An Overview of Diophantine Approximation

Neil Kulkarni

July 2026

Overview

How well can an irrational number be approximated by rationals?

- **Dirichlet's Theorem:** Every irrational has infinitely many rationals p/q with $|\alpha - p/q| < 1/q^2$. Two proofs: pigeonhole and geometry of numbers.
- **Minkowski & Blichfeldt:** Lattice points in convex bodies give the same bound and generalize it.
- **Simultaneous approximation:** Several numbers at once, with the weaker exponent $1 + 1/n$.
- **Hurwitz's Theorem:** The sharp constant $1/\sqrt{5}$; the golden ratio is the extremal case.
- **Irrationality measure:** Quantifying approximability; Liouville numbers.
- **Roth's Theorem:** Algebraic irrationals resist approximation of quality $> 2 + \delta$.

Dirichlet's Approximation Theorem

Theorem (Dirichlet)

Let α be real. For every positive integer N, there is a rational p/q with $1 \leq q \leq N$ and
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq} \leq \frac{1}{q^2}.$$

This Theorem is the basis of approximating rationals and the following sections I cover are essentially generalizations, improvements, or applications of this theorem.

Dirichlet: Pigeonhole Proof

Write $k\alpha = y_k + x_k$ with $y_k \in \mathbb{Z}$ and $x_k \in [0,1)$ for $k = 0, 1, \ldots, N$.

- **Pigeonhole.** There are $N+1$ fractional parts x_k but only N subintervals of $[0,1)$ of length $1/N$.
- So two of them, x_i and x_j with $i < j$, lie in the same subinterval:

$$|x_j - x_i| < \frac{1}{N}.$$

- Rearranging $k\alpha = y_k + x_k$, this says

$$\big|(j-i)\alpha - (y_j - y_i)\big| < \frac{1}{N} \le \frac{1}{j-i}.$$

- Divide through by $(j-i)$:

$$\left|\alpha - \frac{y_j - y_i}{j-i}\right| < \frac{1}{(j-i)N} \le \frac{1}{(j-i)^2}.$$

Minkowski's Convex Body Theorem

Theorem (Minkowski)

Let $C \subset \mathbb{R}^d$ be symmetric, convex, and bounded. If $\mathrm{vol}(C) > 2^d$, then C contains a nonzero lattice point of $\mathbb{Z}^d$.

Three defining properties.

- *Bounded:* some R with $|x| \leq R$ for all $x \in C$.
- *Convex:* for $x, y \in C$, the whole segment $[x, y] \subseteq C$.
- *Symmetric:* $x \in C \implies -x \in C$.

Proof. Let $C' = \frac{1}{2}C$, so $\mathrm{vol}(C') > 1$. The translates $C' + u$ over $u \in \mathbb{Z}^d$ cannot all be disjoint (a volume/packing argument as $N \to \infty$), so two overlap. By symmetry and convexity, the difference of the overlap points is a nonzero lattice point in C.

Minkowski's Convex Body Theorem (Cont'd)

Why translates must overlap. Suppose for contradiction that the translates $C' + u$, $u \in \mathbb{Z}^d$, are pairwise disjoint. Let D be the diameter of C'. Then

$$\bigcup_{u \in [-N,N]^d} (C' + u) \subseteq [-(N+D),\, N+D]^d.$$

Comparing volumes (the translates are disjoint):

$$(2N+1)^d \cdot \mathrm{vol}(C') \leq 2^d (N+D)^d.$$

Rearranging,

$$1 < \mathrm{vol}(C') \leq \left(\frac{2(N+D)}{2N+1} \right)^d.$$

As $N \to \infty$, the right side $\to 1$ contradiction. Hence some translates overlap, and Minkowski follows.

Dirichlet via Minkowski

Apply Minkowski to the strip

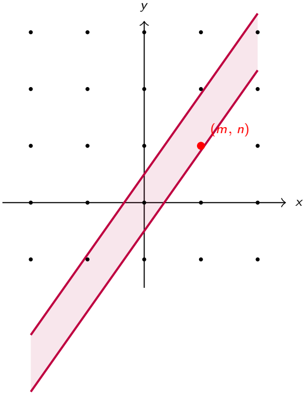
$$P = \left\{(x, y) : |x| \le N + \tfrac{1}{2},\ |y - \alpha x| \le \tfrac{1}{N}\right\}.$$

This P is symmetric, convex, bounded, with

$$|P| = \frac{2(2N+1)}{N} > 2^2.$$

So P contains a nonzero lattice point (m, n) with $|m| \le N$ and $|\alpha m - n| < 1/N$. Dividing by m:

$$\left|\alpha - \frac{n}{m}\right| < \frac{1}{mN}.$$



The strip around $y = \alpha x$ with the guaranteed lattice point.

