$$

Because we sum all $m, n \in \mathbb{Z}$, all our sums are the same. Hence, we have $\wp'(z) = \wp'(z+\omega_1) = \wp'(z+\omega_2)$ for periods ω_1 and ω_2, which gives us:

$$\wp'(z) - \wp'(z+\omega_1) = 0, \wp'(z) - \wp'(z+\omega_2) = 0.$$

Integrating, we get

$$\wp(z) = \wp(z+\omega_1) + C_1, \wp(z) = \wp(z+\omega_2) + C_2.$$

Evaluating $z = -\frac{\omega_1}{2}$ and $z = \frac{\omega_2}{2}$ yields

$$\wp(-\frac{\omega_1}{2}) = \wp(\frac{\omega_1}{2}) + C_1, \wp(-\frac{\omega_2}{2}) = \wp(\frac{\omega_2}{2}) + C_2,$$

which means that integration constants $C_1 = C_2 = 0$ because $\wp(z)$ is an even function, as shown earlier. Therefore, we have

$$\wp(z) = \wp(z+\omega_1) = \wp(z+\omega_2),$$

so we can conclude that $\wp(z)$ and its derivative are both doubly periodic. ∎

Lemma 9.3. *This series for $\wp(z)$ is absolutely convergent.*

Proof. (from [1]). We start with our summed term from above:

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{\omega^2 - \omega^2 + 2z\omega - z^2}{(\omega^2)(z-\omega)^2}\right| = \left|\frac{z(2\omega - z)}{(\omega^2)(z-\omega)^2}\right|.$$

Obviously, for all $|\omega| \leq 2|z|$, the series is bounded because the values are finite. Therefore, we look at the case where $|\omega| > 2|z|$. Using this and the inequality $|a+b| \leq |a|+|b|$ $(\forall a, b \in \mathbb{C})$, we can show that:

$$|2\omega - z| \leq |2\omega| + |-z| < \frac{5|\omega|}{2}$$

Using $|\omega| > 2|z|$ and the inequality $|a+b| \geq |a| - |b|$ $(\forall a, b \in \mathbb{C})$:

$$|z-\omega| \geq |z| - |-\omega| \geq \frac{|\omega|}{2}.$$

Plugging these in, we get the following.

$$\left|\frac{z(2\omega - z)}{(\omega^2)(z-\omega)^2}\right| < \left|\frac{5z|\omega|}{2(|\omega|)^2} \cdot \left(\frac{2}{|\omega|}\right)^2\right| = \frac{10|z|}{|\omega|^3}.$$

From here, all we have to prove is that $\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{|w|^3}$ converges, as z is just our complex argument. We can rewrite $|\omega|$ as

$$|m\omega_1 + n\omega_2| = |\omega_2| \cdot |m| \cdot \left|\frac{\omega_1}{\omega_2} + \frac{n}{m}\right| \geq c_1|m|,$$

for some positive constant c_1. We can factor out $|n|$ and $|\omega_1|$ to also show $|\omega| \geq c_2|n|$, for some positive constant c_2. Hence, we get

$$\begin{aligned}
\frac{|\omega|}{c_1} &\geq |m|, \frac{|\omega|}{c_2} \geq |n| \\
|\omega|\left(\frac{1}{c_1} + \frac{1}{c_2}\right) = |\omega|\left(\frac{c_1+c_2}{c_1c_2}\right) &\geq |m| + |n| \\
|\omega| \geq \left(\frac{c_1c_2}{c_1+c_2}\right)(|m|+|n|) &= d(|m|+|n|),
\end{aligned}$$

where $d = c_1c_2/(c_1+c_2)$.

Now we can group the lattice into various values of $k \in \mathbb{N}$ such that $|m|+|n| = k$. This separates the lattice into concentric diamond rings, as shown in Figure 3.

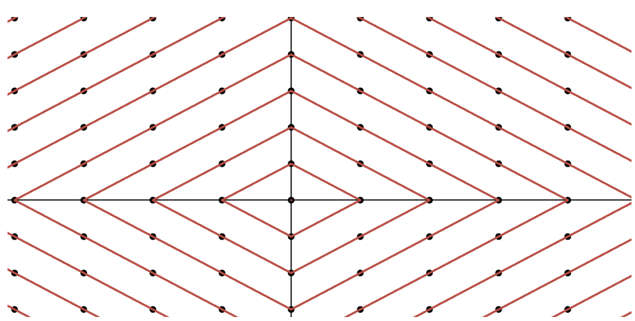


Figure 3. The solid black points above represent all the values of $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$.

