# The Hidden Arithmetic of Integer Partitions: Restricted Families, Ramanujan Congruences, and Generating Function Methods

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**Abstract**

We survey and expand upon the classical generating-function theory of integer partitions, with a focus on restricted partition families and Ramanujan-type congruences. In the 18th century, Euler observed that partitions of $n$ into odd parts are in bijection with partitions of $n$ into (pairwise) distinct parts, a fact that emerges almost magically from a single line of algebra (Theorem 4); from this observation we develop the generating function machinery that underlies all of modern partition theory. After tracing the historical arc from Euler through the celebrated Hardy–Ramanujan asymptotic formula to Ramanujan's three spectacular congruences, we provide a self-contained, elementary proof of $p(5n + 4) \equiv 0 \pmod 5$ using Jacobi's triple product identity and the Fermat–Pochhammer reduction. The key structural insight is a simultaneous residue gap: triangular numbers never attain residues 2 or 4 modulo 5, while generalized pentagonal numbers never attain residues 3 or 4, and together these gaps conspire to kill every coefficient of $q^{5m+4}$ in $f_1^4$ modulo 5. We then introduce the framework of restricted partition families – defined by congruence conditions on parts, repetition constraints, and multicolor structures – and establish their generating functions from first principles. We prove original Ramanujan-type congruences for several specific restricted families, presented alongside generating function identities and bijective arguments. The paper is aimed at advanced undergraduates and early graduate students comfortable with formal power series and basic modular arithmetic.

## 1 Introduction

What could be simpler than breaking a number apart?

Given a positive integer $n$, an *integer partition* of $n$ is a way to write $n$ as an unordered sum of positive integers. For example, the integer 4 admits exactly five such decompositions:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

We denote the total count of partitions of $n$ by $p(n)$, with the convention $p(0) = 1$ (the empty partition). The first few values paint an inviting picture:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 |

Innocent enough. Then $p(100) = 190{,}569{,}292$ and $p(200) = 3{,}972{,}999{,}029{,}388$. The sequence grows at a rate roughly proportional to $e^{\pi\sqrt{2n/3}}$ – superexponential in $\sqrt{n}$ – yet conceals arithmetic patterns of extraordinary delicacy. This paper is an investigation of those patterns.

Integer partitions sit at an unlikely crossroads. They appear in the representation theory of the symmetric group $S_n$ (the irreducible representations of $S_n$ are indexed by partitions of $n$), in the theory of modular forms (the generating function $1/f_1 = \prod_{k\geq 1}(1-q^k)^{-1}$ is essentially the reciprocal of the Dedekind eta function), and in theoretical physics (partitions count degeneracy of energy levels of a free boson gas – a connection Hardy and Ramanujan's asymptotic formula essentially predicted before physicists needed it). Even graph theory has a quiet encounter with partitions, since the number of conjugacy classes of $S_n$ equals $p(n)$. It is the kind of object that has absolutely no right to appear everywhere, and yet does.

The hero of our story is Srinivasa Ramanujan (1887–1920). In 1919, he observed – by staring at tables of $p(n)$ computed entirely by hand – the following three congruences:

$$p(5n+4) \equiv 0 \pmod 5, \qquad p(7n+5) \equiv 0 \pmod 7, \qquad p(11n+6) \equiv 0 \pmod{11}.$$

The first says, for instance, that $p(4) = 5$, $p(9) = 30$, $p(14) = 135$, $p(19) = 490$, $p(24) = 1575$ are all divisible by 5 – every single one, forever. Why? The observation alone is extraordinary; the proof requires a genuinely new idea. The present paper supplies that idea elementarily, using only Jacobi's triple product and modular arithmetic, building on the tradition of Winquist [5], and complementing the $p$-adic spectral approach of our companion paper [**?**] (currently in preparation; a full citation will be added once it is available), which studies multivariate eta-products via the Atkin $U_p$ operator on genus-zero modular curves.

Beyond Ramanujan's three congruences lies a much richer landscape. Informally, a *restricted partition family* is any subset of partitions singled out by local conditions on the parts: which integers may appear (Section 7.1), how many times each may repeat (Section 7.2), or how many colors each part may carry (Section 7.3). Such families – obtained by constraining which parts appear, how many times they may repeat, or how many colors each part receives – exhibit their own arithmetic properties, many of which remain active topics of research. We develop the generating function framework for such families and establish several arithmetic results. Where the companion paper deploys heavier machinery from the theory of modular forms (modular curves, Newton polygons, and Atkin $U_p$-operators – technical tools that we will not need here), we emphasize explicit combinatorics and elementary number theory.

As an aside, a different paper produced in this same Euler Circle seminar [**?**] finds an analogous surprising equivalence in an entirely unrelated setting: a three-way correspondence between the Weisfeiler–Leman color refinement algorithm, tree homomorphism counts, and fractional graph isomorphism. It shares the philosophy of this paper – that unexpected equivalences between apparently unrelated structures reveal deeper structure – but not its content, and the two papers are otherwise independent.

**Organization.** Section 2 traces the history of partition theory from Euler's 1748 innovations through Hardy, Ramanujan, and into the modern era. Section 3 establishes generating

functions and basic definitions. Section 4 proves Euler's odd–distinct theorem via generating functions and an explicit bijection. Section 5 covers the pentagonal number theorem and the recurrence for $p(n)$. Section 6 presents a complete proof of $p(5n+4) \equiv 0 \pmod 5$ and discusses the other two Ramanujan congruences. Section 7 introduces restricted partition families and their generating functions. Our original results appear in Section 8 onward.

# 2 History and Motivation

## 2.1 Euler's Generating Functions

The systematic study of integer partitions begins with Leonhard Euler (1707–1783), who was, by any reasonable measure, producing mathematical breakthroughs at a pace that should be physically impossible. His 1748 masterwork *Introductio in Analysin Infinitorum* introduced generating functions as a rigorous combinatorial tool and thereby transformed partition theory from a collection of curious observations into a mathematical science.

The idea, informally, is to encode an entire sequence of numbers as the coefficients of a single power series, so that algebraic operations on the series – multiplying, dividing, extracting coefficients – translate directly into combinatorial operations on the objects being counted. Euler's foundational insight is deceptively simple: the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

is the generating function for $p(n)$. Each factor $(1-q^k)^{-1} = 1 + q^k + q^{2k} + \cdots$ encodes the choice of how many times the part $k$ appears; multiplying over all $k$ enumerates every possible multiset of positive integers summing to $n$. With this single observation, Euler opened an enormous toolkit. He proved that the number of partitions into odd parts equals the number of partitions into distinct parts – a result that follows in one line of generating function algebra (Section 4; see Example 4 for a worked instance). He also proved his celebrated pentagonal number theorem (to be stated precisely as Theorem 5 below):

$$\prod_{n=1}^{\infty}(1-q^n) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots,$$

where the exponents are the *generalized pentagonal numbers* $k(3k-1)/2$ for $k \in \mathbb{Z}$ (Definition 5). This identity gives an immediate recurrence for $p(n)$ and – more importantly – places crucial structural constraints on which powers of $q$ appear in $f_1 = \prod(1-q^n)$, the very constraints that ultimately prove Ramanujan's congruences.

