ON THE BOUNDARY STABILITY AND CONTROL OF SEMI-DISCRETE HEAT AND WAVE SYSTEMS

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Abstract. In this paper, we examine the controllability of the heat and wave equations under a spatial discretization via finite differences with respect to the mesh-size parameters. We split the paper into two parts, where we devote the first half in showing null-controllability of the semi-discrete heat equation and how unique controls are built by way of an optimization problem which relies on a self-adjoint spatial operator. The second half consists in analyzing the spurious high frequency vibrations caused by the interaction of waves and the discrete mesh and how it induces non-uniform observability of the semi-discrete wave equation. However, by taking a subclass of solutions that satisfy nice regularity properties, provided that enough time in the system has passed, uniform observability can be recovered. As we will see, the proofs and derivations of the theorems and lemmas presented throughout heavily rely on spectral analysis, the theory of nonharmonic Fourier series, and the duality of controllability and observability.

Contents

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Part 1

Null Boundary Controllability of the Semi-Discrete Heat Equation

1. Introduction

1.1. Notation and Prerequisites

We assume that the reader is familiar with elementary functional analysis and topology. Throughout this paper, we use the following notation to classify the standard function spaces:

(1) L^1 is the space of integrable functions
(2) L^2 is the space of square-integrable functions
(3) L^p is the space of $p-$integrable functions with norm $||\cdot||_{L^p}$, i.e., if $f \in L^p(U)$, $||f||_{L^p(U)} = \left(\int_U |f|^p\right)^{1/p} < \infty$
(4) $\mathcal{H}^\infty = W^{1,2}$ is the set of all L^2 functions with 1 derivative in L^2
(5) $\mathcal{H}_0^1 = W_0^{1,2}$ is the set of all $\mathcal{H}^1$ functions that vanish at the boundary. The dual space of $\mathcal{H}_0^1$ is $\mathcal{H}^{-1}$

For more on Hilbert and Sobolev space theory (which is very relevant for deeper topics in the subject), see [9] and [5, Chapter 5]. And for the interested reader, refer to [1] for a short introduction to spectral theory and nonharmonic Fourier series.

Definition 1.1.1. *Controllability* refers to the possibility of being able to steer a system to a given final state at a given final time using a control function.

Definition 1.1.2. *Observability* refers to the measure of how well the internal states of a system can be inferred from knowledge of external outputs.

Remark. *Controllability* and *Observability* are **dual**

For brevity, we use u_t to denote the first-order partial derivative of u with respect to t and likewise u_{xx} as the second-order partial derivative of u with respect to x.

1.2. Outline

We first introduce the one-dimensional continuous controlled heat equation and apply a finite difference method to discretize the state space over the domain $x \in [0,1]$ with respect to the mesh-size parameter h. To analyze the controllability of this discrete system, we invoke a theorem by López and Zuazua in [8], for which Section 2.4 is dedicated to the proof of this theorem. Moreover, using the notion that controllability and observability are dual, we turn our attention to the *adjoint* system, whose solutions are a linear combination of the eigenfunctions of the original system, with coefficients in the family of decaying exponentials. Indeed, if we can show that uniform observability holds in this adjoint system, then we are able to construct controls that steer our original system to the origin at time $t = T$. Fortunately, we can, and it is not that hard to do so.

2. Semi-Discrete Heat Equation

2.1. Finite Difference Approximation

Consider the 1-D controlled heat equation with unit diffusion constant $K = 1$ and Dirichlet boundary conditions:

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1,\ 0 < t < T \\ u(0,t) = 0,\ u(1,t) = v(t) & 0 < t < T \\ u(x,0) = u^0 & 0 < x < 1 \end{cases} \tag{1}$$

We discretize the system with a (central) finite-difference approximation. Define a uniform mesh over $[0,1]$ and $h = \frac{1}{N+1}$ as the mesh size. Then (1) becomes:

$$\begin{cases} u_j'(t) - \left[\frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2}\right] = 0 & 0 < t < T,\ j = 1, \dots, N \\ u_0(t) = 0,\ u_{N+1}(t) = v_h(t) & 0 < t < T \\ u_j(0) = u_j^0, & j = 1, \dots, N \end{cases} \tag{2}$$

To show that (2) is uniformly null-controllable, it is easier instead to prove that the adjoint system of (2) is observable, which invokes the use of the observability inequality (see Appendix A.2) and a lemma that we will state:

Theorem 2.1 (López, Zuazua)**.** Let $T > 0$. For any sequence $\{u_j^0\}_{j=1}^N$ and $h > 0$, there exists a $v_h \in L^2(0,T)$ such that:

$$u_j(T) = 0,\ j = 1, \dots, N \tag{3}$$

And there exists a constant $C > 0$ independent of h satisfying:

$$||v_h||^2_{L^2(0,T)} \leq Ch \sum_{j=1}^{N} \left|u_j^0\right|^2 \tag{4}$$

Then given that $u^0 \in L^2(0,1)$, the controls v_h may be built such that

$$v_h \to v \tag{5}$$

in $L^2(0,T)$ as $h \to 0$. Note that this condition is satisfied only when the u_j^0 in (2) are chosen appropriately.

2.2. **The Discrete Adjoint System**

Let $\mathcal{O}_h$ be the discrete Laplacian operator such that the jth component of $\mathcal{O}_h w$ is:

$$(\mathcal{O}_h w)_j = -\frac{w_{j+1} + w_{j-1} - 2w_j}{h^2}$$

Then the spectral problem in the absence of the control is given by:

$$\begin{cases} \mathcal{O}_h w_j = \lambda w_j, \ \ j = 1, \ldots, N \\ w_0 = w_{N+1} = 0 \end{cases} \tag{6}$$

Using the substitution $w_j = \sin(j\theta)$, which satisfies $w_0 = 0$, system (6) then becomes:

$$-\frac{\sin[(j+1)\theta] + \sin[(j-1)\theta] - 2\sin(j\theta)}{h^2} = \lambda \sin(j\theta)$$

$$-\frac{2\sin(j\theta)\cos(\theta) - 2\sin(j\theta)}{h^2} = \lambda \sin(j\theta)$$

$$\lambda \sin(j\theta) = \frac{2(1-\cos\theta)\sin(j\theta)}{h^2} \iff \lambda = \frac{4}{h^2}\sin^2\left(\frac{\theta}{2}\right)$$

Using the boundary condition $w_{N+1} = 0$,

$$\sin[(N+1)\theta] = 0 \implies (N+1)\theta = k\pi$$

Thus,

$$\theta = \frac{1}{N+1}k\pi = \pi k h$$

Thus, we may write the eigenvalues and eigenvectors of (6) as:

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2}\right) \tag{7}$$

and

$$w_{k,j}(h) = \sin(j\pi k h), \ \ j, k = 1, \ldots, N \tag{8}$$

Consider now the adjoint system of (2) which is given by

$$\begin{cases} \varphi_j' + \left[\frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{h^2}\right] = 0, & 0 < t < T, \, j = 1, \ldots, N \\ \varphi_0 = \varphi_{N+1} = 0, & 0 < t < T \\ \varphi_j(T) = \varphi_j^0, & j = 1, \ldots, N. \end{cases} \tag{9}$$

