On Transcendental Numbers

Nishka Wadhawan

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Abstract

The set of algebraic numbers is defined as the set of real and imaginary numbers that can be represented as the roots of a polynomial with integer coefficients. Transcendental numbers are numbers which do not fall into this set. In this paper we prove the existence of transcendental numbers, Liouville's theorem and the transcendence of two of the most famous transcendental numbers, e and π.

1 Introduction

1.1 Squaring the Circle

We begin our investigation of transcendental numbers within the unlikely realm of Euclidean Geometry. Specifically, in Euclid's *Elements*, he presents the problem of whether it was possible, using only a compass and straightedge, to construct a circle with the same area as a square. Despite the simplicity of the problem's premises, the proof that this was impossible took mathematicians over two thousand years to formulate. The proof was finally completed in 1882 by Ferdinand von Lindemann who proved that π was transcendental.

2 On Classifying Numbers

2.1 Defining the Real Numbers

Before we even begin to establish the existence of transcendental numbers we must first establish what, for the purpose of this paper, a number is. Namely we first note that the set of natural numbers, $\mathbb{N} = \{1, 2, 3 \dots\}$ which is what intuitively arises for the purpose of counting objects is closed under addition

and multiplication. If we were to add a natural number x to another natural number y then their sum would be a third natural number. However for the purposes of subtraction the natural numbers are insufficient, leading to the necessity of including fractions and negative numbers. For the purposes of subtraction, we add zero and the negative numbers, creating a new set of numbers, integers, $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2 \dots\}$. The set of integers, however, has not solved all our problems. Although it is closed under addition, subtraction, and multiplication, it is not closed under division. As such we expand the set to include fractions and name this new set of numbers rational numbers denoted by $\mathbb{Q}$ as the set of all quotients m/n where $m \in \mathbb{Z}$ and $m \in \mathbb{N}$. Within this set we can perform a wide variety of operations. However, certain operations reveal the set's limitations. Primarily, if we consider the issue of repeated multiplication as in the equation

$$x \times x \times x \cdots \times x = r/s$$

we may encounter equations which we cannot solve over $\mathbb{Q}$. It follows that we expand our list of number to include these irrational numbers, denoted with $\sqrt{x}$ where $x \in \mathbb{Q}$ and $x \geq 0$. We have now created a set of numbers we call the real numbers, $\mathbb{R}$ which is closed under addition, subtraction multiplication and division.

2.2 Complex Numbers

What happens when $x < 0$? In order to resolve this we create the set of complex numbers denoted by $a + bi$ where i represents the square root of negative one. Carl Friedrich Gauss proved that this set of complex number, $\mathbb{C}$, contains both real and algebraic numbers, but can solve all algebraic equations even those without rational coefficients. Of course, this necessitates a proper definition for algebraic numbers.

2.3 On Algebraic and Transcendental Numbers

Definition 1 (Algebraic Numbers)**.** *The set of algebraic numbers denoted* $\overline{\mathbb{Q}}$ *is defined as the set of real numbers and complex that are the roots of any nonzero polynomial with rational coefficients.*

From this naturally arises the question of whether all numbers are algebraic. That is, are there any numbers which are not the root of any polynomial with rational coefficients? The answer to that question would be yes; we In fact, in 1872 George Cantor proves that if we were to select a complex or real

number at random the probability of it being transcendental would equal one. Despite that, proving the transcendence of a number is a challenging task and establishing a number's transcendence is quite rare.