As is apparent from the lattice, in each *level set* (k^{th} "ring") , there are $4k$ lattice points contained. Substituting these results into our original expression, instead of summing the entire lattice at once, we can sum it "ring" by "ring":

$$\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{|\omega|^3} = \sum_{k=1}^{\infty} \frac{4k}{(d(|m|+|n|))^3} = \frac{4}{d^3}\sum_{k=1}^{\infty}\frac{k}{k^3} = \frac{4}{d^3}\sum_{k=1}^{\infty}\frac{1}{k^2},$$

which yields the sum from the famous Basel problem and is convergent, which proves the convergence of the infinite series of $\wp(z)$, the Weierstrass elliptic function. ■

10. Laurent Expansion of $\wp(z)$

We can now derive the Laurent Expansion of $\wp(z)$ using its definition and results above:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

We can rewrite the $\frac{1}{(z-\omega)^2}$ term as follows:

$$\frac{1}{(z-\omega)^2} = \frac{1}{(\omega^2)(1-\frac{z}{\omega})^2} = \frac{1}{\omega^2}\cdot\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n.$$

Plugging this back into the original expression, we get

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{\omega^2}\cdot\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n - \frac{1}{\omega^2}\right].$$

The $n = 0$ term of the sum cancels out to 1, so the whole thing simplifies to

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{\omega^2}\cdot\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n\right].$$

Because the series is absolutely convergent (as shown earlier), we can combine the ω terms and switch the order of summation as follows:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n\left[\sum_{\omega\in\Lambda\setminus\{0\}}\frac{1}{\omega^{n+2}}\right].$$

The sum in terms of ω represents the $n+2$ term of the Eisenstein series, G_{n+2}, yielding

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n G_{n+2} \quad , \quad G_n = \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-n}$$

Note the definition of the lattice Λ from above, which is a linear combination of $\omega_1, \omega_2 \in \mathbb{C}$. This means that for any $\lambda \in \Lambda$, $-\lambda$ is also in Λ. This means that for all members of the Eisenstein series that have odd powers will cancel out to 0. For this reason, we can rewrite our sum above as

$$\sum_{n=1}^{\infty}(n+1)z^n G_{n+2} = 2G_3 z + 3z^2 G_4 + 4z^3 G_5 + 5z^4 G_6 + ... = 3z^2 G_4 + 5z^4 G_6 + ...$$

which yields the Laurent Expansion:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)z^{2n}G_{2n+2}.$$

11. Differential Equation of $\wp(z)$

From here, we can derive a differential equation involving $\wp(z)$ and $\wp'(z)$.

For the sake of simplicity, because the higher terms in the Eisenstein series are negligible (ω^{-1} keeps shrinking), we only need to consider the first few terms. For the remainder of the proof, we will use C_1, C_2, etc to group the smaller terms together..

$$\wp(z) = \frac{1}{z^2} + 3z^2G_4 + 5z^4G_6 + Cz^6$$

for whatever C may be. Now we can calculate our differential equation from earlier:

$$\wp(z) = \frac{1}{z^2} + 3z^2G_4 + 5z^4G_6 + C_1z^6$$

$$\wp'(z) = -\frac{2}{z^3} + 6zG_4 + 20z^3G_6 + C_2z^5$$

$$\begin{aligned}(\wp'(z))^2 &= \frac{4}{z^6} - 24\frac{G_4}{z^2} - 80G_6 - 4C_2z^2 + ... = \frac{4}{z^6} - 24\frac{G_4}{z^2} - 80G_6 + C_3z^2 \\ \wp(z)^3 &= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + 3C_1z^2 + 27G_4^2z^2 + ... = \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + C_4z^2\end{aligned}$$

Now we plug in our results:

$$\begin{aligned}(\wp'(z))^2 - 4(\wp(z))^3 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + C_3z^2 - 4\left(\frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + C_4z^2\right) \\ &= -\frac{60G_4}{z^2} - 140G_6 + (C_3 - 4C_4)z^2 \\ (\wp'(z))^2 - 4(\wp(z))^3 + 60G_4(\wp(z)) &= -\frac{60G_4}{z^2} - 140G_6 + (C_3 - 4C_4)z^2 + \frac{60G_4}{z^2} + ... \\ &= -140G_6 + C_5z^2.\end{aligned}$$