The solutions to (9) may be written as

$$\varphi_j(h,t) = \sum_{k=1}^{N} a_k e^{-\lambda_k(h)(T-t)} w_{k,j}(h)$$

Proof. The operator $\mathcal{O}_h$ is a symmetric $N \times N$ matrix and can be written as:

$$\mathcal{O}_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Since its eigenvectors $w_{k,j}(h)$ form a complete basis over $\mathbb{R}^N$, we may represent the solutions of (9) as

$$\varphi_j = \sum_{k=1}^{N} p_k(t) w_{k,j}(h)$$

Substituting this into $\varphi_j' - \mathcal{O}_h \varphi_j = 0$ gives:

$$\sum_{k=1}^{N} p_k' w_{k,j} - \sum_{k=1}^{N} p_k \mathcal{O}_h w_{k,j} = 0$$

Since $\mathcal{O}_h w_{k,j} = \lambda_k w_{k,j}$ by definition,

$$\sum_{k=1}^{N} p_k' w_{k,j} - \sum_{k=1}^{N} p_k \lambda_k w_{k,j} = 0$$

$$\sum_{k=1}^{N} \left(p_k' - p_k \lambda_k\right) w_{k,j} = 0$$

$$\implies p_k' = p_k \lambda_k$$

Thus, upon solving the time-differential equation above,

$$p_k(t) = C_k e^{\lambda_k(h)t}$$

Using the boundary condition of (9), that $\varphi_j(T) = \varphi_j^0$,

$$\sum_{k=1}^{N} C_k e^{\lambda_k(h)T} w_{k,j}(h) = \varphi_j^0$$

Taking the inner product of both sides with respect to some arbitrary eigenvector $w_\ell(h)$,

$$\left\langle \sum_{k=1}^{N} C_k e^{\lambda_k(h)T} w_{k,j}(h), w_\ell(h) \right\rangle_{L^2(0,1)} = \langle \varphi_j^0, w_\ell(h)\rangle_{L^2(0,1)}$$

We note that, under a uniform mesh of $[0,1]$ for which $h = \frac{1}{N+1}$ and $x_j = jh$, the inner product $\langle \cdot, \cdot \rangle_{L^2(0,1)}$ for two functions f and g may be written as

$$\langle f, g\rangle_{L^2(0,1)} \approx h \sum_{j=1}^{N} f_j g_j$$

Thus,

$$\sum_{k=1}^{N} C_k e^{\lambda_k(h)T} \left(h \sum_{j=1}^{N} w_{k,j}(h) w_{\ell,j}(h) \right) \approx h \sum_{j=1}^{N} \varphi_j^0 w_{\ell,j}(h) \tag{10}$$

Proposition 1. The sum

$$h \sum_{j=1}^{N} w_{k,j}(h) w_{\ell,j}(h) = \begin{cases} 1/2 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

Proof of Proposition 2. Let $h \sum_{j=1}^{N} w_{k,j}(h) w_{\ell,j}(h) = \begin{cases} S_{k=\ell} & k = \ell \\ S_{k\neq\ell} & k \neq \ell \end{cases}$.

Recall that $w_{k,j} = \sin(j\pi k h)$:

$$h \sum_{j=1}^{N} w_{k,j}(h) w_{\ell,j}(h) = h \sum_{j=1}^{N} \sin(j\pi k h)\sin(j\pi \ell h)$$

Using the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$h \sum_{j=1}^{N} \sin(j\pi k h)\sin(j\pi \ell h) \iff \frac{h}{2} \sum_{j=1}^{N} \left[\cos(j\pi h(k-\ell)) - \cos(j\pi h(k+\ell))\right]$$

(1) When $k = \ell$,

$$S_{k=\ell} = \frac{h}{2} \sum_{j=1}^{N} \left[1 - \cos(2j\pi \ell h)\right]$$

Now, $\theta = \pi \ell h = \pi \ell/(N+1)$ and it is known that

$$\sum_{j=1}^{N} \cos(2j\theta) = \frac{\sin(N\theta)\cos[(N+1)\theta]}{\sin\theta}$$

Then

$$\theta = \frac{\pi\ell}{N+1} \iff (N+1)\theta = \pi\ell$$

Giving

$$\cos[(N+1)\theta] = \cos(\pi\ell) = (-1)^{\ell}$$

Similarly,

$$\theta = \frac{\pi\ell}{N+1} \iff N\theta = \frac{(N+1)\pi\ell - \pi\ell}{N+1} = \pi\ell - \frac{\pi\ell}{N+1}$$

Giving

$$\sin(N\theta) = \sin(\pi\ell - \theta) = (-1)^{\ell+1}\sin\theta$$

Thus,

$$\sum_{j=1}^{N}\cos(2j\theta) = \frac{(-1)^{2\ell+1}\sin\theta]}{\sin\theta} = -1$$

$$\therefore S_{k=\ell} = \frac{h}{2}(N+1) = \frac{1}{2}$$

(2) We can similarly prove that when $k \neq \ell$,

$$S_{k\neq\ell} = 0 \ \square$$

Using this identity, (10) becomes

$$C_{\ell}e^{\lambda_{\ell}(h)T} \approx 2h\sum_{j=1}^{N}\varphi_j^0 w_{\ell,j}(h)$$

For any C_{ℓ}, we have

$$C_{\ell} = 2he^{-\lambda_{\ell}(h)T}\sum_{j=1}^{N}\varphi_j^0 w_{\ell,j}(h) = e^{-\lambda_{\ell}(h)T}a_{\ell}$$

Giving

$$p_k(t) = e^{-\lambda_k(h)(T-t)}a_k$$

and

$$\varphi_j(h,t) = \sum_{j=1}^{N}a_k e^{-\lambda_k(h)(T-t)}w_{k,j}(h)$$

$\square$

2.3. The Discrete Observability Inequality

The uniform null-controllability for system (2) is equivalent to the uniform observability inequality for (9):

$$h \sum_{j=1}^{N} |\varphi_j(h,0)|^2 \leq C_{obs} \int_0^T \left| \frac{\varphi_N(h,t)}{h} \right|^2 dt \tag{11}$$

Then if we can prove that the adjoint system obeys the observability inequality (11), then we are done.

Proof.

Lemma 2.1 (Fattorini, Russell 1974)**.** Let ρ be a positive number and let $\mathcal{F}(\delta)$ be a positive integer-valued function defined for $\delta > 0$ such that $\mathcal{F}(\delta) \to \infty$ as $\delta \to 0$. Denote by $\mathcal{L}(\rho, \mathcal{F})$ the class of increasing positive numbers $\Lambda = \{\lambda_k\}$ that satisfy:

$$\lambda_{k+1} - \lambda_k \geq \rho \ \forall k \geq 1 \tag{12}$$

and

$$\sum_{k \geq \mathcal{F}(\delta)} \frac{1}{\lambda_k} \leq \delta, \ \forall \delta > 0 \tag{13}$$

Then there is a positive function $\mathcal{K}(\delta)$ defined for $\delta > 0$, determined solely by ρ and $\mathcal{F}$, such that

$$||q_n||_{L^2(0,\infty)} \leq \mathcal{K}(\delta) e^{\delta \lambda_k}, \ \forall k \geq 1 \tag{14}$$

where $\{q_n\}$ is the *biorthogonal* sequence for $\{p_k\}$, where $p_k(t) = e^{-\lambda_k t}$.