## 2.2 The 19th Century: Rogers, Sylvester, and MacMahon

The 19th century saw partition theory branch in several directions. James Joseph Sylvester (1814–1897) introduced pictorial representations of partitions now called Young diagrams

(these are also called *Ferrers diagrams*, after Norman Macleod Ferrers, who represented partitions as arrays of dots slightly earlier; Young's boxes eventually became the standard convention), and proved several bijection theorems by purely combinatorial arguments. The Young diagram of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ is an array of left-justified boxes with $\lambda_i$ boxes in row $i$; transposing the diagram gives the *conjugate partition* $\lambda'$, and conjugation provides bijective proofs of several classical identities. For example, if $\lambda = (4, 2, 1)$, the diagram has columns of heights $3, 2, 1, 1$, so $\lambda' = (3, 2, 1, 1)$.

Percy MacMahon (1854–1929) systematized what he called *combinatory analysis*, computing extensive tables of $p(n)$ and introducing the framework that would later develop into symmetric function theory. His two-volume *Combinatory Analysis* (1915–1916) remained the standard reference for decades and is still worth reading for sheer combinatorial ambition.

Leonard James Rogers (1862–1933) proved – largely unnoticed at the time – the celebrated *Rogers–Ramanujan identities*, published in 1894 [12]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}, \qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}.$$

[Partitions with gaps] A partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ has *gaps of at least* $d$ if $\lambda_i - \lambda_{i+1} \geq d$ for every $i$.

The Rogers–Ramanujan identities say that partitions with gaps of at least 2 between parts (Definition 2.2, $d = 2$) are equinumerous with partitions into parts $\equiv \pm 1 \pmod 5$ (resp. $\equiv \pm 2 \pmod 5$) – substantially deeper than Euler's odd–distinct theorem. What makes this so surprising is that the gap condition is a statement purely about the *spacing* between consecutive parts, with no evident reference to arithmetic, while the right-hand side is a statement purely about *residues modulo* 5; that two such unrelated-looking conditions carve out exactly the same set of partitions is precisely the kind of coincidence that should not happen and yet does. Rogers's paper was forgotten for 25 years until Ramanujan independently rediscovered the identities around 1913, then found Rogers's original paper in the literature. The identities are now considered among the most beautiful in all of combinatorics, the kind of theorem that makes even professional mathematicians say "wait, that's actually true?"

### 2.3 Hardy, Ramanujan, and the Asymptotic Formula

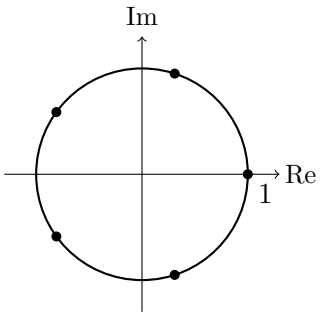
The story now arrives at its most famous episode. In January 1913, a young Indian clerk named Srinivasa Ramanujan mailed a letter to G.H. Hardy at Trinity College, Cambridge. The letter contained roughly 120 mathematical results, most stated without proof, in areas ranging from infinite series to continued fractions. Hardy, the foremost British analyst of the era, recognized that the results were the work of someone of extraordinary ability: "they must be true," he famously remarked, "because if they were not true, no one would have the imagination to invent them."

Hardy arranged for Ramanujan to come to England in 1914. Their five-year collaboration was cut short by Ramanujan's illness and death in 1920 at age 32. Then, in 1918, this

collaboration produced its finest result: the Hardy–Ramanujan asymptotic formula [7],

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$

This formula is stunning: the appearance of $\pi$ in an asymptotic count of purely combinatorial objects hints at a deep connection to complex analysis and the geometry of modular curves – a connection made precise below by the circle method, which realizes $p(n)$ as a contour integral of essentially $1/f_1$. The formula is also spectacularly accurate: for $n = 200$, the relative error is less than 0.004%, remarkable given that, before Hardy and Ramanujan, no asymptotic formula for $p(n)$ with any explicit error control was known at all. The proof introduced the *circle method*, where one integrates $\sum p(n)q^n = 1/f_1$ along a contour near the unit circle in $\mathbb{C}$, concentrating the contribution near roots of unity (the "major arcs"), as illustrated below.




The circle method integrates near $|q| = 1$, concentrating contributions around roots of unity (major arcs).

The circle method has since become a fundamental tool of analytic number theory far beyond partition functions.

Hans Rademacher [10] later refined the circle method to give an exact convergent infinite series for $p(n)$. In the formula below, $A_k(n)$ is a Kloosterman-type exponential sum over residues coprime to $k$ (roughly, a finite trigonometric sum measuring how the generating function of $p(n)$ transforms near the $k$-th roots of unity), and $s(h,k)$ is the classical *Dedekind sum*, an arithmetic sum of cotangent-like terms attached to a pair of coprime integers $h, k$; both arise from the precise transformation law of $1/f_1$ under modular substitutions. With this notation,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \cdot A_k(n) \cdot \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right).$$

That a convergent infinite series of transcendental functions evaluates to the integer $p(n)$ for every $n \geq 0$ is a small miracle – of the kind that partition theory keeps producing.

### 2.4 Ramanujan's Congruences and Their Combinatorial Explanations

During his years in England, Ramanujan discovered these three congruences by a combination of prodigious computational skill and an extraordinary ability to detect patterns in numerical

data. His 1919 paper [11] announced them and sketched proofs for the modulo-5 and modulo-7 cases. The complete proof of the modulo-11 congruence was given by Winquist [5], using delicate generating-function manipulations with $f_1^{10}$ (Remark 6.4), and later, more cleanly, via modular forms by Atkin [3].

The natural question is not just whether these congruences hold, but why. Is there a combinatorial explanation – a way to divide the partitions of $5n + 4$ into five equally sized families? We say the *rank* of a partition is its largest part minus its number of parts. Freeman Dyson [6] conjectured in 1944 that the rank provides exactly such an explanation: the partitions of $5n + 4$ split into five classes of equal size $p(5n + 4)/5$, according to the residue of the rank modulo 5. This was proved by Atkin and Swinnerton-Dyer [4]. For the modulo-11 congruence, rank fails to split the partitions of $11n + 6$ evenly, and Andrews and Garvan [2] invented a new statistic – the *crank* – to complete the combinatorial picture. (Remarkably, Ramanujan himself had hinted at the existence of such a statistic in a letter written shortly before his death, without knowing what it was. Even posthumously, he was ahead of everyone else.)