2.4 Cantor's Diagonal Argument

Cantor's Diagonal Argument states that the set of real numbers is not countable. This may seem intuitive-after all there are infinitely many numbers. However Cantor also proves that the set of rational numbers is countable, at least, countably infinite. What does this mean? We can visualize this by representing the set of rational numbers in a table as such.

$$\begin{aligned} x_1 &= 0.d_{11}d_{12}d_{13}d_{14}\dots \\ x_2 &= 0.d_{21}d_{22}d_{23}d_{24}\dots \\ x_3 &= 0.d_{31}d_{32}d_{33}d_{34}\dots \\ x_4 &= 0.d_{41}d_{42}d_{43}d_{44}\dots \\ &\vdots \\ x_{\text{new}} &= 0.d'_1d'_2d'_3d'_4\dots \end{aligned}$$

In this table we place, in order, every rational number. Cantor shows that this list of numbers does not contain every number. Essentially, if one were to take a number by changing the first digit of the first number, second digit of the second number, and so on, they would create a number that differs from every other number by at least one digit. In doing so Cantor proves that the set of rational numbers is countably infinite.

Definition 2 (Cardinality)**.** *The cardinality of a set S refers to its size, which can be infinite, denoted as $|S|$.*

Theorem 2.1. *Given the set of real numbers $\mathbb{R}$, the probability of choosing a transcendental number is 1.*

Proof. Given a set $U \subset [0,1]$ the *measure* of U is equivalent to the probability $\mu(U)$ of choosing a random number that has in the set. Let A_n be the set of algebraic numbers degree n such that for $x \in \mathbb{R}$ and $a_0, \dots, a_n \in \mathbb{R}$,

$$\{A_n = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0\}.$$

We note that each choice of a gives us between 0 and n values of x. This means A_n is countably infinite. Using a similar logic we can show that $\overline{\mathbb{Q}}$, the set of

algebraic numbers is the countable union $\bigcup_{n=1}^{\infty} A_n$ of countable sets, making it countably infinite. This tells us that $\mu(A \cap [0,1]) = 0$. More generally, given any non-empty interval $[a,b]$, if we define the probability of a randomly chosen number lying in a set $U \subseteq [a,b]$ as:

$$P(U \subset [a,b]) = \frac{\mu(U \cap [a,b])}{\mu([a,b])}$$

and

$$P(Transcendental \subseteq [a,b]) = \frac{\mu(Transcendental \cap [a,b])}{\mu([a,b])} = 1$$

then the given the closed interval $[a,b]$ the probability of choosing a transcendental number is 1 because the set of transcendental numbers is uncountably infinite.

□

3 Liouville and Transcendence

3.1 How to prove a number is transcendental

Proving a number's transcendence is inherently challenging when considering that there are infinitely many polynomials $p(x)$ with integer coefficients. However, proving a number's transcendence consists of an underlying framework where we assume such a nonzero polynomial where $p(\alpha) = 0$ and identify where this assumption violates a fundamental mathematical theory. Before diving into Liouville's contributions, we present a general outline of the general steps in establishing a number's transcendence.

1. Assume there is exists a nonzero polynomial $P(z)$ with integer coefficients such that $P(\alpha) = 0$.
2. We create an integer N using properties of $P(z)$ and α.
3. Prove that $0 < |N|$.
4. Prove that $|N| > 1$.
5. Violating the Fundamental Principle of Number Theory we have produced an integer between 0 and 1. As this is clearly impossible, the only reason for this conclusion is that our initial assumption was false.
6. Since we have just proved that α is not algebraic, it must be transcendental.

3.2 Liouville's Method

Although the existence of transcendental numbers had long since been theorized, it was not until 1844 that Joseph Liouville proved their existence. Essentially, he identified a property shared by algebraic numbers, such that any number not satisfying that property must be transcendental.

In section (2), while approximating rational approximates for $\sqrt(2)$ we notice how the set of number

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

approaches x. In particular, for the approximation $x = \frac{99}{70}$, we have

$$\frac{99}{70}^2 - 2 = \frac{1}{4900},$$

the value of the expression is incredibly close to 0. Knowing this, we can rewrite our definition for irrational numbers in a way that does not require algebraic equations. Instead, a number is a quantity that can be approximated arbitrarily well by rational numbers. By the Cauchy Criterion, this definition means we can identify numbers as the limit of a sequence without knowing what the sequence converges to. With this knowledge we are now equipped with the prerequisites to define and prove Liouville's Theorem.

