Measuring the Infinite: Hausdorff Dimension and the Coastline Paradox

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Abstract

This paper will be an expository paper introducing the Hausdorff dimension, exploring topology, outer measures, open covers, the Moran equation, and the Coastline Paradox.

1 Introduction

In mathematics and physics, the dimension of a mathematical space or object is essentially the minimum number of coordinates needed to specify any point within it. Dimension is an intrinsic property, meaning that the dimension of an object is independent of the dimension of the space it is embedded in. These are the basic principles of Euclidean dimension.

In geometric measure theory, fractal dimension is an index for characterizing fractal patterns by quantifying their complexity as a ration of change in detail to change in scale. For sets describing ordinary shapes, this fractal dimension equals the set's Euclidean or topological dimension. However, unlike topological dimension, the fractal index can take on non-integer values.

The terms "fractal dimension" and "fractal" were both noted by Benoit Mandelbrot in 1975. Mandelbrot was born in Poland in 1924 and emigrated to France in 1936. While he worked at IBM (International Business Machines Corporation) in the United States, he discovered the Mandelbrot set using their computers in 1979.

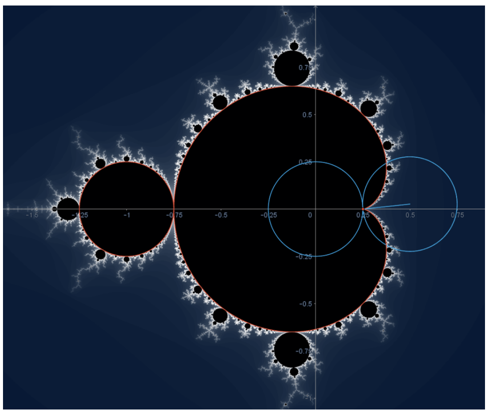

Figure 1: The Mandelbrot Set

Mandelbrot is generally considered to be the father of fractal geometry, and used his notions of fractal dimensions to explore his ”theory of roughness“. This theory of roughness led to his study of the Coastline Paradox, first introduced by Lewis Fry Richardson in the 1950’s.

The Coastline Paradox can be related to the Hausdorff dimension due to its non-integer dimensionality. The Hausdorff dimension, introduced in 1918 by Felix Hausdorff, measures roughness in fractal dimensions, often resulting in a non-integer dimensionality for more complex fractals.

In this paper, we will look at an introduction to these topological topics and dimensions, and explore the connections between them.

2 Preliminaries and Definitions

We begin with an introduction to topology and topological spaces.

Definition:

Let X be a set. A **topology** on X is a collection of subsets satisfying three axioms.

(i) X and ϕ are elements of T

(ii) T is closed under finite intersections

(iii) T is closed under arbitrary unions

If the set X has a topology on it, (X, T) is a **topological space**.

The collection of subsets in the topology will be called open sets. Next, we can introduce the idea of metric spaces, a subclass of topological spaces.

2.1 Metric spaces

The formal definition of a metric space is the following:

A **metric space** on a nonempty set X is a function $d : X \times X \to [0, \infty)$ such that

- $d(x, x) = 0$ for all $x \in X$;
- if $x, y \in X$ and $d(x, y) = 0$, then $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Essentially, we can define a metric space as a pair (X, d), where X is a nonempty set and d is a metric on X.

2.2 Openness

Once the general notion of a topological space has been introduced, the ideas of openness can be defined, including definitions of open sets, balls, and covers. These topics are important because Hausdorff measure and dimension are constructed using coverings by sets whose sizes are measured using a metric.

Definition: Open Ball; $B(x, r)$

Let (X, d) be a metric space, $x \in X$ and $r > 0$. The **open ball** centered at x with radius r, denoted $B(x, r)$ is defined by

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

Given a metric d on a space X, open balls generate a topology. Therefore, every metric space is automatically a topological space, including Hausdorff space. However, not every topological space can be described by a metric function. These topological spaces are called "non-metrizable".

Definition: Open:

A subset Y of a metric space X is called **open** if for every $y \in Y$, there exists $r > 0$ such that

$$B(y, r) \subseteq Y$$

Definition: Open Cover; Finite Subcover

Let $A \subseteq \mathbb{R}$.