Recall that

$$\lambda_k(h) = \frac{4}{h^2} \sin^2 \left(\frac{\pi k h}{2} \right)$$

Then as $h \to 0$ (or as $N \to \infty$),

$$\sin \left(\frac{\pi k h}{2} \right) = \sin \left(\frac{\pi k}{2(N+1)} \right) \approx \frac{\pi k}{2(N+1)}$$

And hence,

$$\lambda_k \approx \pi^2 k^2$$

Giving

$$\lambda_{k+1} - \lambda_k \approx \pi^2 (2k+1) \geq 3\pi^2 > 0$$

for each $k \geq 1$ and

$$\sum_{k \geq \mathcal{F}} \frac{1}{\lambda_k} \leq \frac{1}{\pi^2 (\mathcal{F} - 1)} \leq \delta$$

And we may choose

$$\mathcal{F}(\delta) = \left\lceil 1 + \frac{1}{\pi^2\delta^2} \right\rceil$$

Indeed, as $\delta \to 0$, $\mathcal{F}(\delta) \to \infty$.
Hence, it is sufficient to see that the sequence of eigenvalues $\{\lambda_k(h)\}$ belong to $\mathcal{L}(\rho, \mathcal{F})$ for suitable $\rho > 0$ and $\mathcal{F} : (0, \infty) \to \mathbb{N}$.

Corollary 2.3.1. Given a class of sequences $\mathcal{L}(\rho, \mathcal{F})$ and $T > 0$, there exists a constant $C(\rho, \mathcal{F}, T) > 0$ (that is, C depends on ρ, $\mathcal{F}$, and T) such that:

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \right|^2 \geq \frac{C}{\sum_{k\geq 1} \lambda_k^{-1}} \sum_{k\geq 1} \frac{|a_k|^2 e^{-2\lambda_k T}}{\lambda_k} \tag{15}$$

$\forall \{\lambda_k\} \in \mathcal{L}(\rho, \mathcal{F})$.

Using (2.3.1) on the solution of the adjoint system (9), indeed, we can show that if

$$\varphi_j(h,t) = \sum_{k=1}^{N} a_k e^{-\lambda_k(h)(T-t)} w_{k,j}(h)$$

and hence

$$\frac{\varphi_N(h,t)}{h} = \sum_{k=1}^{N} a_k e^{-\lambda_k(h)(T-t)} \frac{w_{k,N}(h)}{h}$$

then

$$\begin{aligned}
\int_0^T \left| \frac{\varphi_N(h,t)}{h} \right|^2 dt &= \int_0^T \left| \sum_{k=1}^{N} a_k \frac{w_{k,N}(h)}{h} e^{-\lambda_k(h)(T-t)} \right|^2 dt \\
&\geq \frac{C}{\sum_{k=1}^{N} \lambda_k^{-1}} \sum_{k=1}^{N} \frac{|a_k \frac{w_{k,N}(h)}{h}|^2 e^{-2\lambda_k T}}{\lambda_k} \\
&\geq C' \sum_{k=1}^{N} |a_k|^2 e^{-2\lambda_k T}
\end{aligned}$$

Now, note that

$$\varphi_j(h,0) = \sum_{k=1}^{N} a_k e^{-\lambda_k(h)T} w_{k,j}(h)$$

then

$$h \sum_{j=1}^{N} |\varphi_j(h,0)|^2 = \frac{1}{2} \sum_{k=1}^{N} |a_k|^2 e^{-2\lambda_k(h)T}$$

where we have used the identity

$$h\sum_{j=1}^{N} w_{k,j}(h)w_{\ell,j}(h) = \begin{cases} 1/2 & k=\ell \\ 0 & k\neq \ell \end{cases}$$

Thus,

$$\begin{aligned}\int_0^T \left|\frac{\varphi_N(h,t)}{h}\right|^2 dt &\geq 2C'\left(\frac{1}{2}\sum_{k=1}^{N}|a_k|^2 e^{-2\lambda_k T}\right)\\ &= 2C'h\sum_{j=1}^{N}|\varphi_j(h,0)|^2\end{aligned}$$

Define $C_{obs} = 1/2C'$, and hence

$$h\sum_{j=1}^{N}|\varphi_j(h,0)|^2 \leq C_{obs}\int_0^T \left|\frac{\varphi_N(h,t)}{h}\right|^2 dt$$

which proves that there exist discrete controls v_h that drive system (2) back to the origin. □

2.4. The Hilbert Uniqueness Method and the Convergence of Controls

Now that we have shown that such a v_h exists, we will use this section to prove Theorem (2.1) via the **Hilbert Uniqueness Method** assuming that the observability inequality (11) holds.
Assuming that it holds, then for each $j = 1,\ldots,N$, $\exists v_h \in L^2(0,T)$ such that:

$$u_j(T) = 0,\ j = 1,\ldots,N$$

Indeed, by the Hilbert Uniqueness Method, we construct the discrete functional J_h defined as

$$J_h(\varphi^0) = \frac{1}{2}\int_0^T \left|\frac{\varphi_N(h,t)}{h}\right|^2 dt - h\sum_{j=1}^{N}\varphi_j(h,0)u_j^0 \tag{16}$$

where $\varphi = (\varphi_1,\ldots,\varphi_N)$ is the solution of the adjoint system (9) with initial datum $\varphi^0 = (\varphi_1^0,\ldots,\varphi_N^0)$ and $u^0 = (u_1^0,\ldots,u_N^0)$ is to be controlled.
By definition, if $\Phi^0 = (\Phi_1^0,\ldots,\Phi_N^0)$ minimizes (16),

$$DJ_h(\Phi^0) = 0$$

then using the method illustrated in Appendix A.2, we find that there exists a unique control $v_h(t)$ given by

$$v_h = -\frac{\Phi_N(h,t)}{h}$$

And

$$\int_0^T |v_h(t)|^2 dt = h \sum_{j=1}^{N} \Phi_j(h,0) u_j^0$$

$$\implies ||v_h(t)||^2_{L^2(0,T)} = h \sum_{j=1}^{N} \Phi_j(h,0) u_j^0$$

Using the Cauchy-Schwartz inequality, this gives

$$||v_h(t)||^2_{L^2(0,T)} \leq \left(h \sum_{j=1}^{N} |u_j^0|^2 \right)^{\frac{1}{2}} \left(h \sum_{j=1}^{N} |\Phi_j(h,0)|^2 \right)^{\frac{1}{2}}$$

Recall the observability inequality (11)
Using this,

$$||v_h(t)||^2_{L^2(0,T)} \leq \left(h \sum_{j=1}^{N} |u_j^0|^2 \right)^{\frac{1}{2}} \left(C_{obs} ||v_h(t)||^2_{L^2(0,T)} \right)^{\frac{1}{2}}$$

$$||v_h(t)||^2_{L^2(0,T)} \leq \left(h \sum_{j=1}^{N} |u_j^0|^2 \right)^{\frac{1}{2}} (C_{obs})^{\frac{1}{2}} ||v_h(t)||_{L^2(0,T)}$$

$$||v_h(t)||^2_{L^2(0,T)} \leq C_{obs} h \sum_{j=1}^{N} |u_j^0|^2$$

Indeed, since C_{obs} has no h-dependence, then as $h \to 0$,

$$||v_h||_{L^2(0,T)} \to ||v||_{L_2(0,T)}$$

where

$$v = -\partial_x \Phi(1,t)$$

Part 2

Boundary Controllability of the Semi Discrete 1-D Wave Equation

1. Introduction

1.1. Outline

Unlike the semi-discrete heat equation, the discrete scheme of the wave equation does not behave so regularly. In fact, due to spurious high frequencies, the observability constant C blows up as the mesh size h approaches zero. However, if we consider a subclass of the solutions of this discrete system and equip it with nice properties, then uniform observability could be constructed.
We dedicate section 2.1 for introducing the finite difference approximation scheme of the continuous 1-d wave equation with Dirichlet boundary conditions. Section 2.2 is for analyzing eigenvalues and eigenvectors and deriving an important identity that we will find ourselves using often. Section 2.3 shows how uniform observability fails globally. However, Section 2.4 shows how we can reconstruct uniform observability locally by considering a subspace of the solutions.