Theorem 3.1. *labelthm:lvl Let α be a real algebraic number of degree $n \geq 1$ that is irrational. Then there exists a positive constant $c(\alpha) > 0$ such that for all integers p and q with $q > 0$, we have:*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

Proof. In this paper we will prove Liouville's theorem only for the case in which α is a real number. In order to do so we first examine two cases that depend on whether the rational number p/q is close to α.

We first examine the case where p/q is a rational number relatively far from α, specifically

$$1 < |\alpha - \frac{p}{q}|.$$

Since $q \geq a$ we have that

$$\frac{1}{q^d} \leq 1 < |\alpha - \frac{p}{q}|.$$

It follows that this inequality holds for any constant c for which $c \leq 1$. We now consider the case where p/q is close to α. Suppose

$$\alpha - p/q \leq 1$$

. Our goal is to show that p/q is close to α but cannot be *too* close.

Let $f(x)$ be the minimum polynomial with integer coefficients for α such that $f(\alpha) = 0$ that can be expressed as

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

Where $a_d, a_{d-1}, \ldots, a_0 \in \mathbb{Z}$ and $a_d > 0$.

The next step is to create an inequality such that its lower bound supports the inequality in Liouville's theorem. We consider the value $f(\frac{p}{q})$:

$$f(\frac{p}{q}) = a_d \frac{p^d}{q^d} + a_{d-1} \frac{p^{d-1}}{q^{d-1}} + \cdots + a_1 \frac{p}{q} + a_0,$$

which can be expressed with a common denominator as

$$f\left(\frac{p}{q}\right) = \frac{a_d p^d + a_{d-1} p^{d-1} q + \cdots + a_1 p q^{d-1} + a_0 q^d}{q^d}.$$

We let N denote the numerator. It can be inferred that that N is a nonzero integer. The reason for this is that since $f(x)$ is an irreducible polynomial with degree $d > 1$, $f(p/q) \neq 0$. We can simplify the previous expression so that

$$f(\frac{p}{q}) = \frac{N}{q^d}$$

Since $q > 0$, we use absolute values to determine that

$$\frac{|N|}{q^d} = f(\frac{p}{q}).$$

We can now apply the Fundamental Principle of Number Theory to determine that since $N \neq 0, 1 \leq |N|$. This allows us to replace the numerator in the fractional expression with 1, giving us

$$\frac{1}{q^d} = f(\frac{p}{q}).$$

We have now successfully produced a lower bound for the inequality. Our next step is to try to produce the desired expression $|\alpha - \frac{p}{q}|$. In order to do so we

recall that $f(\alpha) = 0$ and subtract zero from both sides of the equation giving us

$$\frac{1}{q^d} = |f(\alpha) - \frac{p}{q}|.$$

Now it appears we are stuck since we need to separate α from $f(\alpha)$. Luckily, we can apply the Mean Value Theorem from calculus which, since we have assumed $\alpha \in \mathbb{R}$.

We assume the polynomial $y = f(x)$ to be a differentiable function. The Mean Value Theorem tells us that give α and p/q there must be a real number φ for between α and p/q such that

$$f(\alpha) - f(\frac{p}{q}) = f'(\varphi)(\alpha - \frac{p}{q}).$$

This identity gives us

$$\frac{1}{q^d} = f'(\varphi)|\varphi - \frac{p}{q}|.$$

Furthermore, we can recall that since $|\alpha - \frac{p}{q}| \leq 1$, and φ is between α and $\frac{p}{q}$ $|\alpha - \varphi| \leq 1$. We can expand this so that we have $\alpha - 1 \leq \varphi \leq \alpha + 1$. We can similarly produce an upper bound for $|f'(\varphi)|$ by noting that

$$|f'(\varphi) \leq \max\{|f'(\theta)| : \theta \in [\alpha - 1, \alpha + 1]\}.$$