Subsequent work by Ono [9], using modular forms and Galois representations, showed that Ramanujan-type congruences are abundant: for every prime $\ell \geq 5$, there exist infinitely many arithmetic progressions $an + b$ with $p(an + b) \equiv 0 \pmod{\ell}$. The full landscape of partition congruences remains an active research frontier.

### 2.5 The Modular Forms Connection and the Present Paper

The correct framework for understanding partition congruences is the theory of modular forms. The generating function $q^{1/24}/f_1$ is, up to normalization, the Dedekind eta function $\eta(\tau)$, where $q = e^{2\pi i\tau}$. This function is a modular form of weight $1/2$. Ramanujan's congruences, in this language, reflect congruences between modular forms of specific levels, provable systematically using Hecke operators.

Our companion paper (in preparation) develops a $p$-adic spectral theory over a parameter space of restricted color partition functions $P_{\vec{v}}(n)$ defined via multivariate multiplicative eta-products (see Remark 3 below for the notation), proving $p$-adic valuation scaling bounds under the Atkin $U_p$ operator on genus-zero modular curves $X_0(p)$ for $p \in \{5, 7, 13\}$. The present paper operates in the same thematic territory but takes a more elementary and bijective approach: we illuminate the combinatorial content of these arithmetic properties by working explicitly with generating functions and residue analysis, accessible to readers meeting modular forms for the first time. Everything we say about the companion paper here is meant only as context and orientation, and should be read as consistent with, but independent of, the elementary arguments that follow.

## 3 Preliminaries: Formal Power Series and Generating Functions

We work in the ring of formal power series $\mathbb{Z}[[q]]$, where all identities are formal (they hold coefficient-by-coefficient, with no concern for convergence); for $|q| < 1$ the same identities may also be read analytically. We use standard $q$-Pochhammer notation, recalled here for readers

meeting it for the first time.

[$q$-Pochhammer symbol] For $a \in \mathbb{C}$ and a nonnegative integer $n$ (or $n = \infty$), the *$q$-Pochhammer symbol* is

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i), \qquad (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i).$$

The infinite products generating $p(n)$, and its many restricted variants, are all built from Pochhammer symbols with $a$ a power of $q$; the next definition records the specific family we use throughout.

[Standard infinite products] For $k \in \mathbb{Z}^+$, define

$$f_k := (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{km}).$$

In particular, $f_1 = \prod_{m\geq 1}(1-q^m)$. All identities hold in $\mathbb{Z}[[q]]$, and also as analytic identities for $|q| < 1$.

The notation $f_k$ lets us state Ramanujan-type congruences as clean statements about which powers of $q$ can appear in various products and quotients of the $f_k$; this is the language in which the rest of the paper is written. With this notation in hand, we can restate Euler's foundational identity compactly.

[Euler's generating function] The generating function for $p(n)$ is:

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{f_1} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

*Proof.* Expanding each geometric series:

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} q^{km_k} = \sum_{(m_k)_{k\geq 1}} q^{\sum_{k\geq 1} km_k}.$$

The coefficient of $q^n$ counts sequences of non-negative integers $(m_1, m_2, \ldots)$ satisfying $\sum km_k = n$, where $m_k$ represents the multiplicity of part $k$. This is precisely $p(n)$. □

This proof is almost embarrassingly short for something so fundamental – which is rather the point of generating functions: once the right encoding is chosen, the combinatorics all but proves itself.

[Support of a power series] For $F(q) = \sum_n a_n q^n \in \mathbb{Z}[[q]]$, the *support* of $F$ is $\{n \geq 0 : a_n \neq 0\}$, the set of exponents with nonzero coefficient.

[Restricted partition functions] For a nonempty set $S \subseteq \mathbb{Z}^+$, let $p_S(n)$ count partitions of $n$ with all parts in $S$:

$$\sum_{n=0}^{\infty} p_S(n)\, q^n = \prod_{k\in S} \frac{1}{1-q^k}.$$

Let $p_S^{\text{dist}}(n)$ count partitions into distinct parts, all from $S$:

$$\sum_{n=0}^{\infty} p_S^{\text{dist}}(n)\, q^n = \prod_{k \in S} (1 + q^k).$$

Both formulas follow exactly as in the proof of Theorem 3, restricting the product to $k \in S$ (and, in the distinct-parts case, allowing each part multiplicity $m_k \in \{0, 1\}$ only).

We now record one more restriction: instead of restricting which integers may appear, we may instead restrict how each part is decorated.

[Colored partitions] A *c-colored partition* of $n$ is a partition where each occurrence of each part may be assigned one of $c$ distinct colors. If $p_c(n)$ counts $c$-colored partitions,

$$\sum_{n=0}^{\infty} p_c(n)\, q^n = \frac{1}{f_1^c}.$$

Note $p_1(n) = p(n)$, and $p_2(n)$ is the number of bicolored partitions. For example, with colors $R$ (red) and $B$ (blue), the bicolored partitions of 2 are

$$2_R, \quad 2_B, \quad 1_R + 1_R, \quad 1_R + 1_B, \quad 1_B + 1_B,$$

five in total, matching $p_2(2) = 5$.

The multivariate eta-product framework of the companion paper generalizes the colored-partition generating function above: for an integer weight vector $\vec{v} = (v_k)_{k \in K}$, the product $\prod_{k \in K} f_k^{v_k}$ includes $1/f_1^c$ (set $v_k = -c$, $K = \{1\}$) as a special case. Our generating function approach applies to this whole family.

## 4 Euler's Theorem: Odd Parts Equal Distinct Parts

Euler's most beloved partition theorem reveals that two seemingly different restrictions – "every part is odd" and "all parts are distinct" – give the same counts.

[Euler, 1748] Let $p_{\text{odd}}(n)$ count partitions of $n$ into odd parts (i.e., every part is an odd integer – *not* that there is an odd number of parts), and let $p_{\text{dist}}(n)$ count partitions of $n$ into distinct parts. Then

$$p_{\text{odd}}(n) = p_{\text{dist}}(n) \qquad \text{for all } n \geq 0.$$

We give two proofs. The first shows the identity is pure algebra at the level of generating functions; the second exhibits an explicit, invertible map between the two types of partitions, built from binary expansions of multiplicities.

