Hyperplane Arrangements

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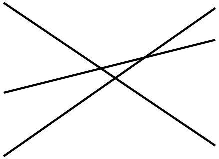
June 2026

Introduction

What is a hyperplane?

A line divides a plane into separate regions, and a plane divides three-dimensional space into two parts.



A **hyperplane** is the higher-dimensional version of this same idea.

What is a Hyperplane?

A hyperplane is an $(n-1)$-dimensional subset of an n-dimensional vector space, K.

$$H = \{x \in K^n : \alpha \cdot x = a\}$$

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- In $\mathbb{R}^2$, a hyperplane is a line.
- In $\mathbb{R}^3$, a hyperplane is an ordinary plane.
- In $\mathbb{R}^4$, we can no longer visualize it, but the same definition is still applied.

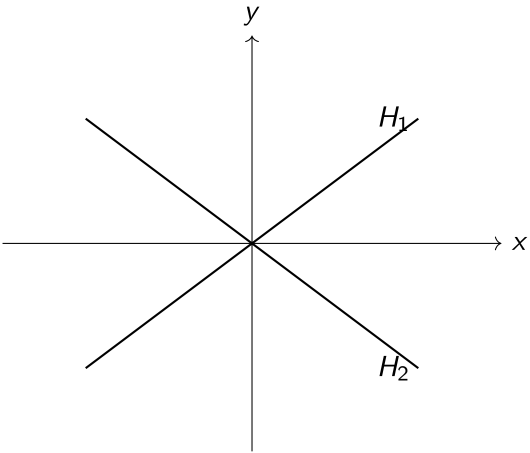
Definition

An arrangement $\mathcal{A}$ is a collection of these hyperplanes:

$$\boxed{\mathcal{A} = \{H_1, H_2, \ldots, H_m\}}$$

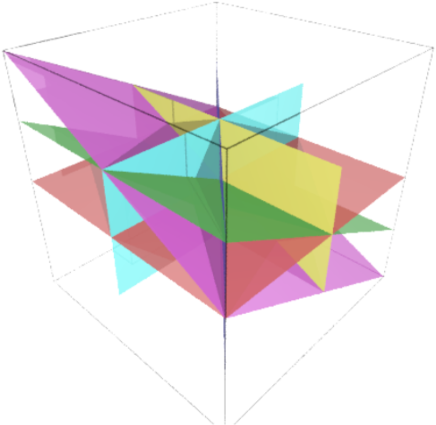
where each H_i is a hyperplane in K^n.

Hyperplanes in Two Dimensions



In $\mathbb{R}^2$: hyperplanes are lines.

Hyperplanes in Three Dimensions



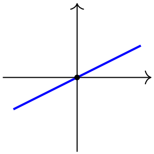
In $\mathbb{R}^3$: hyperplanes are planes.

Linear vs. Affine Hyperplanes

Linear

$$\alpha \cdot x = 0$$

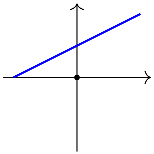
- passes through the origin
- forms a subspace



Affine

$$\alpha \cdot x = a$$

- shifted away from origin
- not a subspace



Affine hyperplanes are simply translated versions of linear hyperplanes.

The Complement

The complement of hyperplanes:

$$M(\mathcal{A}) = V - \bigcup_{H \in \mathcal{A}} H$$

When working over $\mathbb{R}$:

- removing hyperplanes disconnects space
- each connected component is called a **region** (or chamber)

These regions are convex.

What is a Poset?

A **partially ordered set** (or **poset**) is a set together with a relation "$\leq$" satisfying

- Reflexive:

$$x \leq x$$

- Antisymmetric:

$$x \leq y,\ y \leq x \Rightarrow x = y$$

- Transitive:

$$x \leq y,\ y \leq z \Rightarrow x \leq z$$

Unlike ordinary numbers, **not every pair of elements must be comparable.**

$$a \nleq b \qquad \text{and} \qquad b \nleq a$$

This is why it is called a **partial** order.

Example of a Poset

Consider the subsets of

$$\{1,2\}$$

ordered by inclusion.

$$\begin{tikzcd} & \{1,2\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{1\} \arrow[dr, dash] & & \{2\} \arrow[dl, dash] \\ & \emptyset & \end{tikzcd}$$

Notice that order is reversed and larger geometric objects appear lower in the poset.

This picture is called a **Hasse diagram**.

The Intersection Poset

Now we apply the same idea to hyperplane arrangements.

Instead of ordering subsets,
we order the **intersections of hyperplanes**.