Since we know $f(x)$ is a continuous function, it will attain its maximum value M on the closed interval $[\alpha - 1, \alpha + 1]$. Importantly, M is only dependent on α and not p/q. Also, since $f'(\varphi) \neq 0$, $M > 0$. This information produce the inequality

$$\frac{1}{q^d} \leq |f'(\varphi)| \left|\alpha - \frac{p}{q}\right| \leq M \left|\alpha - \frac{p}{q}\right|.$$

Now from this we can prove the existence of the constant c given the rearranged inequality

$$\frac{M^{-1}}{q^d} \leq \left|\alpha - \frac{p}{q}\right|.$$

Combining this with the previous case we can now state that for *any* rational number p/q

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n},$$

where

$$c = \min 1, M^{-}1.$$

We have successfully proved Liouville's Theorem holds for all rational numbers. □

3.3 Liouville's Number

Corollary 3.1.1. *The number*

$$L = \sum_{k=1}^{\infty} 10^{-k!} = 0.11000100000000000000000001000\ldots$$

is transcendental.

Proof. We assume, for sake of contradiction, that L is an algebraic number of degree d. Before applying **??**Liouville's theorem we establish that L is irrational, or that $d > 1$. L has to be irrational as the consecutive zeros in its decimal expansion increase without bound, meaning it can never eventually be periodic. Thus L cannot be rational, and, assuming it is algebraic, must have degree equal or greater than 2. With that out of the way, we can now apply Liouville's theorem which states that there must be a constant c for which

$$\frac{c}{q^d} \leq |L - \frac{p}{q}|$$

for all rational number p/q .

Next we construct a series of rational approximations of L formed by truncating the decimal expansion of L before each run of zeros. We have a positive integer N for which $r_N = 10^{-n!}\sum_{n=1}^{N} 10^{-n!}$. We know that r_N is rational because its decimal expansion terminates. Furthermore $r_N = p_N/q_N$ where

$$p_N = 10^{N!}\sum_{n=1}^{N} 10^{-n!}$$

and

$$q_N = 10^{N!}.$$

By Liouville's theorem we have:

$$\left|\mathcal{L} - \frac{p_N}{q_N}\right| = \sum_{n=1}^{\infty} 10^{-n!} - \sum_{n=1}^{N} 10^{-n!} = \sum_{n=N+1}^{\infty} 10^{-n!}.$$

In particular we take a closer look at the tail of the series.

$$\sum_{n=N+1}^{\infty} 10^{-n!} = 10^{-(N+1)!} + 10^{-(N+2)!} + 10^{-(N+3)!} + \cdots .$$

Note that the powers of 10 in the series are consecutive factorials starting with $(N + 1)!$. We can place an upper bound on the series, that is, the sum of all

values 10^{-m} starting with $m = (N+1)!$. That way, instead of only adding powers of 10^{-1} that are consecutive factorials, we add all powers of 10^{-1} after $10^{-(N+1)!}$. We now have that

$$\sum_{n=N+1}^{\infty} 10^{-n!} < \sum_{n=(N+1)!}^{\infty} 10^{-n} = \frac{10^{-(N+1)!}}{1-10^{-1}} = \frac{10}{9}\,10^{-(N+1)!}.$$

Combining this with **??** we have the following inequality:

$$\left|\mathcal{L} - \frac{p_N}{q_N}\right| < \frac{10}{9}10^{-(N+1)!},$$

Recall that $q_N = 10^{N!}$. Applying Liouville's Theorem we have

$$\left|\mathcal{L} - \frac{p_N}{q_N}\right| < \frac{10}{9}10^{-(N+1)!}.$$

So for all N it must be true that

$$0 < \frac{9}{10}c < 10^{dN!-(N+1)!}.$$

Equivalently,

$$0 < 9 < \left(\frac{10}{c}\right)10^{dN!-(N+1)!}.$$

However, for $N \geq d$, the exponent is negative, meaning that as n approaches infinity,the right hand expression approaches one. In other words the inequality approaches $0 < 9 < 1$, which contradicts the Fundamental Principle of Number Theory. This means our initial assumption that L was algebraic is false and L must be transcendental. □