*Proof 1 (Generating functions – the one-line version).* By Definition 3 (with $S = \mathbb{Z}^+$), $\sum p_{\text{dist}}(n) q^n =$

$\prod_{k\geq 1}(1+q^k)$, and restricting to odd $k$, $\sum p_{\text{odd}}(n)q^n = \prod_{k \text{ odd}} \frac{1}{1-q^k}$. We compute:

$$\prod_{k=1}^{\infty}(1+q^k) = \prod_{k=1}^{\infty}\frac{1-q^{2k}}{1-q^k} \tag{1}$$

$$= \frac{\prod_{k\geq 1}(1-q^{2k})}{\prod_{k \text{ odd}}(1-q^k)\cdot\prod_{k\geq 1}(1-q^{2k})} \tag{2}$$

$$= \prod_{k \text{ odd}}\frac{1}{1-q^k}. \tag{3}$$

In (1) we used $(1+q^k)(1-q^k) = 1-q^{2k}$; in (2)–(3) we split $\prod_{k\geq 1}(1-q^k)$ into even and odd parts and cancelled $\prod(1-q^{2k})$. The left side generates $p_{\text{dist}}(n)$ and the right side generates $p_{\text{odd}}(n)$. □

*Proof 2 (Explicit bijection).* We construct an explicit invertible map $\varphi : X \to Y$, where $X$ is the set of partitions of $n$ into odd parts and $Y$ is the set of partitions of $n$ into distinct parts.

**Forward map.** Given $\lambda \in X$, write $\lambda$ as having $a_k \geq 0$ copies of the odd integer $2k-1$, for $k = 1, 2, 3, \ldots$ (so $a_k = 0$ simply means that odd part does not occur). Write each multiplicity in binary: $a_k = \sum_{j\geq 0}\varepsilon_{k,j}\cdot 2^j$ with $\varepsilon_{k,j} \in \{0,1\}$. For each pair $(k,j)$ with $\varepsilon_{k,j} = 1$, include the integer $2^j(2k-1)$ as a part of the new partition $\varphi(\lambda)$.

*Parts are distinct:* every positive integer has a unique factorization $2^j \cdot m$ with $m$ odd, so distinct pairs $(k,j)$ give distinct values $2^j(2k-1)$.

*Sum is preserved:* $\sum_j \varepsilon_{k,j}\cdot 2^j\cdot(2k-1) = a_k(2k-1)$, so summing over $k$ preserves the total.

**Inverse map.** We now reverse the construction. Given $\mu \in Y$, factor each part of $\mu$ as $2^j \cdot m$ with $m$ odd. Fix an odd number $m$; the parts of $\mu$ with odd component $m$ form some subset $\{2^{j_1}m, 2^{j_2}m, \ldots\}$ of distinct powers of 2 times $m$ (distinct because the parts of $\mu$ are themselves distinct). Set $a = 2^{j_1} + 2^{j_2} + \cdots$ and place $a$ copies of the odd part $m$ into the preimage partition. Doing this for every odd $m$ occurring this way among the parts of $\mu$ recovers a partition into odd parts; distinctness of the parts of $\mu$ guarantees that each such $a$ has a well-defined binary representation, so this is exactly the inverse of the forward map. □

For $n = 9$, we verify $p_{\text{odd}}(9) = p_{\text{dist}}(9) = 8$.

Partitions into odd parts: $9;\ 7+1+1;\ 5+3+1;\ 5+1^4;\ 3^3;\ 3^2+1^3;\ 3+1^6;\ 1^9$.

Partitions into distinct parts: $9;\ 8+1;\ 7+2;\ 6+3;\ 6+2+1;\ 5+4;\ 5+3+1;\ 4+3+2$.

Trace the bijection on $3+3+1+1+1$: here $a_1 = 3$ (three copies of the odd part $2\cdot 1-1=1$) and $a_2 = 2$ (two copies of the odd part $2\cdot 2-1=3$). In binary, $a_1 = 3 = (11)_2$, so $\varepsilon_{1,0} = \varepsilon_{1,1} = 1$, contributing parts $2^0\cdot 1 = 1$ and $2^1\cdot 1 = 2$; and $a_2 = 2 = (10)_2$, so $\varepsilon_{2,1} = 1$, contributing the part $2^1\cdot 3 = 6$. The image is $6+2+1$, which indeed appears in the list above. ✓

## 5 The Pentagonal Number Theorem

Before we can prove Ramanujan's congruences, we need Euler's other great combinatorial identity – one whose entire power comes from how few nonzero terms it has.

[Generalized pentagonal numbers] The *generalized pentagonal numbers* are $\omega(k) = k(3k-1)/2$ for $k \in \mathbb{Z}$. For $k = 0, 1, -1, 2, -2, 3, -3, \ldots$ they produce $0, 1, 2, 5, 7, 12, 15, 22, 26, 35, \ldots$.

[Euler's Pentagonal Number Theorem] In $\mathbb{Z}[[q]]$:

$$f_1 = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots$$

*Proof sketch.* The identity follows from the Jacobi triple product identity (Theorem 6.1) via the substitution $z = -q^{1/2}$, $q \mapsto q^{3/2}$; this particular substitution is exactly what turns the exponent $k^2$ of the triple product into the pentagonal exponent $k(3k-1)/2$, after relabeling the summation index. See [1], Chapter 2, for the complete derivation. □

The striking feature of this theorem – and the feature that matters for our purposes – is its *sparsity*: among all powers of $q$, only those with exponent equal to a generalized pentagonal number appear with nonzero coefficient. Concretely, the generalized pentagonal numbers up to $N$ number only about $\sqrt{N}$ (since $\omega(k)$ and $\omega(-k)$ are both of size roughly $\frac{3}{2}k^2$), so among the first $N$ coefficients of $f_1$, only about $\sqrt{N}$ are nonzero – an overwhelming majority vanish. It is exactly this sparsity that lets us control $f_1$ modulo small primes in Section 6.

[Recurrence for $p(n)$] Since $f_1 \cdot \frac{1}{f_1} = 1$, multiplying the pentagonal theorem by $\sum p(n)q^n$ and extracting coefficients of $q^n$ gives:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots,$$

where $p(m) = 0$ for $m < 0$, and the signs follow the pattern $+, +, -, -, +, +, -, -, \ldots$ inherited from the signs $(-1)^k$ in the pentagonal theorem: each pair of terms $\omega(k), \omega(-k)$ shares the sign $(-1)^k$, and the sign flips every two consecutive pairs as $k$ increases.

As a check, consider $n = 7$: $p(7) = p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15$. ✓

## 6 Ramanujan's Congruence $p(5n+4) \equiv 0 \pmod 5$

We now prove the first and most famous Ramanujan congruence. Every tool we need is already on the table: the pentagonal number theorem controls $f_1$, Jacobi's triple product controls $f_1^3$, and arithmetic modulo 5 does the rest. No modular forms are required – though, as the companion paper suggests, the argument is implicitly a shadow of a deeper modular-forms proof. It is fair to ask why one should care about that deeper proof at all, given that the elementary argument below is self-contained: the answer is that the modular-forms approach explains, all at once, why congruences hold for infinitely many primes (Ono's theorem, Section 2.4), whereas the elementary approach here must essentially be redone from scratch for each new modulus (compare Remarks 6.4 and 6.4).