Now we define a function

$$F(z) = (\wp'(z))^2 - 4(\wp(z))^3 + 60G_4(\wp(z)) + 140G_6 = C_5z^2.$$

$F(z)$ is a polynomial in terms of an elliptic function ($\wp(z)$), which means that $F(z)$ is also holomorphic except at the poles (all the lattice points, including 0). Because it shares the same periods at $\wp(z), \wp'(z)$, it is also an elliptic function. However, $F(0) = 0$, which means that $F(0)$ has no poles, so it is holomorphic. By the extension of Liouville's Theorem (8.2), $F(z)$ is constant; specifically, $F(z) = 0$. This means that

$$(\wp'(z))^2 - 4(\wp(z))^3 + 60G_4(\wp(z)) + 140G_6 = 0.$$

Defining "elliptic invariants" $g_2 = 60G_4$ and $g_3 = 140G_6$, we get the differential equation:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\wp(z)) - g_3. \tag{11.1}$$

This derivation was published by Weierstrass in 1895 ([7]).
We can use this equation to show that $\wp(z)$ is the inverse of the elliptic integral

$$u(z) = \int_z^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}.$$

If $u(z)$ is the inverse of $\wp(z)$, we have

$$z = u(\wp(z)) = \int_{\wp(z)}^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}.$$

From here, we set

$$g(s) = \frac{1}{\sqrt{4s^3 - g_2 s - g_3}}.$$

Then,

$$\frac{d}{dz}(z) = 1 = \wp'(z) \cdot u'(\wp(z)) = -\wp'(z) \cdot g(\wp(z)).$$

From this we can conclude that

$$\wp'(z) = -\frac{1}{g(\wp(z))} = -\sqrt{4(\wp(z))^3 - g_2(\wp(z)) - g_3},$$

which gives us the same differential equation as before (11.1).

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\wp(z)) - g_3.$$

12. Elliptic curves and $\wp(z)$

Elliptic curves in $\mathbb{R}, \mathbb{C}$ can be expressed using the equations in *Weierstrass form*:

$$y^2 = x^3 + Ax + B,\ y^2 = 4x^3 - g_2 x - g_3$$

for constants A, B and the same elliptic invariants g_2, g_3 (as long as the discriminants $\Delta = -16(4A^3 + 27B^2)$ or $\Delta = (g_2)^3 - 27(g_3)^2$ are nonzero). Notice that this equation is in the same form as the differential equation of $\wp(z)$ (11.1). Since plugging in $(x, y) = (\wp(z), \wp'(z))$ yields the differential equation from earlier, we say that $\wp(z)$ and its derivative *parametrize* Weierstrass-form elliptic curves.

It also turns out that elliptic curves have a group structure. "Adding" $P + Q$ yields the reflection across the x-axis of the 3^{rd} point where the line through P and Q intersects the curve. The "point at infinity" is the identity element, or "0". Using this identity, points on an elliptic curve have an Abelian (commutative) group structure over this operation (see appendix, 4).

Using this operation, we can prove the Weierstrass addition formula: Define points $P = (\wp(z), \wp'(z))$, $Q = (\wp(\omega), \wp'(\omega))$, and $R = (\wp(z+\omega), -\wp'(z+\omega))$ on an elliptic curve.

Lemma 12.1. *If P and Q $(\omega \neq z)$ are collinear, then they are also collinear with R.*

Proof. If we define $y = mx + c$ to be the line going through P, Q, then we can define the function

$$f(u) = \wp'(u) - m\wp(u) - c,$$

which has roots z, ω, and since $z \neq \omega$, the third point (by *Bézout's Theorem*) where the line intersects the elliptic curve (call it r_3). Because $\wp'(u)$ has a pole of order 3 at $u = 0$ (see 9.2; the exponent in the denominator is 3) and $\wp(u)$ has a pole of order 2 at 0, $f(u)$ also has a pole of order 3 at $u = 0$. In addition, the sum of zeroes of elliptic functions equals the sum of the poles (see [5]). In this case, $z+\omega+r_3 = 0 \implies r_3 = -(z+\omega)$. Hence, the 3^{rd} intersection point of the elliptic curve is $R = (\wp(-(z+\omega)), \wp'(-(z+\omega))$. Because $\wp(z)$ is even (see 9.1) and $\wp'(z)$ is odd (by properties of a lattice), we have $R = (\wp(-(z+\omega)), \wp'(-(z+\omega))$. ■

Continuing the derivation of the addition formula: the slope of their common line is

$$m = \frac{\wp'(z) - \wp'(\omega)}{\wp(z) - \wp(\omega)}.$$

Therefore, $x = x_P = \wp(z)$, $x = x_Q = \wp(\omega)$, and $x = x_R = \wp(z + \omega)$ satisfy the cubic

$$(mx + c)^2 = 4x^3 - g_2 x - g_3$$

for some constant c. Simplifying, we get

$$4x^3 - m^2x^2 - (2mc + g_2)x - g_3 - c^2 = 0.$$

The sum of the roots of this cubic is $\frac{m^2}{4} = x_P + x_Q + x_R$, yielding the Weierstrass addition formula:

$$\wp(z+\omega) = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(\omega)}{\wp(z) - \wp(\omega)}\right]^2 - \wp(z) - \wp(\omega). \tag{12.1}$$

13. Conclusion

Elliptic integrals have extended far beyond the form from which they started. While initially confined to real space for calculating arc lengths, orbits, etc. their properties and relation to their inverses, elliptic functions, have been used for deep results in complex analysis. The most well known elliptic functions, the Weierstrass $\wp$-function and the 12 Jacobi elliptic functions, exhibit an enormous amount of properties, linking an extension of trigonometric functions to a parametrization of elliptic curves.