4 Proving the Transcendence of e and π

Euler's number is one that lies at the heart of transcendental number theory, appearing not only in the most prominent of theorems but also in many of its deeper aspects. Before proving its transcendence we will establish its irrationality. Afterwards we will explore the Hermite-Lindemann theorem and some of its implications including the transcendence of before diving into its proof. We will also use the Hermite-Lindemann theorem to define and prove the Lindemann-Weierstrass Theorem.

4.1 Proving e is Irrational

Before attempting to prove the transcendence of e we first establish Fourier's proof that e is irrational.

Theorem 4.1. *The number e is irrational.*

Proof. For starters, we can represent e with the following infinite series,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

We first assume e is a rational number such that $e = \frac{r}{s}$ and $s \geq 1$. Using s we are able to approximate e by truncating its infinite series at n = s as such:

$$e = \sum_{n=0}^{s} \frac{1}{n!}.$$

This forces the value $\frac{r}{s} - \sum_{n=0}^{s} \frac{1}{n!}$ to be positive. Multiplying this by $s!$, we eliminate the denominator, which produces the following result,

$$\begin{aligned} s!\left(e - \sum_{n=0}^{s} \frac{1}{n!}\right) &= s!\left(\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n=0}^{s} \frac{1}{n!}\right) \\ &= s!\left(\frac{1}{(s+1)!} + \frac{1}{(s+2)!} + \frac{1}{(s+3)!} + \cdots\right) \\ &= \frac{1}{s+1} + \frac{1}{(s+2)(s+1)} + \frac{1}{(s+3)(s+2)(s+1)} + \cdots . \end{aligned}$$

which we can observe to be a positive integer. In addition, since $s \geq 1$, we can bound this integer by the a geometric series

$$\frac{1}{s+1} + \frac{1}{(s+2)(s+1)} + \frac{1}{(s+3)(s+2)(s+1)} + \cdots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1.$$

Doing so means we have constructed an integer between 0 and 1. This obviously violates the fundamental principle of number theory, meaning that our initial assumption that e is rational must be false. □

4.2 Proving π is Irrational

Theorem 4.2. *The number π is irrational.*

Proof. Assume, for the sake of contradiction that π is a rational number a/b where a and b are positive integers. We define the polynomial

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

and

$$F(x) = f(x) - f''(x) + f''''(x) - \cdots + (-1)^n f^{2n}(x)$$

with the positive integer n being specified later. $n!f(x)$ has integral coefficients and terms of x in degrees not less than n and its derivative $f'(x)$ has integral values for $x = 0$ and $x = \pi = a/b$ because $f(x) = f(a/b - x)$. Using calculus we get that

$$\frac{d}{dx}\{F'(x)\sin(x) - F(x)\cos(x)\} = F''(x)\sin(x) + F(x)\cos(x) = f(x)\sin(x)$$

and

$$\int_0^\pi f(x)\sin x\, dx = [F'(x)\sin x - F(x)\cos x]_0^\pi = F(\pi) + F(0). \tag{1}$$

Since $f^{(j)}$ and $f^{(0)}$ are both integers, $F(\pi) + F(0)$ must also be an integer. However, for $0 < x < \pi$,

$$0 < f(x)\sin(x) < \frac{\pi^n a^n}{n!}.$$

Since the integral in equation (1) must be positive, yet arbitrarily small as n increases, we have constructed an arbitrarily small integer. This violates the Fundamental Principle of Number Theory, meaning our initial assumption that π was rational must be false. □

4.3 The Transcendence of Euler's Constant

Before proving the Lindemann-Weierstrass Theorem, we explore some of its implications in transcendental number theory including pi's transcendence.