2. Semi-Discrete Scheme of the 1-D Wave Equation

2.1. Finite Difference Approximation

The 1-D wave equation with homogeneous Dirichlet boundary conditions is given by:

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < 1, 0 < t < T \\ u(0,t) = u(1,t) = 0 & 0 < t < T \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & 0 < x < 1 \end{cases} \tag{17}$$

Where the total energy is defined as

$$E(t) = \frac{1}{2}\int_0^1 \left(|u_x(x,t)|^2 + |u_t(x,t)|^2\right) dx = E(0),\ \forall 0 \le t \le T \tag{18}$$

In this section, we will derive a finite difference approximation of (17) and its total energy (18). We will also introduce the discrete version

of the observability inequality which is similarly proven using the same methodology (unlike the one we saw for the heat equation, which relies on the solution of the adjoint system). By Semi-Discrete, we mean that only the spatial domain is discretized. For fully discrete boundary control, refer to [3].
The finite-difference approximation of (17) is given by

$$\text{(19)}\qquad \begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \ldots, N \\ u_0(t) = u_{N+1}(t) = 0, & 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \ldots, N \end{cases}$$

Where $u_j(0) = u(x_j, 0)$ and $u_j'(0) = u_j'(x_j, 0)$ are approximations of u_j^0 and u_j^1 in (17).

Proof. Suppose $u(x,t)$ is the sufficiently smooth and regular solution of the 1-D wave equation (where we take $c = 1$ for simplicity), defined on a spatial domain $\Omega = (0,1)$ for time $t > 0$ subject to the Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. Define a uniform mesh over space by partitioning $[0,1]$ into $N+1$ uniform segments, so that the spatial mesh size is given by $h = \frac{1}{N+1}$ and

$$0 = x_0 < x_1 < \ldots < x_{N+1} = 1$$

Using a Taylor series expansion of $u(x,t)$ at neighboring nodes $x_{j-1} = x_j - h$ and $x_{j+1} = x_j + h$, we get:

(20)
$$u(x_{j-1}, t) = u(x_j, t) - hu_x(x_j, t) + \frac{h^2}{2!}u_{xx}(x_j, t) - \frac{h^3}{3!}u_{xxx}(x_j, t) + \mathcal{O}(h^4)$$

(21)
$$u(x_{j+1}, t) = u(x_j, t) + hu_x(x_j, t) + \frac{h^2}{2!}u_{xx}(x_j, t) + \frac{h^3}{3!}u_{xxx}(x_j, t) + \mathcal{O}(h^4)$$

Then adding (20) to (21) yields:

$$\text{(22)}\qquad u(x_{j-1}, t) + u(x_{j+1}, t) = 2u(x_j, t) + h^2 u_{xx}(x_j, t) + \mathcal{O}(h^4)$$

Rearranging for u_{xx} gives:

$$\text{(23)}\qquad u_{xx}(x_j, t) = \frac{u(x_{j+1}, t) + u(x_{j-1}, t) - 2u(x_j, t)}{h^2} - \mathcal{O}(h^2)$$

Dropping the second-degree error $\mathcal{O}(h^2)$ and substituting (23) into (17) gives:

$$u_{tt}(x_j,t) - \frac{u(x_{j+1},t)+u(x_{j-1},t)-2u(x_j,t)}{h^2} = 0 \tag{24}$$

Since time is kept continuous in our approximation scheme, we may use the notational convention $u_{tt} = u''$ and because $u_j(t) \approx u(x_j,t)$, (24) may be written as:

$$u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0, \quad 0 < t < T,\ j = 1,\dots,N \tag{25}$$

□

Corollary 2.1.1. The energy of the solutions of (19) is then given by

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2 + \left|\frac{u_{j+1}(t)-u_j(t)}{h}\right|^2\right] \tag{26}$$

Proof. From $E(t) = \frac{1}{2}\int_0^1 (|u_x(x,t)|^2 + |u_t(x,t)|^2)\,dx$, suppose we define a uniform mesh on $[0,1]$ as before, by partitioning $[0,1]$ into $N+1$ uniform segments with $h = \frac{1}{N+1}$. Using the definition of the Riemann sum, $\Delta x = x_j - x_{j-1} = h$ so that:

$$E(t) \approx \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2 + \left|\frac{u_{j+1}(t)-u_j(t)}{h}\right|^2\right] =: E_h(t)$$

Where we have made the finite-difference approximation $u_x(x,t) \approx \frac{u_{j+1}(t)-u_{j-1}(t)}{h}$. □

The discrete observability inequality for the discrete wave equation is thus given by

$$E_h(0) \le C(T,h)^2 \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \tag{27}$$

Proposition 2. For any $T > 0$,

$$\lim_{h\to 0}\sup_{u}\left[\frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt}\right] \to \infty \tag{28}$$

where u is a solution for (19).

2.2. **The Eigenvalue Problem**

As before, denote $\mathcal{O}_h$ as the discrete Laplacian operator. The eigenvalue problem associated with (19) is given by

$$\begin{cases} \mathcal{O}_h w_j = \lambda w_j & j = 1, \ldots, N \\ w_0 = w_{N+1} = 0 \end{cases} \tag{29}$$

Using a substitution $w_j = \sin(j\theta)$ as before, we get

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2}\right)$$

and

$$w_{k,j} = \sin(j\pi k h)$$

Then since the eigenvectors $w_{k,j}$ form a complete basis over $\mathbb{R}^N$, the state space, every solution $u = (u_1, \ldots, u_N)$ of (19) can be written as

$$u(t) = \sum_{k=1}^{N}\left[a_k \sin\left(\sqrt{\lambda_k t}\right) + b_k \cos\left(\sqrt{\lambda_k t}\right)\right] w_k$$

For $w_k = (w_{k,1}, \ldots, w_{k,N})$.
Or more explicitly,

$$u_j(t) = \sum_{k=1}^{N}\left[a_k \sin\left(\sqrt{\lambda_k t}\right) + b_k \cos\left(\sqrt{\lambda_k t}\right)\right] w_{k,j}$$

Proof. Using $w_{k,j}$ as the basis vectors, we may write each solution u_j in the form

$$u_j(t) = \sum_{k=1}^{N} c_k(t) w_{k,j}(h)$$

Substituting this into (19),

$$u_j'' + \mathcal{O}_h u_j = 0$$

$$\sum_{k=1}^{N} c_k''(t) w_{k,j}(h) + \sum_{k=1}^{N} c_k(t)\mathcal{O}_h w_{k,j}(h) = 0$$