- A collection C of open subsets of $\mathbb{R}$ is an **open cover** of A if A is contained in the union of all sets in C
- An open cover C of A is said to have a **finite subcover** if A is contained in the union of some finite subset of sets in C

Another, more general definition of open covers in the context of arbitrary metric spaces is as follows:

Let (X, d) be a metric space. Let $A \subset (X, d)$. An **open cover** for A is a collection of open sets $\{O_\alpha | \alpha \in \mathbb{N}\}$ such that $A \subset \bigcup_{\alpha \in \mathbb{N}} O_\alpha$.

2.3 Outer Measure

Outer measures are commonly used to determine whether a set is measurable. To first begin discussing the concept of outer measures, the length of an open interval must be explored.

Definition: Length of an open interval; $\ell(I) =$

- $\ell(I) = b - a$ if $I = (a, b)$ for some $(a, b) \in \mathbb{R}$ with $a < b$
- 0 if $I = \phi$
- ∞ if $I = (-\infty, a)$ or $I = (a, \infty)$ for some $a \in \mathbb{R}$
- ∞ if $I = (-\infty, \infty)$

Once the definition of an open interval has been noted, the definition of an outer measure can be stated.

An **outer measure** $|A|$ of $A \subseteq \mathbb{R}$ is defined as $\inf \sum_{K=1}^{\infty} \ell(I_K) : I_1, I_2, \ldots$ such that

$$A \subseteq \bigcup_{K=1}^{\infty} (I_K)$$

It is important to note some key components and properties of outer measures that allow for its importance in calculating non-integer dimensions, such as translation invariance, countable subadditivity, and determining a set's measurability.

2.3.1 Translation Invariance:

Translation invariance is a notable property of outer measure due to its implications about the consistency of the size of a set. Translation invariance allows a set to undergo changes in its position in space without changing the set's size. Essentially, translation invariance allows a set to be "position-independent."

Lemma 2.2.1 (Translation Invariance) If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, the **translation** $t + A$ is defined by

$$t + A := \{t + a : a \in A\}$$

where translation has no effect on outer measure, allowing for

$$|t + A| = |A|$$

2.3.2 Countable Subadditivity:

Arguably one of the most important properties of outer measure, countable subadditivity allows for the handling of an infinite collection of sets without concerns regarding the consistency of the measure's behavior.

Lemma 2.2.2 (Countable Subadditivity)

Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then,

$$|\bigcup_{K=1}^{\infty}(A_K)| \leq \sum_{K=1}^{\infty}|A_K|$$

Countable subadditivity also implies finite subadditivity, meaning that

$$|A_1 \cup \ldots \cup A_n| \leq |A_1| + \ldots + |A_n|$$

2.3.3 Determining Whether a set is Measurable:

In order to understand this property of outer measure, the Carathéodory extension theorem must be mentioned. Carathéodory was a Greek mathematician born in 1873 in Berlin. He made significant advances in the fields of real and complex analysis, measure theory, and the calculus of variations.

Carathéodory's measure determines when a set is Lebesgue measureable. In mathematics, a Lebesgue measure is a standard way of assigning length to Euclidean space.

A subset $E \subseteq X$ is called **$\mu^{\star}$-measurable** (or Carathéodory-measurable) if it satisfies Carathéodory's criterion:

$$\mu^{\star}(C) = \mu^{\star}(C \cap E) + \mu^{\star}(C \setminus E)$$

where $\mu^{\star}$ represents the outer measure. This property ensures that the Hausdorff measure acts as a proper measure, as Hausdorff measure is fundamentally a Carathéodory measure.

3 Definitions of Hausdorff Measure

Felix Hausdorff was a German mathematician born in 1868. He is considered to be one of the founders of modern topology, and mainly studied in Leipzig. His main contributions are to the field of topology and set and measure theory.

The rough application of a Hausdorff measure can be explained using a boundary in a tiled plane. Essentially, counting how many tiles the boundary is contained in and performing a scaling of the tiling allows us to determine how the ratio diverges in order to measure the roughness of the boundary.

