On Transcendental Numbers

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Table of Contents

Squaring the Circle

We begin our exploration of transcendental numbers in the unlikely realm of Euclidean Geometry. This problem originated in Ancient Greece, and consists of only using a compass and straightedge to compose a square with the same area as a circle. Despite the problem's apparent simplicity, the proof that this construction is impossible took mathematicians over 2,000 years to fully formulate. In 1882 Ferdinand von Lindemann completed the proof, which involved stating that π is transcendental.

Classifying Numbers

Definition (Algebraic Numbers)

The set of algebraic numbers denoted $\overline{\mathbb{Q}}$ is defined as the set of real numbers and complex that are the roots of any nonzero polynomial with rational coefficients.

From this naturally arises the question of are there any numbers which are transcendental, and do not fall into this set.

Cantor's Diagonal Argument

We want to answer the question of whether all numbers are algebraic. That is, are there any numbers which are not the root of any polynomial with rational coefficients? The answer to that question would be yes; we In fact, in 1872 George Cantor proves that if we were to select a complex or real number at random the probability of it being transcendental would equal one.

Cantor's Diagonal Argument

A set S is called countably infinite if there is a bijection between S and N. On the other hand, not all infinite sets are countably infinite. Cantor's diagonal argument proved that there are uncountably infinite sets. Essentially, we create a table with all rational numbers in order such that

$$\begin{aligned} x_1 &= 0.d_{11}d_{12}d_{13}d_{14}\dots \\ x_2 &= 0.d_{21}d_{22}d_{23}d_{24}\dots \\ x_3 &= 0.d_{31}d_{32}d_{33}d_{34}\dots \\ x_4 &= 0.d_{41}d_{42}d_{43}d_{44}\dots \\ &\vdots \\ x_{\text{new}} &= 0.d'_1d'_2d'_3d'_4\dots \end{aligned}$$

Cantor's Diagonal Argument

Theorem

Given the set of real numbers $\mathbb{R}$, the probability of choosing a transcendental number is 1.

Proof.

The set of algebraic numbers are the roots of a polynomial with integer coefficients. Since the set of integers is countable, the set of algebraic numbers is also countable. Since we have just proven the set of real numbers is uncountable, the set of transcendental numbers must also be uncountable. Then we have

$$P(Transcendental \subseteq [a,b]) = \frac{\mu(Transcendental \cap [a,b])}{\mu([a,b])} = 1,$$

or, given the closed interval $[a,b]$ the probability of choosing a transcendental number is 1. □

Liouville's Theorem

Theorem

Let α be a real algebraic number of degree $n \geq 1$ that is irrational. Then there exists a positive constant $c(\alpha) > 0$ such that for all integers p and q with $q > 0$, we have:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

Our First Transcendental Number

In 1844 Joseph Liouville proved the existence of transcendental numbers through Liouville's number. Essentially, he indentified a property shared by irrational numbers and and proved it did not apply to transcendental numbers.

Corollary

The number

$$L = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\dots$$

is transcendental.

The Fundamental Principle of Number Theory

Theorem

There is no integer n satisfying

$$0 < n < 1.$$

Equivalently,

$$\nexists\, n \in \mathbb{Z} \text{ such that } 0 < n < 1.$$

Proof

Liouville's proof, which reflects the general proof that a number is transcendental, is composed of six main steps to highlight the general process for proving that any number is transcendental:

- Assume there is exists a nonzero polynomial $P(z)$ with integer coefficients such that $P(\alpha) = 0$.
- We create an integer N using properties of $P(z)$ and α.
- Prove that $0 < |N|$.
- Prove that $|N| > 1$.
- Violating the Fundamental Principle of Number Theory we have produced an integer between 0 and 1. As this is clearly impossible, the only reason for this conclusion is that our initial assumption was false.
- Since we have just proved that α is not algebraic, it must be transcendental.

The Hermite-Lindemann Theorem

Theorem

For some algebraic number α the number e^{α} is transcendental.

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Corollary

If α is a nontrivial algebraic number, $\ln \alpha$ is transcendental.

Proof.

If $\ln \alpha$ is algebraic, by then Hermite-Lindemann $e^{\ln \alpha}$ should be transcendental. However, we know that $e^{ln\alpha} = \alpha$, which is algebraic, so $\ln \alpha$ must be transcendental. $\square$

The Hermite-Lindemann Thereom

Theorem

For some algebraic number α the number e^{α} is transcendental.

Corollary

If α is a nonzero real and algebraic number then $\sin\alpha$, $\cos\alpha$, *and* $\tan\alpha$ *is transcendental.*

Proof.