By definition, $\mathcal{O}_h w_{k,j} = \lambda_k w_{k,j}$
Thus,

$$\sum_{k=1}^{N} c_k''(t) w_{k,j}(h) + \sum_{k=1}^{N} c_k(t)\lambda_k(h) w_{k,j}(h) = 0$$

$$\sum_{k=1}^{N}\left[c_k''(t) + c_k(t)\lambda_k(h)\right] w_{k,j}(h) = 0$$

Since the $w_{k,j}$ are independent and pairwise orthogonal, then the only way the sum evaluates to zero is if each coefficient for $k = 1, \dots, N$ equals zero (linear algebra 101):

$$c_k''(t) + c_k(t)\lambda_k(h) = 0$$

From physics 1, we know that the solution for c_k is thus

$$c_k(t) = a_k \sin\left(\sqrt{\lambda_k}t\right) + b_k \cos\left(\sqrt{\lambda_k}t\right)$$

Giving

$$u_j(t) = \sum_{k=1}^{N} \left[a_k \sin\left(\sqrt{\lambda_k}t\right) + b_k \cos\left(\sqrt{\lambda_k}t\right)\right] w_{k,j}$$

$\square$

Lemma 2.1. For any eigenvector $w_k = (w_{k,1}, \dots, w_{k,N})$ of (29),

$$h \sum_{j=0}^{N} \left| \frac{w_{k,j+1} - w_{k,j}}{h} \right|^2 = \frac{2}{4 - \lambda_k h^2} \left| \frac{w_{k,N}}{h} \right|^2 \tag{30}$$

Proof 1 (Direct Substitution).
Recall that

$$w_{k,j} = \sin(j\pi kh)$$

Thus,

$$\begin{aligned} w_{k,j+1} - w_{k,j} &= \sin\left[(j+1)\pi kh\right] - \sin(j\pi kh) \\ &= 2\sin\left(\frac{\pi kh}{2}\right)\cos\left(j\pi kh + \frac{\pi kh}{2}\right) \end{aligned}$$

Substituting this into the LHS of (30),

$$\begin{aligned} h \sum_{j=0}^{N} \left| \frac{w_{k,j+1} - w_{k,j}}{h} \right|^2 &= \frac{1}{h} \sum_{j=0}^{N} 4\sin^2\left(\frac{\pi kh}{2}\right)\cos^2\left(j\pi kh + \frac{\pi kh}{2}\right) \\ &= \frac{4}{h}\sin^2\left(\frac{\pi kh}{2}\right) \sum_{j=0}^{N} \left[\frac{1}{2} + \frac{1}{2}\cos(2j\pi kh + \pi kh)\right] \\ &= \frac{4}{h}\sin^2\left(\frac{\pi kh}{2}\right)\left(\frac{1}{2h}\right) = \frac{2}{h^2}\sin^2\left(\frac{\pi kh}{2}\right) \end{aligned}$$

Now,

$$\begin{aligned}
\frac{2}{h^2}\sin^2\left(\frac{\pi k h}{2}\right) &= \frac{4\sin^2\left(\frac{\pi kh}{2}\right)\cos^2\left(\frac{\pi kh}{2}\right)}{2h^2\cos^2\left(\frac{\pi kh}{2}\right)} \\
&= \frac{1}{2\cos^2\left(\frac{\pi kh}{2}\right)}\frac{\sin^2(\pi kh)}{h^2} \\
&= \frac{1}{2-2\sin^2\left(\frac{\pi kh}{2}\right)}\left|\frac{\sin\left[(N+1)\pi kh - \pi kh\right]}{h^2}\right|^2 \\
&= \frac{2}{4-\frac{4}{h^2}\sin^2\left(\frac{\pi kh}{2}\right)h^2}\left|\frac{\sin(N\pi kh)}{h}\right|^2 \\
&= \frac{2}{4-\lambda_k h^2}\left|\frac{w_{k,N}}{h}\right|^2
\end{aligned}$$

□

Proof 2 (Multiplier Method).
Roughly speaking, the *Multiplier Method* is a technique used to simplify long, convoluted sums into nicer looking ones.
So, suppose we multiply (29) by a *multiplier* $j\left(\frac{w_{k,j+1}-w_{k,j-1}}{2}\right)$ and sum from $j = 1, \dots, N$:

$$\begin{aligned}
&\sum_{j=1}^{N} j\left(\frac{w_{k,j+1}+w_{k,j-1}-2w_{k,j}}{h^2}\right)\left(\frac{w_{k,j+1}-w_{k,j-1}}{2}\right) \\
&\quad = -\sum_{j=1}^{N} j\lambda_k w_{k,j}\left(\frac{w_{k,j+1}-w_{k,j-1}}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2h^2}\sum_{j=1}^{N} j\left[|w_{k,j+1}|^2 - |w_{k,j-1}|^2\right] - \frac{1}{h^2}\sum_{j=1}^{N} jw_{k,j}(w_{k,j+1}-w_{k,j-1}) \\
&\quad = -\frac{\lambda_k}{2}\sum_{j=1}^{N} jw_{k,j}(w_{k,j+1}-w_{k,j-1})
\end{aligned}$$

Evaluating the telescoping series gives

$$\begin{aligned}
&\frac{1}{h^2}\left[\sum_{j=1}^{N}|w_{k,j}|^2 - \frac{N+1}{2}|w_{k,N}|^2 - \sum_{j=1}^{N} w_{k,j}w_{k,j+1}\right] \\
&\quad = -\frac{\lambda_k}{2}\sum_{j=1}^{N} w_{k,j+1}w_{k,j}
\end{aligned}$$

On the other hand, multiplying (29) by $w_{k,j}$ gives

$$\sum_{j=1}^{N} w_{k,j}\left(\frac{w_{k,j+1}+w_{k,j-1}-2w_{k,j}}{h^2}\right) = -\lambda_k \sum_{j=1}^{N} |w_{k,j}|^2$$

Where

$$\sum_{j=1}^{N}(w_{k,j+1}+w_{k,j-1}-2w_j)w_{k,j} = 2\left(\sum_{j=1}^{N} w_{k,j+1}w_{k,j} - \sum_{j=1}^{N}|w_{k,j}|^2\right)$$

Thus,

$$\frac{2}{h^2}\sum_{j=1}^{N}\left(w_{k,j+1}w_{k,j} - |w_{k,j}|^2\right) = -\lambda_k \sum_{j=1}^{N}|w_{k,j}|^2$$

And this may be rearranged to give

$$\sum_{j=1}^{N} w_{k,j+1}w_{k,j} = \left(1-\frac{\lambda_k h^2}{2}\right)\sum_{j=1}^{N}|w_{k,j}|^2$$

Substituting this into where we left off previously, we get

$$\begin{aligned}
&\frac{1}{h^2}\left[\sum_{j=1}^{N}|w_{k,j}|^2 - \frac{N+1}{2}|w_{k,N}|^2 - \left(1-\frac{\lambda_k h^2}{2}\right)\sum_{j=1}^{N}|w_{k,j}|^2\right] \\
&\quad = -\frac{\lambda_k}{2}\left(1-\frac{\lambda_k h^2}{2}\right)\sum_{j=1}^{N}|w_{k,j}|^2 \\
&\frac{\lambda_k}{2}\sum_{j=1}^{N}|w_{k,j}|^2 - \frac{N+1}{2h^2}|w_{k,N}|^2 = \left(-\frac{\lambda_k}{2}+\frac{\lambda_k^2 h^2}{4}\right)\sum_{j=1}^{N}|w_{k,j}|^2
\end{aligned}$$