Blichfeldt's Generalization

Theorem (Blichfeldt)

Let $A \subset \mathbb{R}^n$ be measurable with $\mathrm{vol}(A) > k$. Then some translate $A + x$ contains at least $k + 1$ points of $\mathbb{Z}^n$.

Proof. Count lattice points via $f(x) = \sum_{y \in \mathbb{Z}^n} \mathbf{1}_{A+x}(y)$. Integrating over the unit cube and unfolding:

$$\int_{[0,1]^n} f(x)\,dx = \int_{\mathbb{R}^n} \mathbf{1}_A(t)\,dt = \mathrm{vol}(A) > k.$$

If the average of the integer-valued f exceeds k, then $f(x) \geq k + 1$ for some x.

Minkowski is the case $k = 1$ applied to $\frac{1}{2}C$: this is the engine behind the whole geometric approach.

Simultaneous Dirichlet Approximation

Theorem

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. For any integer $Q \geq 1$, there exist integers $p_1, \ldots, p_n$ and q with $1 \leq q \leq Q$ such that

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q\, Q^{1/n}} \quad \text{for all } i.$$

Proof Apply Minkowski in $\mathbb{R}^{n+1}$ to the box

$$C = \left\{|x_0| < Q + 1,\ |\alpha_i x_0 - x_i| < Q^{-1/n}\right\}.$$

This region is symmetric, convex, and bounded, with

$$\mathrm{vol}(C) = 2(Q+1)\,(2Q^{-1/n})^n = \frac{2^{n+1}(Q+1)}{Q} > 2^{n+1}.$$

A nonzero lattice point $(q, p_1, \ldots, p_n)$ with $q > 0$ gives the result.

Table

The single-variable exponent 2 degrades as the number of simultaneous targets grows:

n	Exponent $1 + 1/n$	Bound
1	2	$1/q^2$
2	$3/2$	$1/q^{3/2}$
3	$4/3$	$1/q^{4/3}$
n	$1 + 1/n$	$1/q^{1+1/n}$

As $n \to \infty$, the exponent $\to 1$: approximating many numbers with a *shared* denominator gets progressively harder. Yet the same geometric application in Minkowski's theorem delivers all of these at once.

Continued Fractions

Definition

A continued fraction is

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0; a_1, a_2, \ldots].$$

The truncations h_n/k_n are the *convergents*.

- Finite continued fractions $\leftrightarrow$ rationals; infinite ones $\leftrightarrow$ irrationals.
- Convergents are the *best* rational approximations, improving monotonically:

$$\left|\xi - \frac{h_n}{k_n}\right| < \left|\xi - \frac{h_{n-1}}{k_{n-1}}\right|.$$

- **Legendre:** any r/s (lowest terms) with $|\xi - r/s| < 1/(2s^2)$ *must* be a convergent.
- Slowly growing partial quotients have poor approximability. The golden ratio $\phi = [1; 1, 1, 1, \ldots]$ is the extreme case.

Hurwitz's Theorem

Theorem (Hurwitz)

For any irrational ξ, there are infinitely many rationals h/k with

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}\,k^2}.$$

The constant $1/\sqrt{5}$ is the best possible.

Proof strategy. Set $\gamma_n = k_n^2\,|\xi - h_n/k_n|$. Among any three consecutive convergents,

$$\min(\gamma_{n-1}, \gamma_n, \gamma_{n+1}) < \frac{1}{\sqrt{5}}.$$

Using the identity $|\xi - h_n/k_n| + |\xi - h_{n-1}/k_{n-1}| = 1/(k_n k_{n-1})$ and the convergent recurrences, one shows $5\gamma^2 < 1$ for the minimum. Since this holds infinitely often, infinitely many convergents satisfy the bound. ■

The Irrationality Measure

Definition

The irrationality measure $\mu(x)$ is the infimum of exponents μ for which

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^{\mu}}$$

has only finitely many rational solutions.