This notion gives the same answer as classical dimension, but extends the idea to rougher boundaries.

Definition:

Let X be a metric space. If $S \subset X$ and $d \in [0, \infty)$, the d-dimensional **Hausdorff measure** is defined as:

$$H_\delta^d(S) = \inf\{\sum_{i=1}^{\infty}(diamU_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, diamU_i < \delta\}$$

Where the infimum is taken over all countable covers U of S, with inf representing the infimum, $diam$ representing diameter, and U_i representing an element of a finite cover of the set S.

The inf, or the infimum, of a subset S of a partially ordered set P is the greatest element in P that is less than or equal to each element of S, if such an element exists. The infimum is also denoted the "greatest lower bound."

Conversely, the supremum, (sup), of a subset S of a partially ordered set P is the least element in P that is greater than or equal to each element of S, if such an element exists.

The diameter, $diamU_i$, is the supremum of distances between any two points in U_i, with $diam\phi = 0$.

Definition:

The d-dimensional **Hausdorff outer measure** can then be defined as:

$$H^d(S) := \lim_{\delta \to \infty} H_\delta^d(S)$$

The Hausdorff measure is the restriction of the Hausdorff outer measure to Carathéodory-measureable sets. Essentially, a measure must satisfy the property

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

However, an outer measure is only guaranteed to satisfy

$$\mu^\star(A \cup B) \leq \mu^\star(A) + \mu^\star(B)$$

By applying Carathéodory's criterion, we can obtain the class of Carathéodory-measurable sets, and the restriction of the outer measure to this class is called the d-dimensional Hausdorff measure.

Because the Hausdorff measure is therefore an outer measure, the previously stated properties of an outer measure can be applied. For example, the Hausdorff measure is translation invariant (Lemma 2.2.1):

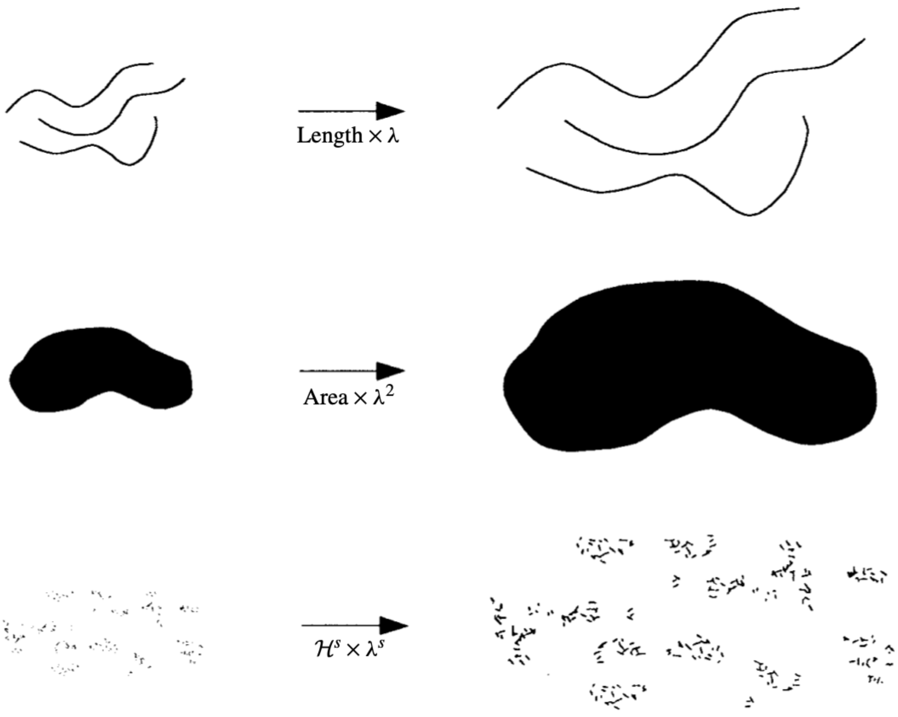

Figure 2: The area, length, and Hausdorff Measure are all equally scaled by a factor of λ^2

where λ acts as a scaling factor.