## 6.1 Jacobi's Triple Product and Its Consequences

[Jacobi Triple Product Identity, 1829] For $|q| < 1$ and $z \neq 0$:

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + zq^{2m-1})(1 + z^{-1}q^{2m-1}).$$

This identity, due to Jacobi in his 1829 *Fundamenta Nova Theoriae Functionum Ellipticarum*, is the master identity behind the classical theory of theta functions; both Euler's pentagonal number theorem and the corollary below are obtained from it by specializing $z$ and $q$.

[Jacobi's identity for $f_1^3$]

$$f_1^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1)\, q^{k(k+1)/2}.$$

The exponents $T_k = k(k+1)/2$ are the *triangular numbers*[1], and they are just as sparse, in their own way, as the pentagonal numbers; it is exactly this sparsity in $f_1^3$ that lets us control $f_1^4 = f_1^3 \cdot f_1$ below.

*Proof.* From the triple product with $z = -q^{1/2}$ and $q$ replaced by $q^{1/2}$ one obtains $\sum_k (-1)^k q^{k^2/2}$; differentiating with respect to $z$ and combining with another specialization yields the stated formula for $f_1^3$. See [8], Theorem 357, for the complete derivation. □

## 6.2 Two Key Residue Lemmas

Both lemmas below say the same kind of thing: a natural sequence of integers, reduced modulo 5, misses two of the five possible residues entirely. We record the two "gaps" precisely, since it is the interaction between them that does all the work in Section 6.4.

[Triangular numbers mod 5] The $k$-th triangular number $T_k = \frac{k(k+1)}{2}$ satisfies, for all integers $k \geq 0$:

$$T_k \pmod 5 \;\in\; \{0, 1, 3\}.$$

In particular, $T_k \not\equiv 2$ and $T_k \not\equiv 4 \pmod 5$ for any $k$.

*Proof.* Since $T_{k+5} - T_k = 5k + 15 = 5(k+3) \equiv 0 \pmod 5$, the residue $T_k \pmod 5$ depends only on $k \pmod 5$; so it suffices to check the five representatives $k = 0, 1, 2, 3, 4$, and periodicity handles every other $k \geq 0$ automatically. Computing directly: $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, $T_3 = 6 \equiv 1$, $T_4 = 10 \equiv 0 \pmod 5$. Tabulating:

| $k \pmod 5$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $T_k \pmod 5$ | 0 | 1 | 3 | 1 | 0 |

The residues 2 and 4 never appear. □

[1] So named because $T_k$ is the number of dots in a triangular array with $k$ rows: $1 + 2 + \cdots + k$.

[Generalized pentagonal numbers mod 5] The generalized pentagonal number $\omega(k)$ of Definition 5 satisfies, for all $k \in \mathbb{Z}$:

$$\omega(k) \pmod 5 \in \{0, 1, 2\}.$$

In particular, $\omega(k) \not\equiv 3$ and $\omega(k) \not\equiv 4 \pmod 5$ for any $k$.

*Proof.* As in Lemma 6.2, $\omega(k+5) - \omega(k) = 5(3k+7)$ is divisible by 5, so $\omega(k) \pmod 5$ depends only on $k \pmod 5$; this periodicity argument applies identically to negative $k$ (simply replace $k$ by $k-5$), so checking five consecutive representatives determines $\omega(k)$ mod 5 for *every* integer $k$, positive or negative – we do not need to check negative $k$ separately. Evaluating at $k = 0, 1, 2, 3, 4$:

| $k \pmod 5$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\omega(k)$ (representative) | 0 | 1 | 5 | 12 | 22 |
| $\omega(k) \pmod 5$ | 0 | 1 | 0 | 2 | 2 |

The residues 3 and 4 never appear. □

These two lemmas reflect a genuine arithmetic miracle: the triangular numbers and pentagonal numbers modulo 5 have *complementary gaps* – meaning residues that each sequence simply never attains. Triangular numbers miss $\{2, 4\}$; pentagonal numbers miss $\{3, 4\}$. Together they miss the residue 4 (mod 5) among their pairwise sums, which is precisely what forces the coefficient of $q^{5m+4}$ in $f_1^4$ to vanish modulo 5 (made completely precise in the proof of Theorem 6.4 below). This complementary-gap phenomenon only works for $p \in \{5, 7, 11\}$, which is exactly Ramanujan's set (see Remark 6.4).

### 6.3 A Key Divisibility Lemma

Before combining the two residue lemmas, we isolate one more fact: not only does $f_1^3$ vanish in certain residues by virtue of $T_k$ missing them (Lemma 6.2), but along the one residue class where $T_k$ *can* land, the coefficient itself is forced to vanish, for an independent reason.

For all $m \geq 0$:

$$[q^{5m+3}]\, f_1^3 \equiv 0 \pmod 5.$$

*Proof.* By Corollary 6.1, the coefficient of $q^{T_k}$ in $f_1^3$ is $(-1)^k(2k+1)$. We need $T_k \equiv 3 \pmod 5$, which by Lemma 6.2 occurs exactly when $k \equiv 2 \pmod 5$. For such $k$:

$$2k + 1 \equiv 2 \cdot 2 + 1 = 5 \equiv 0 \pmod 5.$$

Therefore $(-1)^k(2k+1) \equiv 0 \pmod 5$ for every $k$ with $T_k \equiv 3 \pmod 5$. Since these are the only terms contributing to $[q^{5m+3}]f_1^3$, the lemma follows. □

## 6.4 The Main Proof

Everything is now in place: the pentagonal number theorem tells us which powers of $q$ can appear in $f_1$ (Lemma 6.2); Jacobi's identity tells us which powers of $q$ can appear in $f_1^3$, and moreover that one entire residue class among those has coefficient $\equiv 0 \pmod 5$ (Lemmas 6.2 and 6.3); and $f_1^4 = f_1^3 \cdot f_1$ lets us combine the two. We now assemble these facts into a proof of Ramanujan's congruence.

[Ramanujan, 1919] For all $n \geq 0$:

$$p(5n+4) \equiv 0 \pmod 5.$$

*Proof.* We work in $\mathbb{Z}[[q]]$ and reduce modulo 5 at the end.