That is,

$$\begin{aligned}
\left(\lambda_k - \frac{\lambda_k^2 h^2}{4}\right)\sum_{j=1}^{N}|w_{k,j}|^2 &= \frac{N+1}{2h^2}|w_{k,N}|^2 \\
\frac{\lambda_k}{4}(4-\lambda_k h^2)\sum_{j=1}^{N}|w_{k,j}|^2 &= \frac{1}{2h^3}|w_{k,N}|^2 \\
\sum_{j=1}^{N}|w_{k,j}|^2 &= \frac{2}{\lambda_k h^3(4-\lambda_k h^2)}|w_{k,N}|^2
\end{aligned}$$

Shifting indices and using all the results that we have discovered using the multiplier method above, we finally prove the statement of the lemma

$$\begin{aligned}
h\sum_{j=0}^{N}\left|\frac{w_{k,j}-w_{k,j+1}}{h}\right|^2 &= \frac{1}{h}\sum_{j=0}^{N}\left(|w_{k,j}|^2+|w_{k,j+1}|^2-2w_{k,j}w_{k,j+1}\right)\\
&= \frac{2}{h}\left(\sum_{j=1}^{N}|w_{k,j}|^2-\sum_{j=1}^{N}w_{k,j}w_{k,j+1}\right)\\
&= \frac{2}{h}\left(\sum_{j=1}^{N}|w_{k,j}|^2-\left(1-\frac{\lambda_k h^2}{2}\right)\sum_{j=1}^{N}|w_{k,j}|^2\right)\\
&= \frac{2}{h}\left(\frac{\lambda_k h^2}{2}\sum_{j=1}^{N}|w_{k,j}|^2\right)\\
&= \lambda_k h\sum_{j=1}^{N}|w_{k,j}|^2\\
&= \lambda_k h\frac{2}{\lambda_k h^3(4-\lambda_k h^2)}|w_{k,N}|^2\\
&= \frac{2}{4-\lambda_k h^2}\left|\frac{w_{k,N}}{h}\right|^2
\end{aligned}$$

□

2.3. **Discrete Nonuniform Observability**

We will dedicate this subsection to proving Proposition (2), which is a direct consequence of Lemma (2.1). That is,

$$\lim_{h\to 0}\sup_{u}\left[\frac{E_h(0)}{\int_0^T\left|\frac{u_N(t)}{h^2}\right|^2 dt}\right]\to\infty$$

Recall that the solution of (19) may be written as

$$u_j(t)=\sum_{k=1}^{N}\left[a_k\sin\left(\sqrt{\lambda_k}t\right)+b_k\cos\left(\sqrt{\lambda_k}t\right)\right]w_{k,j}$$

Then let u be the solution associated with the Nth eigenfunction,

$$\begin{aligned}
u_j(t) &= e^{i\sqrt{\lambda_N}t}w_{N,j}\\
\Longrightarrow u_j'(t) &= i\sqrt{\lambda_N}e^{i\sqrt{\lambda_N}t}w_{N,j}
\end{aligned}$$

Recall that the energy of the solution u is given by

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2 + \left|\frac{u_{j+1}(t)-u_j(t)}{h}\right|^2\right]$$

Then at $t=0$,

$$\begin{aligned}
E_h(0) &= \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(0)|^2 + \left|\frac{u_{j+1}(0)-u_j(0)}{h}\right|^2\right]\\
&= \frac{h}{2}\sum_{j=0}^{N}\left[\lambda_N|w_{N,j}|^2 + \left|\frac{w_{N,j}-w_{N,j+1}}{h}\right|^2\right]\\
&= h\sum_{j=0}^{N}\left|\frac{w_{N,j}-w_{N,j+1}}{h}\right|^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt &= \int_0^T \left|\frac{e^{i\sqrt{\lambda_N}t}w_{N,N}}{h^2}\right|^2 dt\\
&= \int_0^T \left|\frac{w_{N,N}(h)}{h^2}\right|^2 dt\\
&= T\left|\frac{w_{N,N}}{h^2}\right|^2
\end{aligned}$$

Using the identity

$$\frac{2}{4-\lambda_N h^2}\left|\frac{w_{N,N}}{h}\right|^2 = h\sum_{j=0}^{N}\left|\frac{w_{N,j+1}-w_{N,j}}{h}\right|^2$$

$$\left|\frac{w_{N,N}}{h}\right|^2 = \frac{h}{2}(4-\lambda_N h^2)\sum_{j=0}^{N}\left|\frac{w_{N,j}-w_{N,j+1}}{h}\right|^2$$

Hence,

$$\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt = \frac{Th}{2}(4-\lambda_N h^2)\sum_{j=0}^{N}\left|\frac{w_{N,j}-w_{N,j+1}}{h}\right|^2$$

We deduce that

$$\frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt} = \frac{2}{T(4-\lambda_N h^2)}$$

Recall that

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2}\right)$$
$$\implies \lambda_N(h)h^2 = 4\sin^2\left(\frac{\pi N h}{2}\right)$$
$$= 4\cos\left(\frac{\pi h}{2}\right) \to 4$$

as $h \to 0$. However, this implies that

$$\frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h^2}\right|^2 dt} \to \infty$$

which shows that the 1-d semi-discrete wave equation is globally non-observable.

2.4. **Recovering Boundary Observability**

As we have seen, the discrete wave equation does not behave nicely globally. However, it is well known that if we can methodically choose some $\gamma < 4$ such that $\lambda h^2 \le \gamma$, provided that T is large enough, then uniform observability holds in this subclass of solutions u!
Formally, we denote this subclass as $C_h(\gamma)$, where

$$C_h(\gamma) := \left\{ u = \sum_{\lambda_k \le \gamma h^{-2}} \left[a_k \sin\left(\sqrt{\lambda_k} t\right) + b_k \cos\left(\sqrt{\lambda_k} t\right)\right] w_k,\ \ a_k, b_k \in \mathbb{R} \right\}$$

Theorem 2.1. Let $\gamma < 4$. Then there exists $T(\gamma) \ge 2$ such that for all $T > T(\gamma)$, there exists a $C = C(T, \gamma)$ such that the observability inequality (27) holds for every solution u of (19) in the class $C_h(\gamma)$.