It is important that these properties of an outer measure apply to the Hausdorff measure, as they allow the Hausdorff measure to behave sensibly as a notion of size.

4 Hausdorff Dimension

The Hausdorff dimension is used to provide a numerical value that describes how "rough" or "intricate" a set or space is, and allows us to understand the complexity of a space by measuring its level of detail. It is built using the Hausdorff measure.

Definition:

The **Hausdorff dimension** $dim_H(X)$ of X is defined by:

$$dim_H(X) := \inf\{d \geq 0 : H^d(X) = 0\}$$

This is the same as the supremum of the set $d \in [0, \infty)$ such that $H^d(X) = \infty$.

Essentially, the Hausdorff dimension generalizes the dimension of a vector space.

In the book "Fractal Geometry" by Falconer, his proof is as follows:

Proof

For every set X, there exists a unique number d such that

if $s < d$, then

$$H^s(X) = \infty$$

if $s > d$, then

$$H^s(X) = 0$$

This theorem justifies defining $dim_H(X) = d$.

As the dimension parameter increases, the Hausdorff measure decreases. This means the behavior of $H^d(X)$ acts as follows:

Dimension d	Hausdorff Measure
$d < dim_H(X)$	∞
$d = dim_H(X)$	finite value
$d > dim_H(X)$	0

This allows us to see that the Hausdorff dimension is a unique transition point where the measure drops from infinity to zero. The critical exponent is characterized by the behavior around it, not necessarily at the value.

An excellent example that expands on the idea of this transition point is the Koch curve.

The Hausdorff dimension essentially covers some set X with small balls and sums the diameter of these balls. In terms of the Koch curve, we can represent this using covering sets with a small diameter of, for example, $diam(U_i) = 0.001$.

Now, let $d = 1$. This gives

$$0.001^1 = 0.001$$

Because we need to sum infinitely many of these diameters to cover the Koch curve, the total sum intuitively approaches infinity. Therefore, the Hausdorff measure is

$$H^1(K) = \infty$$

If we let $d = 2$, this gives

$$0.001^2 = 0.000001$$

Each diameter now covers much less of the curve, so when this cover is refined, the total sum shrinks towards 0. Therefore, we get a Hausdorff measure of

$$H^2(K) = 0$$

This shows us that the Hausdorff dimension of the Koch curve must be between 1 and 2.

Now, consider the set

$$\{d : H^d(X) = 0\}$$

This set consists of all sufficiently large exponents d such that $H^d(X) = 0$. For the Koch curve, this set looks like

$$\{1.2619, 1.3, 1.5, 2, 3, \ldots\}$$

The smallest point where the Hausdorff measure is equal to 0 is at 1.2619.

Therefore, we can define

$$dim_H(X) := \inf\{d \geq 0 : H^d(X) = 0\}$$

We use the infimum of the zero set - or the supremum of the infinity set - to capture the behavior of the boundary rather than the measure of the critical exponent itself.

4.1 The Moran Equation

The Moran Equation can be used to calculate the Hausdorff dimension of self-similar sets. It acts as the bridge between the similarity-dimension equation and the Hausdorff equation.

For self-similar fractals, a common approach to find this value is utilizing the similarity-dimension equation:

$$d = \frac{\log(N)}{log(L)}$$

Where d represents the Euclidean dimension, N represents the self-similar objects needed to cover the original object, and L represents the reduction factor.

This equation gives the same answer as Hausdorff dimension for fractals that are self-similar, such as the Koch snowflake and Sierpiński triangle.

The Hausdorff dimension d can also be determined for self-similar sets using the **Moran Equation**.

If a fractal is composed of m self-similar sets with contraction ratios $r_1, r_2, \dots, r_m$, the dimension d is the unique, positive solution to the equation:

$$\sum_{i=1}^{m} r_i^d = 1$$

We can use the Koch curve as an example here once more. Each iteration of the Koch curve creates 4 times as many pieces, each 1/3 of the original size. Using the Moran equation, this gives:

$$4(\frac{1}{3})^d = 1$$

Solving it yields

$$d = \frac{ln4}{ln3}$$

or 1.2619, the same value as the Hausdorff equation.

