Measuring the Infinite: Hausdorff Dimension and the Coastline Paradox

Miriam Wells

Euler Circle IRPW

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Order of topics

The material will be presented in the following order:

- History of fractals
- Preliminaries (metric spaces and outer measures)
- Hausdorff Measure
- Hausdorff Dimension and examples
- The Coastline Paradox

Fractal History

The terms "fractal dimension" and "fractal" were both coined by Benoit Mandelbrot in 1975. Mandelbrot's theory of roughness led to his study of the Coastline Paradox, first introduced by Lewis Fry Richardson in the 1950's.

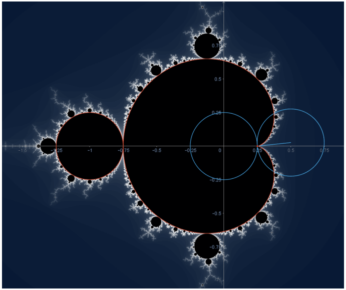

Figure: The Mandelbrot Set

Metric Spaces

A **metric space** on a nonempty set X is a function $d : X \times X \to [0, \infty)$ such that

- $d(x, x) = 0$ for all $x \in X$;
- if $x, y \in X$ and $d(x, y) = 0$, then $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Essentially, we can define a metric space as a pair (X, d), where X is a nonempty set and d is a metric on X.

Preliminaries

Definition: Open Cover; Finite Subcover

Let $A \subseteq \mathbb{R}$.

- A collection C of open subsets of $\mathbb{R}$ is an **open cover** of A if A is contained in the union of all sets in C
- An open cover C of A is said to have a **finite subcover** if A is contained in the union of some finite subset of sets in C

Outer Measure

An **outer measure** $|A|$ of $A \subseteq \mathbb{R}$ is defined as $\inf \sum_{K=1}^{\infty} \ell(I_K) : I_1, I_2, \ldots$ such that

$$A \subseteq \bigcup_{K=1}^{\infty} (I_K)$$

Translation invariance

Translation invariance allows a set to undergo changes in its position in space without changing the set's size.

Property

If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, the translation $t + A$ is defined by

$$t + A := \{t + a : a \in A\}$$

where translation has no effect on outer measure, allowing for

$$|t + A| = |A|$$

Countable subadditivity

Countable subadditivity allows for the handling of an infinite collection of sets without concerns regarding the consistency of the measure's behavior.

Property

Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then,

$$|\bigcup_{K=1}^{\infty}(A_K)| \leq \sum_{K=1}^{\infty} |A_K|$$

Determining a set's measurability
Carathéodory's measure determines when a set is Lebesgue measureable. In mathematics, a Lebesgue measure is a standard way of assigning length to Euclidean space.

Property

A subset $E \subseteq X$ is called $\mu^\star$-**measurable** (or Carathéodory-measurable) if it satisfies Carathéodory's criterion:

$$\mu^\star(C) = \mu^\star(C \cap E) + \mu^\star(C \setminus E)$$

where $\mu^\star$ represents the outer measure.

Hausdorff Measure

The rough application of a Hausdorff measure can be explained using a boundary in a tiled plane.

- Count how many tiles the boundary is contained in
- Perform a scaling of the tiling
- Determine how the ratio diverges in order to measure the roughness of the boundary

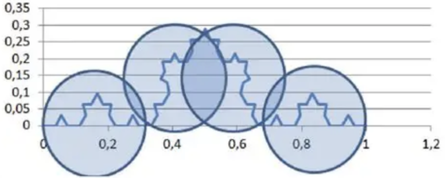

Figure: Coverings on the Koch Curve

Quick Definitions

- The infimum (inf), of a subset S of a partially ordered set P is the greatest element in P that is less than or equal to each element of S, if such an element exists. The infimum is also denoted the "greatest lower bound."
- Conversely, the supremum, (sup), of a subset S of a partially ordered set P is the least element in P that is greater than or equal to each element of S, if such an element exists.

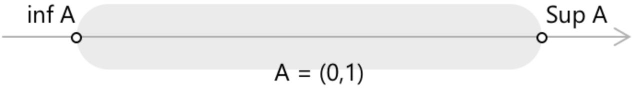

Figure: Infimum and Supremum

Hausdorff Measure

Definition:

Let X be a metric space. If $S \subset X$ and $d \in [0, \infty)$, the d-dimensional **Hausdorff measure** is defined as:

$$H_\delta^d(S) = \inf\{\sum_{i=1}^{\infty}(diam U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, diam U_i < \delta\}$$

Where the infimum is taken over all countable covers U of S, with inf representing the infimum, *diam* representing diameter, and U_i representing an element of a finite cover of the set S.

Hausdorff Measure

Definition:

The d-dimensional **Hausdorff outer measure** can then be defined as:

$$H^d(S) := \lim_{\delta \to \infty} H^d_\delta(S)$$

A measure must satisfy the property

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

However, an outer measure is only guaranteed to satisfy

$$\mu^\star(A \cup B) \leq \mu^\star(A) + \mu^\star(B)$$

Hausdorff Dimension

The Hausdorff dimension provides a numerical value that describes how "rough" or "intricate" a set or space is.

Definition

The **Hausdorff dimension** $dim_H(X)$ of X is defined by:

$$dim_H(X) := \inf\{d \geq 0 : H^d(X) = 0\}$$

This is the same as the supremum of the set $d \in [0, \infty)$ such that $H^d(X) = \infty$.