In fact, $T(\gamma) \to \infty$ as $\gamma \to 4$ since the gap between consecutive eigenvalues become indistinguishable as k increases.
But also, $T(\gamma) \to 2$ as $\gamma \to 0$ where $T(\gamma) = 2$ is the observability time of (19).
As [4] intuitively puts it, uniform observability is achieved provided the high frequencies are filtered. In fact, the spurious high frequency vibrations that are caused by the interaction of waves between the discrete mesh are not observed as $h \to 0$ which causes non-uniform observability.
To prove Theorem (2.1), we need the following lemma:

Lemma 2.2. Let

$$\gamma(\varepsilon) = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right) \tag{31}$$

for $\varepsilon \in [0, 1)$.
Indeed, $0 \leq \gamma = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right) < 4$ for such ε.
Then

$$\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j} \geq \pi \cos\left(\frac{\pi\varepsilon}{2}\right) \tag{32}$$

for all eigenvalues in the range

$$\lambda h^2 \leq \gamma \tag{33}$$

Proof. Recall that $\lambda_j = \frac{4}{h^2}\sin^2\left(\frac{\pi j h}{2}\right)$. Using (33), using the fact that $\lambda_{j+1} > \lambda_j$ and the set of eigenvalues $\Lambda = \{\lambda_k\}$ is an increasing sequence,

$$\begin{aligned}
\frac{4}{h^2}\sin^2\left(\frac{\pi(j+1)h}{2}\right) h^2 &\leq \gamma(\varepsilon) \\
\sin^2\left(\frac{\pi(j+1)h}{2}\right) &\leq \sin^2\left(\frac{\pi\varepsilon}{2}\right) \\
\implies (j+1)h \leq \varepsilon &
\end{aligned}$$

Then the gap between consecutive eigenvalues can be bounded as follows:

$$\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j} = \frac{2}{h}\left[\sin\left(\frac{\pi(j+1)h}{2}\right) - \sin\left(\frac{\pi j h}{2}\right)\right]$$

Using the mean value theorem on $f(x) = \sin x$ over the interval $\left[\frac{\pi j h}{2}, \frac{\pi(j+1)h}{2}\right]$, we get:

$$\sin\left(\frac{\pi(j+1)h}{2}\right) - \sin\left(\frac{\pi j h}{2}\right) = \frac{\pi h}{2}\cos(\xi), \quad \xi \in \left[\frac{\pi j h}{2}, \frac{\pi(j+1)h}{2}\right]$$

Thus,

$$\begin{aligned}
\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j} &= \frac{2}{h}\left[\frac{\pi h}{2}\cos(\xi)\right] \\
&= \pi\cos(\xi)
\end{aligned}$$

Now, $(j+1)h \leq \varepsilon$ gives

$$0 \leq \xi \leq \frac{\pi(j+1)h}{2} \leq \frac{\pi\varepsilon}{2}$$

and finally, we prove (32):

$$\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j} = \pi\cos(\xi) \geq \pi\cos\left(\frac{\pi\varepsilon}{2}\right)$$

□

Now, in order to prove that

$$E_h(0) \le C(T,\gamma)\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt$$

We use Inhgham's inequality, and show that for a solution $u \in C_h(\gamma)$, the observability inequality above holds true:

Lemma 2.3 (Ingham)**.** For any $0 < \varepsilon < 1$ and $T > 2/\cos(\frac{\pi\varepsilon}{2})$, there exists a constant $C_1(T,\varepsilon) > 0$ such that

$$C_1(T,\varepsilon)\sum_{\lambda h^2 \le \gamma(\varepsilon)} |a_k|^2 \le \int_0^T \left|\sum_{\lambda h^2 \le \gamma(\varepsilon)} a_k e^{i\sqrt{\lambda}t}\right|^2 dt \tag{34}$$

Indeed, for a solution of (19) $u \in C_h(\gamma)$, we may write u in the form

$$u_j(t) = \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k t} w_{k,j}$$

where $|\mu| = \sqrt{\lambda}$.
Moreover,

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \frac{1}{h^2}\int_0^T \left|\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k t} w_{k,N}\right|^2 dt$$

Thus, applying (34) gives

$$\int_0^T \left|\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k t} w_{k,N}\right|^2 dt \ge C_1(T,\varepsilon)\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} |a_k|^2 |w_{k,N}|^2$$

Recall now the identity

$$h\sum_{j=0}^N \left|\frac{w_{k,j+1}-w_{k,j}}{h}\right|^2 = \frac{2}{4-\lambda_k h^2}\left|\frac{w_{k,N}}{h}\right|^2$$

For eigenvalues in the range $\lambda h^2 \le \gamma(\varepsilon)$,

$$h\sum_{j=0}^N \left|\frac{w_{k,j+1}-w_{k,j}}{h}\right|^2 \le \frac{2}{4-\gamma(\varepsilon)}\left|\frac{w_{k,N}}{h}\right|^2 = \frac{1}{2\cos^2\left(\frac{\pi\varepsilon}{2}\right)}\left|\frac{w_{k,N}}{h}\right|^2$$

Thus,

$$|w_{k,N}|^2 \ge 2h^3\cos^2\left(\frac{\pi\varepsilon}{2}\right)\sum_{j=0}^N \left|\frac{w_{k,j+1}-w_{k,j}}{h}\right|^2$$

Giving

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \geq \frac{C_1(T,\varepsilon)}{h^2} \sum_{|\mu_k|h \leq \sqrt{\gamma(\varepsilon)}} |a_k|^2 |w_{k,N}|^2$$

$$\geq \frac{C_1(T,\varepsilon)}{h^2} \sum_{|\mu_k|h \leq \sqrt{\gamma(\varepsilon)}} |a_k|^2 \left(2h^3 \cos^2\left(\frac{\pi\varepsilon}{2}\right) \sum_{j=0}^{N} \left|\frac{w_{k,j+1} - w_{k,j}}{h}\right|^2\right)$$

$$= 2C_1(T,\varepsilon) \cos^2\left(\frac{\pi\varepsilon}{2}\right) \sum_{|\mu_k|h \leq \sqrt{\gamma(\varepsilon)}} \left(|a_k|^2 h \sum_{j=0}^{N} \left|\frac{w_{k,j+1} - w_{k,j}}{h}\right|^2\right)$$

Note that

$$E_h(0) = \sum_{|\mu_k|h \leq \sqrt{\gamma(\varepsilon)}} \left(|a_k|^2 h \sum_{j=0}^{N} \left|\frac{w_{k,j+1} - w_{k,j}}{h}\right|^2\right)$$

Thus,

$$2C_1(T,\varepsilon) \cos^2\left(\frac{\pi\varepsilon}{2}\right) E_h(0) \leq \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt$$

$$E_h(0) \leq \frac{1}{2C_1(T,\varepsilon)\cos^2\left(\frac{\pi\varepsilon}{2}\right)} \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt$$

Since $\gamma(\varepsilon) = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right)$, then

$$\cos^2\left(\frac{\pi\varepsilon}{2}\right) = 1 - \frac{\gamma}{4}$$

Giving

$$E_h(0) \leq \frac{1}{2\left(1 - \frac{\gamma}{4}\right) C_1(T,\varepsilon)} \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt$$

Thus, Theorem (2.1) holds iff we choose

$$T(\gamma) = \frac{2}{\cos\left(\frac{\pi\varepsilon}{2}\right)}$$

$$C(T,\gamma) = \frac{1}{2\left(1 - \frac{\gamma}{4}\right) C_1(T,\varepsilon)}$$

provided that $\gamma = \gamma(\varepsilon)$.

Appendices

A. The Hilbert Uniqueness Method

A.1. **The N-Dimensional Control Problem**

Consider the finite-dimensional system

$$\dot{x} = Ax + Bv, \quad 0 \leq t \leq T \tag{35}$$

Which has initial datum $x(0) = x^0$ and where A is an $N \times N$ square matrix with real, constant coefficients and B an $N \times M$ matrix.
Here, $v = v(t) \in \mathbb{R}^M$ is the $M-$dimensional control where $M \leq N$, a function which steers the system to any final state x^T at time T.
In the context of null-controllability, we ask ourselves the question:

"does there exist a control $v : [0, T] \to \mathbb{R}^M$ so that $x(T) = 0$?"