4.2 Behavior Under Unions and Products

The Hausdorff dimension's behavior under unions and products tells us that Hausdorff dimension behaves sensibly.

4.2.1 Unions

If $X = \bigcup_{i=1}(X_i)$ is a finite or countable union, then

$$dim_{Haus}(X) = \sup_{i \in I} dim_{Haus}(X_i)$$

Example:

Let:

- A be a line segment with $dim_H = 1$
- B be a square with $dim_H = 2$

Then,

$$dim_H(A \bigcup B) = 2$$

Essentially, the line does not increase the dimension, and the more complicated dimension dominates. This is intuitive because the union of two lines would not have a dimension of 2, showing that the Hausdorff dimension acts sensibly under unions.

Interestingly, the Coastline paradox can be explored using this principle, as it shows that the Hausdorff dimension measures the most complex scaling behavior.

4.2.2 Products

If X and Y are non-empty metric spaces, then the Hausdorff dimension of their product satisfies

$$dim_{Haus}(X \times Y) \geq dim_{Haus}(X) + dim_{Haus}(Y)$$

The product property of the Hausdorff dimension shows us that independent sources of complexity accumulate, essentially combining two spaces into an object with higher dimensionality.

5 Common Applications of Hausdorff

The Hausdorff dimension is commonly used to find non-integer dimensions, especially for objects like fractals, with complex dimensionality.

Countable sets have a dim_H equal to 0, and the Euclidean space $\mathbb{R}^n$ has Hausdorff dimension of n.

Some commonly cited examples of the concept of non-integer dimensionality are the Koch Snowflake and Sierpiński Triangle, two self-similar fractals.

5.1 Koch Snowflake

The Koch Snowflake is constructed beginning with an equilateral triangle. In each iteration, three steps occur:

1. Each of the line segments are divided into three equal parts.
2. An equilateral triangle is constructed on the middle third, pointing outward.
3. The base of the new triangle is removed.

This will create four smaller, equal segments where there was previously one straight line segment.

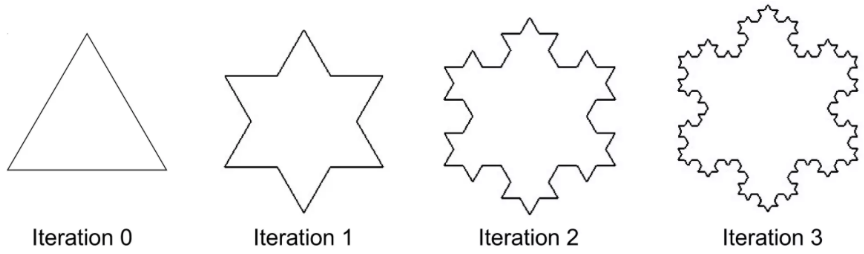

Figure 3: Koch Snowflake Construction

The Snowflake's iterations consist of $N = 4$ self-similar copies, each scaled by a factor of $r = \frac{1}{3}$. Let the Hausdorff dimension be d. With a scale factor of $\frac{1}{3}$, the d-dimensional Hausdorff measure scales by

$$(\frac{1}{3})^d$$

With 4 copies, the total measure, which must be equal to the measure of the original curve, is

$$H^d(K) = 4(\frac{1}{3})^d H^d(K)$$

Since the Hausdorff measure is nonzero at the critical exponent, we can divide by $H^d(K)$, which gives us the Moran equation:

$$4(\frac{1}{3})^d = 1$$

After solving for d (4.1, The Moran Equation), we get a result of

$$dim_H(K) = \frac{\ln 4}{\ln 3} \approx 1.2619$$

The Koch Snowflake's dimension can also be calculated using the previously stated similarity dimension formula (4.1, The Moran Equation), as the fractal is exactly self-similar. L can be determined to be $\frac{1}{r}$, or 3. Using the similarity dimension formula, this yields a dimensionality of

$$d = \frac{\ln(4)}{\ln(3)}$$

or ≈ 1.26.

This formula is the same as the one we reached in our earlier calculations, because the Koch Snowflake is exactly self-similar. This allows us to simply use the quicker similarity dimension formula without using the Hausdorff formula.