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14. Appendix

Lemma .1. *For a complex-valued $f(z)$ that is analytic on and within the region bounded by closed contour C, the value of $f(a)$ (where a inside C) is $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$.*

Proof. Recall that for any analytic function $f(z)$ in closed contour C, the following holds true:

$$\oint_C f(z)dz = 0.$$

Now, for contour C with center a, we examine the contour integral

$$\oint_C \frac{f(z)}{z-a}dz.$$

Since $f(z)$ is analytic everywhere, we know that the integrand is analytic everywhere except when $z = a$. Since by the fundamental property above analytic points in a contour integral sum up to 0, we can define a new contour C_ε, a circle with a tiny radius centered at $z = a$. That means that

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_\varepsilon} \frac{f(z)}{z-a}dz.$$

Using basic contour integration, we can calculate the right hand side by parameterizing z in terms of θ, where $0 \le \theta \le 2\pi$, as follows:

$$z = a + \varepsilon e^{i\theta} \implies z - a = \varepsilon e^{i\theta}$$

$$\frac{dz}{d\theta} = i\varepsilon e^{i\theta} \implies dz = (i\varepsilon e^{i\theta})d\theta$$

$$\oint_{C_\varepsilon} \frac{f(z)}{z-a}dz = \oint_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}(i\varepsilon e^{i\theta})d\theta = i\oint_0^{2\pi} f(a+\varepsilon e^{i\theta})d\theta$$

Now, because ε is tiny and its value doesn't affect the initial integral, and $f(a)$ is a constant relative to θ, we have:

$$\oint_{C_\varepsilon} \frac{f(z)}{z-a}dz = \lim_{\varepsilon\to 0}\left[i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})d\theta\right] = i\int_0^{2\pi} f(a)d\theta = (i\cdot f(a))(2\pi - 0) = 2\pi i f(a).$$

Hence, using our result from earlier when defining C_ε, we have:

$$f(a) = \frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz.$$

■

Lemma .2. *Elliptic curves are an Abelian group over addition.*

Proof. As stated earlier, for any two points P, Q on an elliptic curve, $P+Q$ is the reflection across the x-axis of the third point (including multiplicity as shown below) where the line PQ intersects the curve. Now, we go through all the criteria for being a group:

(1) **Closure**: Because elliptic curves are symmetric about the x-axis, a reflection of the third point of the intersection of line PQ will be on the curve as well.
(2) **Inverse**: As shown in Case 3 above, the reflection of P across the x-axis is its inverse.
(3) **Identity Element**: The "point at infinity", "0",. where all non-horizontal lines intersect. If we take a vertical line (with points $P, -P, 0$, $P+0$ is the reflection of the 3^{rd} point their line intersects (a reflection of $-P$, which is just P. Hence, $P+0=P$.

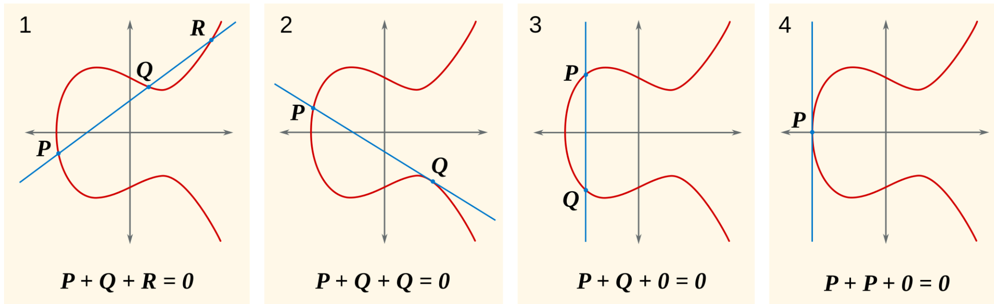

Figure 4. (source) In this operation, intersection points are counted with multiplicity (tangent points are duplicated), and a reflection across the x-axis is the inverse.

(4) **Associativity**: Please look at [4] for the full proof, as it is too intricate to summarize in this paper.

∎

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