Theorem 4.3. *The number e^α is transcendental for any nonzero algebraic number α.*

An equivalent expression of this theorem would be that the number e^α is *not* algebraic. In other words, $e^\alpha \neq \beta$ for any algebraic number β. In addition, these two quantities would remain unequal when multiplied by any nonzero algebraic number. Consequently we produce the expression:

$$\beta_0 + \beta_1 e^\alpha \neq 0.$$

Lemma 1. *If α is a nonzero algebraic number and β_0, β_1 are algebraic numbers that are both not equal to zero, then*

$$\beta_0 e^0 + \beta_1 e^\alpha \neq 0.$$

This theorem states that e^0 and e^α are linearly independent over algebraic numbers since there is no linear combination of nonzero algebraic coefficients that produces a sum of 0. Taking this a step further, we have the Lindemann-Weirstrass Theorem:

Lemma 2. *Let $\alpha_0, \alpha_1, \ldots \alpha_m$ be distinct algebraic numbers. Then $e^{\alpha_0}, e^{\alpha_1}, \ldots e^{\alpha_m}$ are linearly independent over algebraic numbers and*

$$\sum_{m=0}^{M} \beta_m e^{\alpha_m} \neq 0.$$

Corollary 4.3.1. *If α is a nontrivial algebraic number, $\ln \alpha$ is transcendental.*

Proof. If $\ln \alpha$ is algebraic, by then Hermite-Lindemann $e^{\ln \alpha}$ should be transcendental. However, we know that $e^{ln\alpha} = \alpha$, which is algebraic, so $\ln \alpha$ must be transcendental. □

Corollary 4.3.2. *If α is a nonzero real and algebraic number then $\sin \alpha$, $\cos \alpha$, and $\tan \alpha$ is transcendental.*

Proof. From Euler's Identity we know that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta),$$

and

$$\cos^2(\alpha) + \sin^2(\alpha) = 1.$$

We assume that $\cos(\alpha)$ is algebraic. This would mean that $\sin(\alpha)$ is also algebraic which means $i \sin(\alpha)$ is algebraic. By Euler's Identity this means that $e^{i\alpha}$ is algebraic. However, we know because of Lindemann-Weierstrass that e^α is transcendental if α is algebraic. Since we have clearly reached a contradiction the assumption that $\cos(\alpha)$ is algebraic is false. It is simple to prove this holds for $\sin(\alpha)$ as well by swapping sine and cosine.

In order to prove this holds for $\tan(\theta)$ we will use the identity

$$cos(\theta) = \frac{1}{\sqrt{\tan^2(\theta) - 1}}.$$

Assuming $tan(\alpha)$ is algebraic, $\tan(\alpha)$ will also be algebraic. However, we have just proved that if α is a nonzero real and algebraic number $\cos(\theta)$ is transcendental. Therefore $\tan(\theta)$must also be transcendental.

□

Corollary 4.3.3. *π is transcendental.*

Proof. The proof of this function is surprisingly simple. We consider Euler's identity which states that

$$e^{i\pi} + 1 = 0.$$

Assuming π is algebraic, and keeping in mind that the product of two algebraic numbers must be algebraic, $i\pi$ is a nonzero algebraic number. However, knowing that $e^{i\pi} = -1$, a value which is algebraic, means that our initial assumption must be wrong and pi is transcendental. □

4.4 Some Algebraic Preliminaries

Before we attempt to prove e's transcendence and the Lindemann-Weierstrass Theorem we establish the necessary background required for the proofs.

4.4.1 On Symmetric Polynomials

Let $F(x_1, x_2, x_3, \ldots, x_L)$ be a function in L variables. We say F is symmetric if the rearrangement of the variables does not change the function.