Theorem A.1 (Kalman Control Condition)**.**
System (35) is controllable at some time $T > 0$ if and only if

$$\operatorname{rank}\big(\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{N-1}\mathbf{B}\big) = N$$

Proof. Using an integrating factor, we may express the solution of (35) at time $t = T$ as:

$$x(T) = e^{\mathbf{A}T}x(0) + \int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}v(t)dt$$

By null-controllability, $x(T) = 0$:

$$\int_0^T e^{-\mathbf{A}t}\mathbf{B}v(t)dt = -x(0)$$

By Cayley-Hamilton, since A is a square $N \times N$ matrix, we may express

$$e^{-\mathbf{A}t} = \sum_{k=0}^{N-1} c_k(t)\mathbf{A}^k$$

Let

$$\int_0^T c_k(t)v(t)dt = v_k$$

$$\therefore -x(0) = \sum_{k=0}^{N-1} \mathbf{A}^k\mathbf{B}v_k = \begin{bmatrix}\mathbf{B}, & \mathbf{AB}, & \ldots, & \mathbf{A}^{N-1}\mathbf{B}\end{bmatrix}\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1}\end{bmatrix}$$

Then system (35) is controllable iff $x(0)$ lies within the column space of

$$\begin{bmatrix}\mathbf{B}, & \mathbf{AB}, & \ldots, & \mathbf{A}^{N-1}\mathbf{B}\end{bmatrix}$$

where if each column vector is linearly independent so as to span the N-dimensional state space, we have

$$\text{rank}\begin{bmatrix}\mathbf{B}, & \mathbf{AB}, & \ldots, & \mathbf{A}^{N-1}\mathbf{B}\end{bmatrix} = N$$

□

A.2. **The Adjoint System**

More interestingly, the dual problem of *observability* of the adjoint system to (35) is related to the *controllability* of (35). Consider the adjoint system:

$$\dot{\varphi} = -A^*\varphi, \;\; 0 \le t \le T; \;\; \varphi(T) = \varphi^T \tag{36}$$

Taking the inner product of (35) with φ and integrating on $(0, T)$, then $\forall \varphi^T \in \mathbb{R}^N$:

$$\int_0^T \langle \dot{x}, \varphi\rangle dt = \int_0^T \langle Ax + Bv, \varphi\rangle dt$$

$$\langle x^T, \varphi^T\rangle - \langle x^0, \phi^0\rangle - \int_0^T \langle x, \dot{\varphi}\rangle dt = \int_0^T \langle Ax, \varphi\rangle dt + \int_0^T \langle Bv, \varphi\rangle dt$$

Since $\dot{\varphi} = -A^*\varphi$:

$$\langle x^T, \varphi^T\rangle - \langle x^0, \varphi^0\rangle + \int_0^T \langle Ax, \varphi\rangle dt = \int_0^T \langle Ax, \varphi\rangle dt + \int_0^T \langle Bv, \varphi\rangle dt$$

$$\iff \langle x^T, \varphi^T\rangle_{\mathbb{R}^N} - \langle x^0, \varphi^0\rangle_{\mathbb{R}^N} = \int_0^T \langle v, B^*\varphi\rangle_{\mathbb{R}^M} dt$$

So v is a control function for (35) iff $\forall \varphi^T \in \mathbb{R}^N$:

$$\int_0^T \langle v, B^*\varphi\rangle_{\mathbb{R}^M} dt + \langle x^0, \varphi^0\rangle_{\mathbb{R}^N} = 0 \tag{37}$$

Theorem A.2 (Observability Inequality (Lions, 1988))**.**
System (35) is controllable in time $T > 0$ iff the adjoint system is observable in time T if there exists a constant $C_{obs}(T) > 0$ such that

$$|\varphi(0)|^2 \le C_{obs}^2 \int_0^T |B^*\varphi|^2_{\mathbb{R}^M} dt \tag{38}$$

The proof that (cite zuazua numerics) presents assumes that if (A.2) holds, then by constructing a strictly convex and coercive functional J, there exists a unique control V_{hum} that is built by the minimizer of J. Moreover, we will also show the equivalence of (A.2) and (A.1):

Proof.

(i) Assume that (A.2) holds and construct a strictly convex and coercive functional

$$J(\varphi^T) = \frac{1}{2}\int_0^T |B^*\varphi(t)|^2_{\mathbb{R}^M} + \langle x_0, \varphi^0\rangle_{\mathbb{R}^N} \tag{39}$$

which depends on the initial datum of the adjoint system. Since J is strictly convex and coercive, there exists a unique solution Φ of the adjoint system such that

$$DJ(\Phi^T) = 0$$

To make something useful out of $DJ(\Phi^T)$, we apply the direct method of the calculus of variations. Since Φ minimizes J, consider an infinitesimal variation $\varepsilon\varphi^T$, $\varphi \in \mathbb{R}^N$ such that

$$J(\Phi^T + \varepsilon\varphi^T) = \frac{1}{2}\int_0^T |B^*\Phi|^2 dt + \int_0^T \varepsilon\langle B^*\Phi, B^*\varphi\rangle_{\mathbb{R}^M} dt + \frac{1}{2}\int_0^T \varepsilon^2|B^*\varphi|^2 dt + \langle x^0, \Phi^0\rangle_{\mathbb{R}^N} + \langle x^0, \varepsilon\varphi^0\rangle_{\mathbb{R}^N}$$

Since

$$DJ(\Phi^T + \varepsilon\varphi^T)\Big|_{\varepsilon=0} = 0$$

Giving

$$DJ(\Phi^T) = \int_0^T \langle B^*\Phi, B^*\varphi\rangle_{\mathbb{R}^M} dt + \langle x^0, \varphi^0\rangle_{\mathbb{R}^N} = 0$$

However, notice that this expression is equivalent to (37)! Thus, using the minimizer of the functional J, we have constructed a unique control input V_{hum} which is given by

$$V_{hum} = B^*\Phi$$

Hence, the solution of (35), x, influenced by this control input satisfies $x(T) = 0$.

(ii) To prove that (38) implies the Kalman condition and hence controllability of (35), we need to show that if $B^*\varphi = 0$ for all $t \in [0, T]$, this implies that $\varphi \equiv 0$. Solving the adjoint problem $\dot{\varphi} = -A^*\varphi$ with initial condition $\varphi(T) = \varphi^T$ gives

$$\varphi(t) = e^{-A^*(t-T)}\varphi^T$$

Since φ is analytic in time, then $B^*\varphi$ vanishes iff the derivatives of $B^*\varphi$ is zero at $t = T$:

$$B^*(A^*)^k\varphi^T = 0 \tag{40}$$

Now, we invoke Cayley-Hamilton (again!!) so that we can write $(A^*)^k = \sum_{j=0}^{k-1} c_j (A^*)^j$.
Note that (40) is equivalent to saying that φ^T *lies* in the kernel of $B^*(A^*)^k$ for $k = 0, 1, \ldots, N-1$:

$$\varphi^T \in \bigcap_{k=0}^{N-1} \text{Ker}\left[B^*(A^*)^k\right] = \{0\}$$

or that

$$\text{rank}[B, AB, \ldots, BA^{N-1}] = N$$

which gives $\varphi^T = 0 \implies \varphi(t) = e^{-A^*(t-T)}(0) = 0$

$\square$

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