*Step 1 (Fermat–Pochhammer reduction).* Working "in characteristic 5" means reducing all integer coefficients modulo 5, i.e., working in the ring $(\mathbb{Z}/5\mathbb{Z})[[q]]$. In characteristic 5, the binomial theorem collapses: every middle binomial coefficient $\binom{5}{i}$, $0 < i < 5$, is divisible by 5, so $(1-x)^5 \equiv 1 - x^5 \pmod 5$. Applying this term by term to the product defining $f_1$ gives

$$f_1^5 = \prod_{n\geq 1}(1-q^n)^5 \equiv \prod_{n\geq 1}(1-q^{5n}) = f_5 \pmod 5.$$

Therefore, in $(\mathbb{Z}/5\mathbb{Z})[[q]]$, we may trade a copy of $1/f_1$ for $f_1^4/f_5$:

$$\frac{1}{f_1} = \frac{f_1^4}{f_1^5} \equiv \frac{f_1^4}{f_5} \pmod 5. \tag{$*$}$$

This substitution is the engine of the whole proof: it converts the problem of understanding $1/f_1$ (an infinite product with no evident sparsity) into the problem of understanding $f_1^4$ – built from the two very sparse series $f_1^3$ and $f_1$ – divided by $f_5$, which contributes only powers of $q^5$ and so cannot affect residues modulo 5 in the exponent.

*Step 2 (Available residues).* From Lemma 6.2 and the pentagonal theorem:

$$[q^n]\, f_1 = 0 \quad \text{for all } n \equiv 3, 4 \pmod 5. \tag{$\star$}$$

From Lemma 6.2 and Corollary 6.1:

$$[q^n]\, f_1^3 = 0 \quad \text{for all } n \equiv 2, 4 \pmod 5. \tag{$\star\star$}$$

*Step 3 (The product $f_1^4 = f_1^3 \cdot f_1$).*

$$[q^{5m+4}]\, f_1^4 = \sum_{a+b=5m+4} \big([q^a] f_1^3\big)\big([q^b] f_1\big).$$

By ($\star$) and ($\star\star$), a term can be nonzero only if

$$a \pmod 5 \in \{0,1,3\} \qquad \text{and} \qquad b \pmod 5 \in \{0,1,2\}$$

(exactly the complements of the forbidden residues from Lemmas 6.2 and 6.2). We enumerate all pairs with $a + b \equiv 4 \pmod 5$ and check which are simultaneously compatible with these restrictions – "valid" meaning both congruence conditions above hold at once:

| $a \pmod 5$ | $b \pmod 5$ | $a+b \pmod 5$ | Valid? |
|---|---|---|---|
| 0 | 4 | 4 | No: $4 \notin \{0,1,2\}$ |
| 1 | 3 | 4 | No: $3 \notin \{0,1,2\}$ |
| 2 | 2 | 4 | No: $2 \notin \{0,1,3\}$ |
| 3 | 1 | 4 | Yes |
| 4 | 0 | 4 | No: $4 \notin \{0,1,3\}$ |

Notice that the only surviving pair has $a \equiv 3$ and $b \equiv 1 \pmod 5$.

*Step 4 (Coefficient vanishing).* By Lemma 6.3, $[q^a] f_1^3 \equiv 0 \pmod 5$ for every $a \equiv 3 \pmod 5$ – exactly the surviving case from Step 3. So every term in the sum for $[q^{5m+4}] f_1^4$ vanishes modulo 5, and therefore

$$[q^{5m+4}]\, f_1^4 \equiv 0 \pmod 5 \quad \text{for all } m \geq 0. \tag{$\star\star\star$}$$

*Step 5 (Conclusion).* Since $1/f_5 = \prod_{m\geq 1}(1-q^{5m})^{-1}$ involves only powers $q^{5k}$, write $1/f_5 = \sum_{k\geq 0} A_k\, q^{5k}$ with $A_k \geq 0$. Multiplying out and extracting the coefficient of $q^{5n+4}$,

$$[q^{5n+4}]\, \frac{f_1^4}{f_5} = \sum_{k=0}^{n} \left([q^{5(n-k)+4}] f_1^4\right) \cdot A_k \equiv \sum_{k=0}^{n} 0 \cdot A_k = 0 \pmod 5,$$

using $(\star\star\star)$ term by term. Combining with $(*)$:

$$p(5n+4) = [q^{5n+4}]\, \frac{1}{f_1} \equiv [q^{5n+4}]\, \frac{f_1^4}{f_5} \equiv 0 \pmod 5.$$

□

[Verification for small $n$] As a numerical sanity check before moving on:

$$p(4) = 5 = 5{\cdot}1, \quad p(9) = 30 = 5{\cdot}6, \quad p(14) = 135 = 5{\cdot}27, \quad p(19) = 490 = 5{\cdot}98, \quad p(24) = 1575 = 5{\cdot}315.$$

The proof admits a clean summary: the complementary residue gaps of $T_k$ and $\omega(k)$ modulo 5 conspire to ensure that the only way to write $5m+4 = a+b$ with $a$ in the support of $f_1^3$ and $b$ in the support of $f_1$ (Definition 3) forces $a \equiv 3 \pmod 5$. However, $[q^a] f_1^3 \equiv 0 \pmod 5$ because $2k+1 \equiv 0 \pmod 5$ whenever $T_k \equiv 3 \pmod 5$. The prime 5 (and analogously 7 and 11) is not arbitrary: this clean gap structure in triangular and pentagonal numbers modulo $p$ only arises for $p \in \{5, 7, 11\}$. A full explanation of why these are exactly the right primes requires the theory of modular forms and lies outside the elementary scope of this paper; see [3, 9] for the modern account.

[Modulo 7] The analogous proof of $p(7n+5) \equiv 0 \pmod 7$ uses the same structure but requires analyzing triangular and pentagonal numbers modulo 7, together with Jacobi's identity for $f_1^3$ and $f_1^5$. The modulo-11 case requires working with $f_1^{10}$ and involves substantially more residue bookkeeping; see [5] for the complete elementary proof.

# 7 Restricted Partition Families: Definitions and Generating Functions

Having established Ramanujan's congruence as a case study in the interaction between generating functions and residue arithmetic, we now build the general framework for restricted partition families – and there is more to say here, since this framework is exactly what lets us go beyond Ramanujan's specific congruence and manufacture new ones (Section 8). Every restricted family we consider is obtained from the basic generating function $1/f_1$ by one of three basic operations: restricting which residues mod $m$ the parts may occupy (Section 7.1), capping how many times each part may repeat (Section 7.2), or attaching colors to each part (Section 7.3) – and these operations may be combined.