For instance, let $F(x_1, x_2)$ be a function of two variables. F is symmetric iff $(Fx_1, x_2) = F(x_2, x_1)$. A function with three variables is symmetric iff

$$F(x_1, x_2x_3) = F(x_1, x_3x_2) = F(x_2, x_1x_3) = F(x_3, x_2x_1) = F(x_3, x_1x_2).$$

Alternatively a symmetric function L is symmetric if it is unchanged after $L!$ permutations of the variables. We will observe a symmetric polynomial $P(x_1, x_2) = 3x_1^3x_2^3 - x_1^2 - x_2^2$. In order to generate a symmetric polynomial in $x_1, x_2, \ldots, x_L$ is to consider them as zeroes of a polynomial and consider the coefficients such that:

$$F(z) = (z - x_1)(z - x_2)\cdots(z - x_L) = z^L - \sigma_1 z^{L-1} + \sigma_2 z^{L-2} - \cdots + (-1)^L \sigma_L$$

in which

$$\begin{aligned}
\sigma_1(x_1, x_2, \ldots, x_L) &= x_1 + x_2 + \cdots + x_L, \\
\sigma_2(x_1, x_2, \ldots, x_L) &= x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots + x_{L-1}x_L, \\
&\vdots \\
\sigma_l(x_1, x_2, \ldots, x_L) &= \text{the sum of all products of } l \text{ different } x_n\text{'s}, \\
&\vdots \\
\sigma_L(x_1, x_2, \ldots, x_L) &= x_1x_2\cdots x_L.
\end{aligned}$$

We refer to the symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_L$ as elementary symmetric functions in $x_1, x_2, \ldots, x_L$. The reason these functions are called elementary functions are because any symmetrical function can be decomposed into a polynomial expression consisting of them. Take for example the function:

$$P(x_1, x_2) = 3x_1^3x_2^3 - 2x_1x_2 = ((x_1 + x_2)^2 - 2x_1x_2) = 3\sigma_2^3 - \sigma_1 - 2\sigma_2.$$

where $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1x_2$.

Theorem 4.4 (Fundamental Theorem of Symmetric Polynomials)**.** *Let $P(x_1, x_2, \ldots, x_L)$ be a symmetric polynomial with rational coefficients. We can express P as a polynomial function in the elementary symmetric functions in $x_1, x_2, \ldots, x_L$with rational coefficients. In other words there exists a polynomial F with rational coefficients such that*

$$P(x_1, x_2, \ldots, x_L = F(\sigma_1, \sigma_2, \ldots, \sigma_L).$$

4.5 Gamma Function

Theorem 4.5 (The Gamma Function)**.**

$$\Gamma(z) = \int_0^\infty x^{n-1}e^{-n}\, dt = (n-1)!$$

For a polynomial $f \in \mathbb{Z}[x]$ we have

$$\int_0^\infty f(x)e^{-x}dx \in \mathbb{Z}$$

Since the polynomial has integer coefficients we can write it as a sum of gamma functions meaning the integral has to be an integer.

For a large prime we have p

$$\frac{1}{(p-q)!}\int_0^\infty x^k e^{-x}dx = \frac{k!}{(p-1)!} = \begin{cases} 1 & \text{if } k = p-1 \\ \text{a multiple of } p & \text{if } k < p-1. \end{cases} \tag{2}$$

We will later use these cases to prove divisibility. The case $k < p - 1$ is not mentioned as irrelevant to the proof.

Next we combine integrals to obtain

$$\frac{1}{(p-1)!}\int_0^\infty f(x)e^{-x}dx.$$

We now construct an equivalent expression for e^k, $k \in \mathbb{Z}$:

$$e^k = \frac{\frac{e^k}{(p-1)!} \cdot \int_0^\infty f(x)e^{-x}\,dx}{\frac{1}{(p-1)!} \cdot \int_0^\infty f(x)e^{-x}\,dx}. \tag{3}$$

Recall that $\frac{d}{dk}e^k = e^k$. Move e^k into the integral so that

$$e^k = \frac{\frac{1}{(p-1)!} \cdot \int_0^\infty f(x)e^{k-x}\,dx}{\frac{1}{(p-1)!} \cdot \int_0^\infty f(x)e^{-x}\,dx}. \tag{4}$$

The goal is to define e^k since any polynomial in e will have terms $a_k e^k$. We have now defined all the necessary prerequisites to prove the transcendence of e.