- $\mu(x) = 1$ if x is rational.
- $\mu(x) = 2$ for almost all reals, including all algebraic irrationals (by Roth) and e.
- $\mu(x) = \infty$ for *Liouville numbers* approximable beyond any polynomial rate.

Via continued fractions:

$$\mu(x) = 1 + \limsup_{n\to\infty} \frac{\ln q_{n+1}}{\ln q_n} = 2 + \limsup_{n\to\infty} \frac{\ln a_{n+1}}{\ln q_n}.$$

Known Measures

Constant x	**Upper bound on** $\mu(x)$	**Method**
ϕ (golden ratio)	2	Hurwitz, convergents
e	2	Hermite–Lindemann
$\ln 2$	3.8914	Hata (1993)
$\zeta(3)$	5.5139	Rhin–Viola (1999)
π	7.1032	Hata (1992)
Liouville L	∞	direct construction

The Liouville constant $L = \sum_{n=1}^{\infty} 10^{-n!} = 0.11000100\ldots$ was the first proven transcendental: its factorial-spaced digits make it approximable to any order, forcing $\mu(L) = \infty$.

Roth's Theorem

Theorem (Roth, 1955)

Let α be an irrational algebraic number and $\delta > 0$. Then there are only finitely many integer pairs (x, y), $y > 0$, with

$$\left| \alpha - \frac{x}{y} \right| \leq y^{-2-\delta}.$$

- *Dirichlet:* **infinitely many** approximations of quality ≤ 2 for every irrational.
- *Roth:* for **algebraic** irrationals, quality $> 2 + \delta$ happens only **finitely** often.

The Subspace Theorem

Definition 1 (Linear forms)

A *linear form* in n variables is

$$L(x) = \alpha_1 x_1 + \cdots + \alpha_n x_n, \qquad \alpha_i \in \mathbb{C}.$$

Forms $L_1, \ldots, L_n$ are *linearly independent* if $c_1 L_1 + \cdots + c_n L_n \equiv 0$ (with $c_i \in \mathbb{C}$) forces every $c_i = 0$; equivalently $\det(\alpha_{ij}) \neq 0$. The norm of $x \in \mathbb{Z}^n$ is $\|x\| = \max(|x_1|, \ldots, |x_n|)$.

Theorem 2 (Subspace Theorem, Schmidt)

Let $L_1, \ldots, L_n$ be linearly independent linear forms with algebraic coefficients, and $\delta > 0$. Then the integer solutions of

$$|L_1(x) \cdots L_n(x)| \leq \|x\|^{-\delta}$$

lie in **finitely many** *proper linear subspaces of $\mathbb{Q}^n$.*

Not finitely many *solutions*: for $n \geq 3$ a whole subspace (e.g. $x_3 = 0$) can carry infinitely many. The theorem localizes them, it does not count them.

Subspace Theorem Proving Roth's Theorem

Take integers (x, y), $y > 0$, with $\left|\alpha - \frac{x}{y}\right| \leq y^{-2-\delta}$.

1. Reshape as a form product. Multiply by y^2:

$$|y(x - \alpha y)| \leq y^{-\delta}.$$

The forms Y and $X - \alpha Y$ are linearly independent.

2. Replace y by $\|x\|$. Since $|x/y| \leq |\alpha| + y^{-2} \leq |\alpha| + 1$, we get $\|x\| \leq Cy$ with $C := |\alpha| + 1$. Hence

$$|y(x - \alpha y)| \leq y^{-\delta} \leq C^{\delta} \|x\|^{-\delta}.$$

3. Apply the theorem. Solutions lie in finitely many proper subspaces of $\mathbb{Q}^2$ i.e. finitely many lines T through the origin.

4. Each line is finite. Write $T = \{\lambda(x_0, y_0) : \lambda \in \mathbb{Q}\}$ with $\gcd(x_0, y_0) = 1$. A solution in $T \cap \mathbb{Z}^2$ is $(x, y) = \lambda(x_0, y_0)$, $\lambda \in \mathbb{Z}$, and

$$0 < \left|\alpha - \frac{x_0}{y_0}\right| = \left|\alpha - \frac{x}{y}\right| \leq |\lambda|^{-2} y_0^{-2},$$

bounding $|\lambda|$. So each T holds finitely many solutions, and there are finitely many T.

Thanks for listening!

Questions?