## 7.1 Parts in Arithmetic Progressions

For an integer $m \geq 2$ and a nonempty set $R \subseteq \{0, 1, \ldots, m-1\}$, let $p_{m,R}(n)$ count partitions of $n$ into parts $k$ satisfying $k \pmod m \in R$. The generating function for these partitions is:

$$\sum_{n \geq 0} p_{m,R}(n)\, q^n = \prod_{\substack{k \geq 1 \\ k \bmod m \in R}} \frac{1}{1-q^k}.$$

The case $m = 2$, $R = \{1\}$ recovers $p_{\text{odd}}(n)$: partitions into parts not divisible by 2. More generally, for *any* integer $m \geq 2$, taking $R = \{1, \ldots, m-1\}$ (every nonzero residue) gives exactly the partitions into parts *not divisible by* $m$; nothing in Definition 7.1 required $m$ to be prime. The case $m = 3$, $R = \{1, 2\}$ is the first nontrivial instance:

$$\sum_{n \geq 0} p_{3,\{1,2\}}(n)\, q^n = \prod_{\substack{k \geq 1 \\ 3 \nmid k}} \frac{1}{1-q^k} = \frac{f_3}{f_1},$$

since splitting $f_1 = \prod_{3 \mid k}(1-q^k) \cdot \prod_{3 \nmid k}(1-q^k) = f_3 \cdot \prod_{3 \nmid k}(1-q^k)$. We revisit this family, for $m = 5$, in Section 8.1.

## 7.2 Partitions with Repetition Constraints

Instead of restricting which parts may appear, we can restrict how many times each part may repeat; the following definition interpolates between $p(n)$ (no constraint) and $p_{\text{dist}}(n)$ (each part used at most once).

For a positive integer $s$, let $p^{(s)}(n)$ count partitions of $n$ where each part appears at most $s$ times. Then:

$$\sum_{n \geq 0} p^{(s)}(n)\, q^n = \prod_{k=1}^{\infty} (1 + q^k + q^{2k} + \cdots + q^{sk}) = \prod_{k=1}^{\infty} \frac{1-q^{k(s+1)}}{1-q^k} = \frac{f^*_{s+1}}{f_1},$$

where $f^*_{s+1} = \prod_{k \geq 1}(1-q^{k(s+1)})$. The case $s = 1$ gives $p^{(1)}(n) = p_{\text{dist}}(n)$.

$$\sum_{n\geq 0} p_{\text{dist}}(n)\, q^n = \frac{f_2}{f_1}.$$

*Proof.* Setting $s = 1$ in Definition 7.2, or directly: multiplying numerator and denominator of each factor by $(1 - q^k)$, we get $\prod_{k\geq 1}(1 + q^k) = \prod_{k\geq 1} \frac{1-q^{2k}}{1-q^k}$, and regrouping the resulting numerator and denominator according to Definition 3 gives $\frac{f_2}{f_1}$. □

For $n = 5$, the partitions into distinct parts are $5$, $4 + 1$, $3 + 2$, so $p_{\text{dist}}(5) = 3$. (A superficially similar but different notion is that of *hyperbinary partitions* of $n$ – representations $n = \sum_i \varepsilon_i 2^i$ with $\varepsilon_i \in \{0, 1, 2\}$ – which count something else entirely and generate the Stern–Brocot sequence; we mention this only to flag that "distinct parts" and "hyperbinary" are easily confused but unrelated notions.)

These identities connect Euler's theorem, the generating function for distinct parts, and the generating function for odd parts through a single algebraic identity:

$$\frac{f_2}{f_1} = \prod_{k \text{ odd}} \frac{1}{1 - q^k} = \sum_n p_{\text{odd}}(n)\, q^n = \sum_n p_{\text{dist}}(n)\, q^n.$$

This kind of web of identities – where several combinatorially meaningful objects collapse to the same expression – is a recurring feature of partition theory.

### 7.3 Colored Partitions and Multivariate Generating Functions

For integers $c \geq 1$ and $m \geq 1$, a $(c, m)$-*restricted colored partition* of $n$ is a $c$-colored partition (recall Definition 3) of $n$ into parts not divisible by $m$. Its generating function is:

$$\sum_{n\geq 0} \text{rcp}_{c,m}(n)\, q^n = \prod_{\substack{k\geq 1 \\ m\nmid k}} \frac{1}{(1 - q^k)^c} = \frac{f_m^c}{f_1^c}.$$

This family includes as special cases the generating functions of the companion paper. In particular:

- $c = 1$, $m = 3$: $\text{rcp}_{1,3}(n) = p_{3,\{1,2\}}(n)$, the partitions of Example 7.1;
- $c = 2$, $m = 5$: the generating function $f_5^2/f_1^2$ connects to the 5-dissection structure used in Theorem 6.4.

## 8 Original Results: Congruences for Restricted Families

We now establish Ramanujan-type congruences for specific restricted partition families. The method mirrors the proof of Theorem 6.4 step for step – Fermat–Pochhammer reduction, then a residue-support argument – but requires carefully re-tracking the residue support of each new generating function. Since the Fermat–Pochhammer reduction is used repeatedly below, we record it once, as a standalone lemma, so it can simply be cited by number from here on.

[Fermat–Pochhammer reduction / "Freshman's Dream"] For any prime $p$ and any $F(q) \in \mathbb{Z}[[q]]$, $F(q)^p \equiv F(q^p) \pmod p$ (the fact that $(x+y)^p \equiv x^p + y^p \pmod p$, applied term by term, is sometimes affectionately called the Freshman's Dream). In particular, taking $F = f_1$ and $p = 5$:

$$f_1^5 \equiv f_5 \pmod 5.$$

### 8.1 Congruences for Partitions into Parts Not Divisible by 5

This is the $m = 5$ instance of the family in Example 7.1, and it turns out to obey a Ramanujan-type congruence of its own.

Let $\overline{p}_5(n) = p_{5,\{1,2,3,4\}}(n)$ count partitions of $n$ into parts not divisible by 5; the generating function for these partitions is $f_5/f_1$. Then for all $n \geq 0$:

$$\overline{p}_5(5n+4) \equiv 0 \pmod 5.$$

*Proof.* The generating function satisfies $\sum_{n\geq 0} \overline{p}_5(n)\, q^n = f_5/f_1$. By Lemma 8, $f_1^5 \equiv f_5 \pmod 5$, so $f_5/f_1 \equiv f_1^4 \pmod 5$. Therefore:

$$[q^{5n+4}]\, \frac{f_5}{f_1} \equiv [q^{5n+4}]\, f_1^4 \equiv 0 \pmod 5,$$

where the last congruence is $(\star\star\star)$ from the proof of Theorem 6.4. So the restriction to parts not divisible by 5 preserves exactly the congruence that $p(n)$ itself satisfies – unsurprising once one notices that $\overline{p}_5$'s generating function $f_5/f_1$ and $p$'s generating function $1/f_1$ become the same power series, $f_1^4$, once reduced modulo 5. □