The elements are

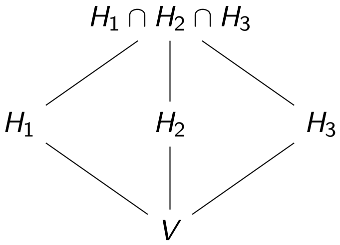
$$V,\ H_1,\ H_2,\ H_3,\ H_1 \cap H_2,\ H_1 \cap H_3,\ \dots$$

ordered by

$$x \leq y \quad \Longleftrightarrow \quad x \supseteq y.$$

Notice the order is reversed:
larger geometric objects appear lower in the poset.

Intersection Poset of Three Lines



Every node represents an intersection.
Edges show immediate containment relationships.
This combinatorial object is called the

$$L(\mathcal{A}),$$

the **intersection poset**.

Rank

Definition

The **rank** of $x \in L(\mathcal{A})$ is the codimension of the flat x.

For example in $\mathbb{R}^3$

Object	Dimension	Rank
$\mathbb{R}^3$	3	0
Plane	2	1
Line	1	2
Point	0	3

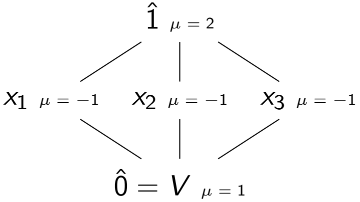
Higher rank means a "smaller" intersection.

Characteristic Polynomial

Definition

$$\chi(\mathcal{A}, t) = \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x)\, t^{\dim x}.$$

- Packages the Möbius function of the whole poset into one polynomial.
- Purely combinatorial invariant of $\mathcal{A}$.



$$\chi(\mathcal{A}, t) = t^2 - 3t + 2 = (t-1)(t-2)$$

Zaslavsky's Theorem

Theorem (Zaslavsky, 1975)

For a real arrangement $\mathcal{A}$ in $\mathbb{R}^n$:

$$\#\{\text{regions}\} = (-1)^n \chi(\mathcal{A}, -1),$$

$$\#\{\text{bounded regions}\} = (-1)^{\mathrm{rank}(\mathcal{A})} \chi(\mathcal{A}, 1).$$

- Plug in $t = -1$: get the number of regions.
- Plug in $t = 1$: get the number of bounded regions.

Our example: $(-1)^2 \chi(\mathcal{A}, -1) = (-1-1)(-1-2) = 6$ ✓

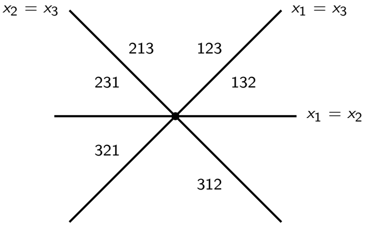
The Braid Arrangement

Definition

$$\mathcal{A}_n = \{x_i = x_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n.$$

- Regions correspond to orderings $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$.

Restricting to the hyperplane $x_1 + x_2 + x_3 = 0$, the three hyperplanes $x_1 = x_2,\ x_1 = x_3,\ x_2 = x_3$ become three concurrent lines.



each of the $3! = 6$ regions is labeled by the ordering it represents

Braid Arrangement

- So the number of regions must be $n!$.
- Characteristic polynomial:

$$\chi(\mathcal{A}_n, t) = t(t-1)(t-2)\cdots(t-n+1).$$

- Check with Zaslavsky: $(-1)^n \chi(\mathcal{A}_n, -1) = n!$. ✓
- Intersection poset $L(\mathcal{A}_n) =$ the **partition lattice** Π_n

$$\begin{tikzcd}[column sep=small]
 & 123 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
12|3 \arrow[dr, dash] & 13|2 \arrow[d, dash] & 23|1 \arrow[dl, dash] \\
 & 1|2|3 &
\end{tikzcd}$$

$L(\mathcal{A}_3) \cong \Pi_3$, the partition lattice on $\{1, 2, 3\}$

Shi Arrangements

Definition

The *Shi arrangement* is a collection of hyperplanes in $\mathbb{R}^n$ defined by:

$$\mathcal{S}_n = \{x_i - x_j = 0,\ x_i - x_j = 1 \mid 1 \leq i < j \leq n\}.$$

- The equations $x_i - x_j = 0$ represent equal coordinates:

$$x_i = x_j$$

- The equations $x_i - x_j = 1$ represent coordinates differing by exactly 1.

The Shi arrangement divides space into regions based on the relative ordering and differences between coordinates.

Applications of Hyperplane Arrangements

- **Optimization**
  - Partition feasible regions in linear programming.
  - Analyze solution spaces and constraint boundaries.
- **Machine Learning**
  - Decision boundaries for linear classifiers such as Support Vector Machines (SVMs).
  - Understanding classification regions in high-dimensional data.
- **Robotics and Motion Planning**
  - Model obstacle-free configuration spaces.
  - Plan collision-free paths.