The Hidden Arithmetic of Integer Partitions

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What is a Partition?

Definition

An *integer partition* of a positive integer n is a way of writing n as an unordered sum of positive integers.

Example

The integer 4 has exactly five partitions:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

We write $p(n)$ for the number of partitions of n, with the convention $p(0) = 1$.

How Fast Does $p(n)$ Grow?

n	0	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	1	2	3	5	7	11	15	22	30	42

- Innocent enough... but then

$$p(100) = 190{,}569{,}292, \qquad p(200) = 3{,}972{,}999{,}029{,}388.$$

- Hardy and Ramanujan later showed

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\tfrac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

- Super-exponential growth – yet it hides extraordinary arithmetic patterns.

Ramanujan's Three Congruences

In 1919, Ramanujan discovered – by hand! –

Theorem (Ramanujan, 1919)

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

- $p(4) = 5,\ p(9) = 30,\ p(14) = 135, \ldots$ – every single one divisible by 5, forever.
- The observation alone is extraordinary. **Why** should this be true?
- This talk builds the tools to prove the mod 5 case, elementarily.

History: Leonhard Euler (1707–1783)

- Euler began the systematic study of partitions.
- His 1748 masterwork *Introductio in Analysin Infinitorum* introduced **generating functions** as a combinatorial tool.
- A single idea transformed partition theory from a collection of curious observations into a mathematical science.

Euler's Generating Function

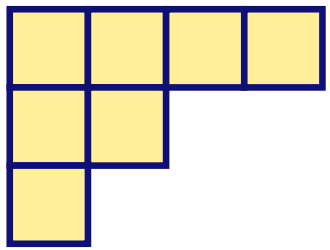
Theorem (Euler's generating function)

$$\sum_{n=0}^{\infty} p(n)\, q^n \;=\; \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

- Each factor $(1-q^k)^{-1} = 1 + q^k + q^{2k} + \cdots$ encodes how many times the part k is used.
- Multiplying over all k enumerates every multiset of positive integers summing to n.
- One observation $\Rightarrow$ an enormous toolkit.

The 19th Century: Sylvester's Young Diagrams

- James Joseph Sylvester (1814–1897) introduced **Young diagrams** to represent partitions pictorially.
- Percy MacMahon (1854–1929) systematized "combinatory analysis," computing vast tables of $p(n)$.



The Young diagram of $\lambda = (4, 2, 1) \vdash 7$

Rogers–Ramanujan Identities

Theorem (Rogers, 1894)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

- "Partitions with gaps of at least 2" $\longleftrightarrow$ "partitions into parts $\equiv \pm 1 \pmod 5$".
- Forgotten for 25 years – until Ramanujan independently rediscovered it around 1913.

Hardy and Ramanujan: The Letter

- January 1913: a young Indian clerk, Srinivasa Ramanujan, mailed a letter to G.H. Hardy at Cambridge.
- Roughly 120 results, most stated without proof.
- Hardy: "they must be true, because if they were not true, no one would have the imagination to invent them".
- Ramanujan came to England in 1914; their collaboration lasted until his death in 1920, at age 32.

The Hardy–Ramanujan Asymptotic Formula

Theorem (Hardy–Ramanujan, 1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

- π appears in a formula about pure integer combinatorics!
- Astonishingly accurate: for $n = 200$ the relative error is under 0.004%.
- The proof introduced the **circle method** – now a cornerstone of analytic number theory.

Why? The Search for a Combinatorial Proof

- Ramanujan's 1919 paper announced the congruences, with sketches for mod 5 and mod 7.
- Dyson (1944) conjectured that the **rank** (largest part minus number of parts) splits $p(5n+4)$ into 5 equal classes.
- Proved by Atkin–Swinnerton-Dyer (1954). Rank fails for mod 11; Andrews–Garvan invented the **crank**.
- Our route today is different: elementary, via generating functions.

Preliminaries: Notation

We work in the ring of formal power series $\mathbb{Z}[[q]]$.

Definition

For $k \in \mathbb{Z}^+$, define

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty} (1 - q^{km}).$$

In particular, $f_1 = \prod_{m \geq 1}(1 - q^m)$.

All identities hold formally, and also analytically for $|q| < 1$.

Euler's Generating Function, Revisited

Theorem

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

Proof

Expand each geometric series:

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} q^{km_k} = \sum_{(m_k)} q^{\sum_k km_k}.$$

The coefficient of q^n counts multiplicities $(m_1, m_2, \dots)$ with $\sum km_k = n$ – exactly $p(n)$. □

Euler's Theorem: Odd Parts Equal Distinct Parts

Theorem (Euler, 1748)

Let $p_{\text{odd}}(n)$ count partitions of n into odd parts, and $p_{\text{dist}}(n)$ count partitions of n into distinct parts. Then

$$p_{\text{odd}}(n) = p_{\text{dist}}(n) \quad \text{for all } n \geq 0.$$

Two seemingly unrelated restrictions... give exactly the same count!