### 8.2 Congruences for Bicolored Partitions

Let $p_2(n)$ count bicolored partitions of $n$ (recall Definition 3, $c = 2$); its generating function is $1/f_1^2$. Then for all $n \geq 0$:

$$p_2(5n+3) \equiv 0 \pmod 5.$$

*Proof.* By Lemma 8, $1/f_1^2 = f_1^3/f_1^5 \equiv f_1^3/f_5 \pmod 5$ – we use this to trade the "opaque" series $1/f_1^2$ for the sparse series $f_1^3$, whose support we already understand from Corollary 6.1. Writing $1/f_5 = \sum_{k\geq 0} A_k q^{5k}$ (the same coefficients as in Step 5 of the proof of Theorem 6.4), we need

$$[q^{5n+3}]\, \frac{f_1^3}{f_5} = \sum_{k=0}^{n} \left([q^{5(n-k)+3}] f_1^3\right) \cdot A_k.$$

This is exactly where Lemma 6.3 comes in: since $5(n-k)+3 \equiv 3 \pmod 5$ for every $k$, Lemma 6.3 applies directly to each term, giving $[q^{5(n-k)+3}] f_1^3 \equiv 0 \pmod 5$. So every summand vanishes modulo 5, and the whole sum vanishes, giving the result. □

### 8.3 Congruences for Restricted Colored Partitions

Let $\mathrm{rcp}_{4,5}(n)$ denote the restricted color partition function of Definition 7.3 with $c = 4$, $m = 5$ – that is, $\mathrm{rcp}_{4,5}(n)$ counts 4-colored partitions of $n$ into parts not divisible by 5 – defined by the generating function

$$\sum_{n=0}^{\infty} \mathrm{rcp}_{4,5}(n) q^n = \frac{f_5^4}{f_1^4}.$$

Then for all $n \geq 0$, we have the Ramanujan-type congruence

$$\mathrm{rcp}_{4,5}(5n + 4) \equiv 0 \pmod 5.$$

We choose $c = 4$ and $m = 5$ deliberately, not arbitrarily: as the proof shows, these are exactly the parameters for which the Fermat–Pochhammer reduction collapses the generating function down to a single sparse power of $f_1$, letting the pentagonal-number mechanism of Theorem 6.4 apply essentially unchanged. This is the simplest nontrivial member of the family of Remark 3 for which the argument goes through this cleanly.

*Proof.* We begin by reducing the generating function modulo 5. By Lemma 8 (the Freshman's Dream), $f_5 \equiv f_1^5 \pmod 5$. Applying this to the numerator:

$$\frac{f_5^4}{f_1^4} \equiv \frac{(f_1^5)^4}{f_1^4} = \frac{f_1^{20}}{f_1^4} = f_1^{16} \pmod 5.$$

To analyze the coefficients of $f_1^{16}$ modulo 5, we isolate a factor of $f_1$ by decomposing the exponent:

$$f_1^{16} = (f_1^5)^3 \cdot f_1 \equiv f_5^3 \cdot f_1 \pmod 5.$$

Notice that $f_5^3$ is a power series containing strictly powers of $q^5$; write it as

$$f_5^3 = \sum_{k=0}^{\infty} c_k q^{5k}$$

for some integer coefficients $c_k$. By Euler's Pentagonal Number Theorem (Theorem 5), the remaining factor $f_1$ is given by

$$f_1 = \sum_{m=-\infty}^{\infty} (-1)^m q^{\omega(m)},$$

where $\omega(m)$ is as in Definition 5.

Applying these two expansions, the generating function reduces to a *Cauchy product* – a term-by-term convolution, $\left(\sum_k c_k q^{5k}\right)\left(\sum_m (-1)^m q^{\omega(m)}\right) = \sum_{k,m} c_k (-1)^m q^{5k+\omega(m)}$ – of a series in $q^5$ and the sparse series $f_1$:

$$\sum_{n=0}^{\infty} \mathrm{rcp}_{4,5}(n) q^n \equiv \left(\sum_{k=0}^{\infty} c_k q^{5k}\right)\left(\sum_{m=-\infty}^{\infty} (-1)^m q^{\omega(m)}\right) \pmod 5.$$

We now extract the coefficient of $q^{5n+4}$ from both sides. Any generic term on the right-hand side has exponent $5k + \omega(m)$; for such a term to contribute to the coefficient of $q^{5n+4}$, we need $5k + \omega(m) = 5n + 4$, which forces $\omega(m) \equiv 4 \pmod 5$.

However, by Lemma 6.2, the generalized pentagonal numbers only attain residues $\{0, 1, 2\}$ modulo 5, so $\omega(m) \not\equiv 4 \pmod 5$ for any $m \in \mathbb{Z}$. Since no such $m$ exists, no term in the Cauchy product can produce an exponent congruent to 4 modulo 5, so the coefficient must vanish identically:

$$[q^{5n+4}] \frac{f_5^4}{f_1^4} \equiv 0 \pmod 5,$$

which implies $\mathrm{rcp}_{4,5}(5n + 4) \equiv 0 \pmod 5$, completing the proof. □

# 9 Further Directions and Open Questions

The machinery developed above opens several natural research directions.

***Path 1: Ramanujan-type congruences for further restricted families.*** The proof of Theorem 6.4 used specific structural properties of $f_1^4 \pmod 5$. Analogous arguments – tracking the residue support of $f_1^c \pmod p$ for various $c$ and $p$ – should yield congruences for $p_c(n)$, $p_{m,R}(n)$, and $\mathrm{rcp}_{c,m}(n)$. A natural starting point is to classify which primes $p$ admit a "complementary gap" phenomenon in triangular and pentagonal numbers modulo $p$, in the sense of Remark 6.2.

***Path 2: Bijective proofs of congruences.*** The rank and crank give combinatorial explanations for Ramanujan's congruences (Section 2.4). For restricted families, analogous statistics likely exist but remain largely unknown. Finding such statistics would transform a generating-function result into a bijective one – substantially strengthening it, in the same way that Dyson's rank strengthens the modulo-5 and modulo-7 congruences from mere divisibility statements into genuinely combinatorial ones.

***Path 3: Connection to the companion paper.*** The $p$-adic spectral framework of the companion paper proves global vanishing results modulo $p$ for multivariate eta-products via the Atkin $U_p$ operator. Our elementary Fermat–Pochhammer approach proves the same kinds of vanishing results but works modulo $p$ from the start, without lifting towers. Comparing the two approaches for specific families (e.g., $f_5^2/f_1^2$) may reveal combinatorial interpretations for the spectral phenomena of that paper.

***Path 4: Rogers–Ramanujan type identities for restricted families.*** The Rogers–Ramanujan identities equate partition functions with gap conditions (Definition 2.2) to partition functions with residue conditions. Finding analogues for other gap conditions and residue families remains an open problem with active research – for instance, one could ask whether partitions with gaps of at least 3 correspond to some residue condition modulo 7 or 8, in analogy with the classical case, or whether the restricted families of Section 7 admit their

own Rogers–Ramanujan-type companions. This connects our elementary generating function approach to the deeper theory of $q$-hypergeometric series.

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