From Euler's Identity we know that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

and

$$\cos^2(\alpha) + \sin^2(\alpha) = 1.$$

$\square$

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□

We assume that $\cos(\alpha)$ is algebraic. This would mean that $\sin(\alpha)$ is also algebraic which means $i\sin(\alpha)$ is algebraic. By Euler's Identity this means that $e^{i\alpha}$ is algebraic. However, we know because of Lindemann-Weierstrass that e^{α} is transcendental if α is algebraic. Since we have clearly reached a contradiction the assumption that $\cos(\alpha)$ is algebraic is false. We can prove that this holds for $\sin(\alpha)$ as well by swapping sine and cosine.

Corollary

If α is a nonzero real and algebraic number then $\tan\alpha$ is transcendental.

In order to prove this holds for $\tan(\theta)$ we will use the identity

$$cos(\theta) = \frac{1}{\sqrt{\tan^2(\theta) - 1}}.$$

Assuming α is algebraic, $\tan(\alpha)$ will also be algebraic. However, we have just proved that if α is a nonzero real and algebraic number $\cos(\theta)$ is transcendental. Therefore $\tan(\theta)$must also be transcendental.

Corollary

π is transcendental.

Proof.

The proof of this function is surprisingly simple. We consider Euler's identity which states that

$$e^{i\pi} + 1 = 0.$$

Assuming π is algebraic, and keeping in mind that the product of two algebraic numbers must be algebraic, $i\pi$ is a nonzero algebraic number. However, knowing that $e^{i\pi} = -1$, a value which is algebraic, means that our initial assumption must be wrong and pi is transcendental. $\square$

Corollary

If α is a nontrivial algebraic number, $\ln \alpha$ *is transcendental.*

Proof.

If $\ln \alpha$ is algebraic, by then Hermite-Lindemann $e^{\ln \alpha}$ should be transcendental. However, we know that $e^{ln \alpha} = \alpha$, which is algebraic, so $\ln \alpha$ must be transcendental. $\square$

The Lindemann-Weierstrass Theorem

Theorem

The number e^{α} is transcendental for any nonzero algebraic number α.

An equivalent expression of this theorem would be that the number e^{α} is *not* algebraic. In other words, $e^{\alpha} \neq \beta$ for any algebraic number β. In addition, these two quantities would remain unequal when multiplied by any nonzero algebraic number. Consequently we produce the expression:

$$\beta_0 + \beta_1 e^{\alpha} \neq 0.$$

The Lindemann Weierstrass Theorem

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Lemma

If α is a nonzero algebraic number and β_0, β_1 are algebraic numbers that are both not equal to zero, then

$$\beta_0 e^0 + \beta_1 e^\alpha \neq 0.$$

The Lindemann Weierstrass Theorem

This theorem states that e^0 and e^α are linearly independent over algebraic numbers since there is no linear combination of nonzero algebraic coefficients that produces a sum of 0. Taking this a step further, we have the Lindemann-Weirstrass Theorem:

Lemma

Let $\alpha_0, \alpha_1, \dots \alpha_m$ be distinct algebraic numbers. Then $e^{\alpha_0}, e^{\alpha_1}, \dots e^{\alpha_m}$ are linearly independent over algebraic numbers and

$$\sum_{m=0}^{M} \beta_m e^{\alpha_m} \neq 0.$$

The Gamma function

Before going over an overview of the proof of the Lindemann-Weierstrass function, it it necesary to define the gamma function:

$$\int_0^\infty x^n e^{-x} dx = n! \tag{1}$$

Proof

Suppose, for the sake of contradition, that e is algebraic such that there exists an equation $a(x)$ such that

$$a(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

We write a second equation:

$$f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p}{(p-1)!}$$

Such that given a prime p the polynomial has roots of multiplicity p at each positive integer up to p.

Proof

We integrate $f(x)$ from 0 to each root of the polynomial using the following integral:

$$S = \sum_{j=0}^{n} a_j \int_0^j e^{-x} f(x)\, dx.$$

We can apply the gamma function to $f(x)$ since it is the sum of powers of x.

Proof

We integrate $f(x)$ from 0 to each root of the polynomial using the following integral:

$$S = \sum_{j=0}^{n} a_j \int_0^j e^{-x} f(x)\, dx.$$

The sum S is an integer due to the polynomial's factorial structure. In addition, we choose a value p such that $p > N$. This means S cannot be divisible by P. Furthermore, S must be a nonzero integer.

Proof

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$$S = \sum_{j=0}^{n} a_j \int_0^j e^{-x} f(x)\, dx.$$

The sum S is an integer due to the polynomial's factorial structure. In addition, we choose a value p such that $p > N$. This means S cannot be divisible by P. Furthermore, S must be a nonzero integer. However, as we increase P, the remainder approaches 0. S cannot be an integer between 0 and 1 since that violates the fundamental principle of number theory. Therefore, our assumption that e is algebraic must be false.

Thanks for listening! Any questions?