4.6 Proving e is Transcendental

Assume, for the sake of contradiction, that e is an algebraic number such that there exists a polynomial $a(x)$ such that

$$a(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

. We first construct an integer N. Let

$$f(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-d)^p$$

for a large prime p. We split the integral into two parts:

$$\frac{1}{(p-1)!}\int_0^\infty = f(x)e^{k-x}dx$$
$$= \frac{1}{(p-1)!}\int_0^k + \frac{1}{(p-1)!}\int_k^\infty$$

$$\delta_k = \frac{1}{(p-1)!}\int_0^k f(x)e^{k-x}\,dx,$$
$$R_k = \frac{1}{(p-1)!}\int_k^\infty f(x)e^{k-x}\,dx,$$
$$S = \frac{1}{(p-1)!}\int_0^\infty f(x)e^{-x}\,dx,$$

We can substitute these equations into the previous fraction so that:

$$e^k = \frac{R_k + \delta_k}{S}.$$

By the properties of the gamma function S must be an integer. Also, $p \neq S \pmod{p}$. For this reason we include the power $p-1$. This is because when we expand f there is only one term that is not divisible by p

$$f(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-d)^p = x^{p-1}(-1)^p(-2)^p \cdots (-d)p + n$$

where n is a large multiple of p. With a large enough p the integral cannot be a multiple of $p!$. However, p is a multiple of R. $t = x - k$. With that substitution we obtain

$$R_k = \frac{1}{(p-1)!}\int_k^\infty f(x)e^{k-x}\,dx = \frac{1}{(p-1)!}\int_0^\infty f(t+k)e^{-t}\,dt.$$

We know that there is a term $(x-k)^p$ in $f(x)$ because $1 \leq k \leq d$. Extending that reasoning, there must exist a term t^p for $f(t+k)$. All the terms must have a common factor of t^p so R_k is divisible by p. Having established that R_k and S are integers, we have rational approximations for e^k of R_k/S. For every value of k we note that rational approximation of e^k has the same denominator. Going back to Liouville's method we can construct an integer $S \times a(e)$:

$$S \cdot a(e) = S\left(a_0 + \sum_{k=1}^{d} a_k \frac{R_k + \delta_k}{S}\right).$$

$$= Sa_0 + S\sum_{k=1}^{d} a_k \frac{R_k + \delta_k}{S}.$$

$$= Sa_0 + \sum_{k=1}^{d} a_k R_k + \sum_{k=1}^{d} a_k \delta_k.$$

Since $a(e) = 0$, $S \times a(e)$ must also equal zero. We let

$$R = S\sum_{k=1}^{d} a_k R_k,$$

and

$$N = S\sum_{k=1}^{d} a_k \delta_k,$$

where N must be an integer since $S \times a(e)$, Sa_0, and R are all integers. By extension, since an p times an integer minus an integer which is not a multiple of p cannot equal 0.

$$N| = |Sa(e) - Sa_0 - R| > 0,$$

We next establish that $|N| < 1$. In order to do this recall that every term in f is in the form $x - d$ for $0 \leq d \leq k$. Also, since $|x - k| \leq d$,

$$f(x) \leq d^{p(d+1)-1}$$

since that is the degree of f. Substituting this into our equation for δ_k we get

$$\delta_k \leq \frac{e^d d^{p(d+1)-1}}{(p-1)!}$$

For a large enough p we can make $\delta_k < \epsilon$ given d and $\epsilon > 0$. N is a linear combination of δ_k which means $|N|$ must also be greater than one. With a large enough p N is an inetger between 0 and 1, contradicting the funadamental principle of number theory. And at long last, we have proved e's transcendence.

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