Hausdorff Dimension

As the dimension parameter increases, the Hausdorff measure decreases. This means the behavior of $H^d(X)$ acts as follows:

Dimension d	Hausdorff Measure
$d < dim_H(X)$	∞
$d = dim_H(X)$	finite value
$d > dim_H(X)$	0

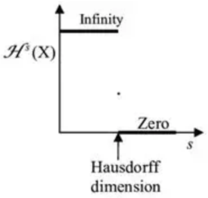

Figure: Hausdorff Dimension Transition Point

Hausdorff Dimension

Moran Equation.

Definition:

If a fractal is composed of m self-similar sets with contraction ratios $r_1, r_2, \ldots, r_m$, the dimension d is the unique, positive solution to the equation:

$$\sum_{i=1}^{m} r_i^d = 1$$

Similarity-dimension equation:

Definition:

$d = \frac{\log(N)}{\log(L)}$

Where d represents the Euclidean dimension, N represents the self-similar objects needed to cover the original object, and L represents the reduction factor.

The Moran Equation and the Koch Curve
Each iteration of the Koch curve creates 4 times as many pieces, each 1/3 of the original size. Using the Moran equation, this gives:

$$4(\frac{1}{3})^d = 1$$

Solving it yields

$$d = \frac{ln4}{ln3}$$

or 1.2619.

Examples

The Koch Snowflake

The Koch Snowflake is constructed beginning with an equilateral triangle. In each iteration, three steps occur:

1. Each of the line segments are divided into three equal parts.
2. An equilateral triangle is constructed on the middle third, pointing outward.
3. The base of the new triangle is removed.

This will create four smaller, equal segments where there was previously one straight line segment.

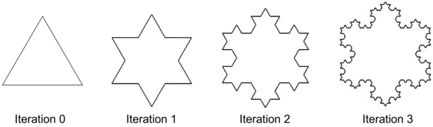

Figure: Koch Snowflake Construction

Examples

The Snowflake's iterations consist of $N = 4$ self-similar copies, each scaled by a factor of $r = \frac{1}{3}$. Let the Hausdorff dimension be d. The d-dimensional Hausdorff measure scales by

$$(\frac{1}{3})^d$$

With 4 copies, the total measure is

$$H^d(K) = 4(\frac{1}{3})^d H^d(K)$$

The Hausdorff measure is nonzero at the critical exponent, we can divide by $H^d(K)$:

$$4(\frac{1}{3})^d = 1$$

After solving for d, we get a result of

$$dim_H(K) = \frac{\ln 4}{\ln 3} \approx 1.2619$$

The Coastline Paradox

A coastline does not have a defined perimeter; instead, the length of a coastline changes depending on the scale with which it is measured. When increasingly short measurements are used, the sum approaches the true length, the supremum.

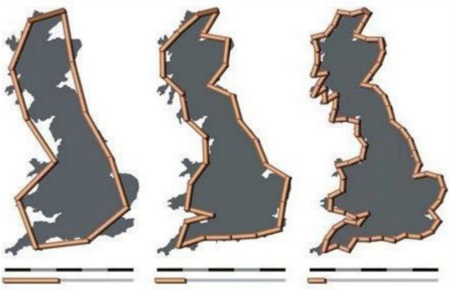

Figure: The Coastline Paradox

The Coastline Paradox

The prevailing method of determining the length of a coastline is to lay out n equal straight-line segments of length l with dividers on a map or aerial photograph.

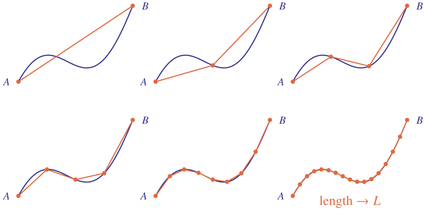

Figure: Measuring a Curve

The "Richardson effect:" the sum of the segments monotonically increases when the common length of the segments decreases. Under certain circumstances, as l approaches 0, the length of the coastline approaches infinity.

The Coastline Paradox

Mandelbrot's development of the Hausdorff dimension does not depend on length l in the same way. Mandelbrot's statement of the Richardson effect is:

$$L(\epsilon) \sim F\epsilon^{1-D}$$

where L represents the length of the coastline, ϵ represents the measurement unit, F is a constant, and D is a parameter that Richardson found depends on the coastline approximated by L.
A rearranged form of this expression gives us

$$F\epsilon^{-D} \cdot \epsilon$$

where $F\epsilon^{-D}$ is the number of units ϵ needed to obtain L.

Summary

- Classical Euclidean geometry is insufficient for measuring irregular natural objects and fractals
- Metric spaces are used to define diameters and coverings
- These ideas lead to the Hausdorff measure and dimension
- The Hausdorff dimension can take on non-integer values and quantify the complexity of fractals

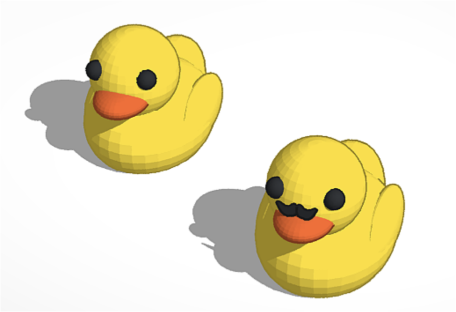

Figure: 3D Modeling of Ducks