Proof: The One-Line Version

Proof

Since $(1+q^k)(1-q^k) = 1-q^{2k}$, splitting $\prod_{k\geq 1}(1-q^k)$ into its even and odd parts and cancelling gives

$$\prod_{k=1}^{\infty}(1+q^k) \;=\; \prod_{k \text{ odd}} \frac{1}{1-q^k}.$$

- Left side generates $p_{\mathrm{dist}}(n)$ (each part used 0 or 1 times).
- Right side generates $p_{\mathrm{odd}}(n)$ (only odd parts, any multiplicity). $\square$

Example: $n = 9$

Example

$p_{\text{odd}}(9) = p_{\text{dist}}(9) = 8.$

Odd parts: $9;\ 7{+}1{+}1;\ 5{+}3{+}1;\ 5{+}1^4;\ 3^3;\ 3^2{+}1^3;\ 3{+}1^6;\ 1^9.$

Distinct parts:

$9;\ 8{+}1;\ 7{+}2;\ 6{+}3;\ 6{+}2{+}1;\ 5{+}4;\ 5{+}3{+}1;\ 4{+}3{+}2.$

There is even an explicit, invertible bijection between the two lists.

The Pentagonal Number Theorem

Definition

The *generalized pentagonal numbers* are $\omega(k) = \frac{k(3k-1)}{2}$ for $k \in \mathbb{Z}$: $0, 1, 2, 5, 7, 12, 15, \ldots$

Theorem (Euler)

$$f_1 = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2}$$

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots$$

f_1 is remarkably *sparse*: only exponents equal to a pentagonal number survive.

A Recurrence for $p(n)$

Corollary

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

Example

$$p(7) = p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15. \checkmark$$

This sparsity is the seed of Ramanujan's congruences.

Proving $p(5n+4) \equiv 0 \pmod 5$

Theorem (Ramanujan, 1919)

For all $n \geq 0$: $\quad p(5n+4) \equiv 0 \pmod 5$.

Our elementary route (no modular forms!):

1. Jacobi's triple product $\Rightarrow$ a formula for f_1^3.
2. Two residue lemmas: triangular & pentagonal numbers mod 5.
3. A "complementary gap" forces a key coefficient to vanish.

Jacobi's Triple Product

Theorem (Jacobi Triple Product)

For $|q| < 1$, $z \neq 0$:

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + zq^{2m-1})(1 + z^{-1}q^{2m-1}).$$

Corollary

$$f_1^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1)\, q^{k(k+1)/2}.$$

The exponents $T_k = k(k+1)/2$ are the **triangular numbers**.

Two Key Residue Lemmas

Lemma (Triangular numbers mod 5)

$T_k \pmod 5 \in \{0, 1, 3\}$ for all k. (2 and 4 never appear.)

Lemma (Pentagonal numbers mod 5)

$\omega(k) \pmod 5 \in \{0, 1, 2\}$ for all k. (3 and 4 never appear.)

Both proofs: the residue depends only on $k \pmod 5$ – just check five cases!

The Arithmetic Miracle

Remark

Triangular numbers miss $\{2, 4\}$; pentagonal numbers miss $\{3, 4\}$, modulo 5.

- These **complementary gaps** conspire so that writing $5m + 4$ as (triangular residue) + (pentagonal residue) forces a coefficient to vanish.
- This clean gap structure occurs only for $p \in \{5, 7, 11\}$ – exactly Ramanujan's set!

Sketch of the Main Proof

- **Step 1 (Fermat–Pochhammer):** in char. 5, $(1-x)^5 \equiv 1-x^5$, so $f_1^5 \equiv f_5 \pmod 5$.
- **Step 2:** write $f_1^4 = f_1^3 \cdot f_1$. Nonzero terms $q^a \cdot q^b$ need

$$a \pmod 5 \in \{0,1,3\}, \qquad b \pmod 5 \in \{0,1,2\},$$

$$a + b \equiv 4 \pmod 5.$$

- **Step 3:** checking all cases – only $a \equiv 3,\ b \equiv 1$ survives.
- **Step 4:** but $[q^a] f_1^3 \equiv 0 \pmod 5$ whenever $a \equiv 3 \pmod 5$! Every term vanishes.

$\therefore \quad p(5n+4) \equiv [q^{5n+4}] f_1^4 / f_5 \equiv 0 \pmod 5.$ □

Restricted Partition Families

Definition

For a prime m and $\emptyset \neq R \subseteq \{0, \ldots, m-1\}$, let $p_{m,R}(n)$ count partitions into parts k with $k \pmod m \in R$:

$$\sum_{n \geq 0} p_{m,R}(n)\, q^n = \prod_{\substack{k \geq 1 \\ k \bmod m \in R}} \frac{1}{1-q^k}.$$

Definition (Colored partitions)

A c-colored partition assigns one of c colors to each occurrence of each part:

$$\sum_{n \geq 0} p_c(n)\, q^n = \frac{1}{{f_1}^c}.$$

Original Results: New Congruences

Using the same complementary-gap machinery, we prove:

Proposition

$\overline{p}_5(5n+4) \equiv 0 \pmod 5$, where $\overline{p}_5(n)$ counts partitions into parts $\not\equiv 0 \pmod 5$.

Proposition

$p_2(5n+3) \equiv 0 \pmod 5$, for bicolored partitions.

Theorem

$\mathrm{rcp}_{4,5}(5n+4) \equiv 0 \pmod 5$, for 4-colored partitions into parts $\not\equiv 0 \pmod 5$.

Further Directions

- **More primes, more families:** track the residue support of $f_1^c \pmod{p}$ for other c, p.
- **Bijective proofs:** rank/crank-style statistics for restricted families?
- **Rogers–Ramanujan analogues:** gap conditions $\leftrightarrow$ residue conditions, for new families.
- Connections to the modular-forms / p-adic perspective of the companion paper.

Acknowledgments

Thank you for your attention!

Questions?