5.2 Sierpiński Triangle

The Sierpiński Triangle is another widely known example of an exactly self-similar fractal with a non-integer dimension.

In each iteration, an equilateral triangle is divided into four, smaller, identical equilateral triangles using the midpoints of each side. Then, the middle, upside-down triangle is removed, leaving three smaller triangles at the corners. This is repeated with each remaining triangle, creating a self-similar fractal.

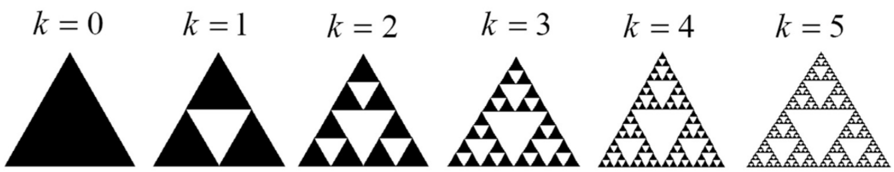

Figure 4: Sierpiński Triangle Construction

The triangle is a self-similar fractal, and so the similarity dimension formula can be used once again. The fractal is a union of $N = 3$ copies of itself, each shrunk by a factor of

$r = 1/2$. Therefore, L can be determined to be $\frac{1}{r}$, or 2. Using the similarity dimension formula, this yields a dimensionality of

$$d = \frac{\ln(3)}{\ln(2)}$$

or ≈ 1.58.

6 The Coastline Paradox

The Coastline Paradox was first systematically studied by Lewis Fry Richardson in the 1950's, and expanded upon by Benoit Mandelbrot.

The main idea of this paradox is that, counterintuitively, a coastline does not have a defined perimeter; instead, the length of a coastline changes depending on the scale with which it is measured. Because a landmass has features at every scale, there is no obvious measure that should be used to determine its perimeter. When increasingly short measurements are used, the sum approaches the true length, the supremum.

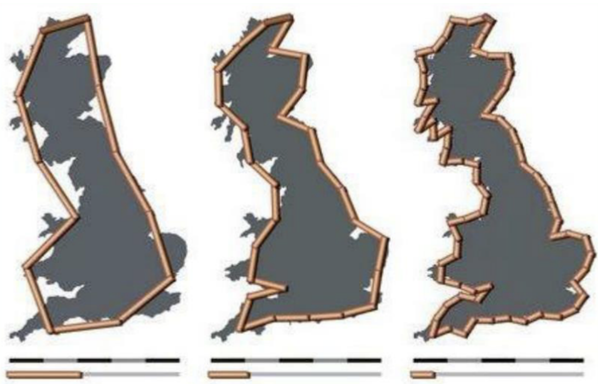

Figure 5: The Coastline Paradox

In 1967, Mandelbrot published his paper "*How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*," in which he discusses curves with Hausdorff dimensions between 1 and 2. This paradox is an example of a fractal that has a non-integer dimension, but is not entirely self-similar. However, a key factor of fractals is self-similarity, at least to some degree. This similarity appears in coastlines due to their bays and promontories (a point of land that juts out into the water). At any point on a coastline, no matter how great the scale of magnification, a similar pattern of bays and promontories should appear. Mandelbrot characterized a coastline with this property as a curve "whose fractal dimension is greater than 1."

Just as the measured value of a fractal curve diverges to infinity, the length of a coastline measured with near-infinite resolution would add up to infinity.

Put in simple terms, the commonly used equation to find the length of a coastline looks something like this:

Length of the coastline = length of the ruler $\times$ number of rulers used.

When Lewis Richardson began to study coastlines, the prevailing method of determining the length of a coastline was as follows: lay out n equal straight-line segments of length l with dividers on a map or aerial photograph. While investigating the problems with this method, Richardson discovered the "Richardson effect:" the sum of the segments monotonically increases when the common length of the segments decreases. This essentially means that as the length of the ruler gets shorter, the measured border gets longer. In addition, under certain circumstances, as l approaches 0, the length of the coastline approaches infinity.

Based on Euclidean geometry, a coastline should approach a fixed length. This led to Mandelbrot's development of the Hausdorff dimension, which does not depend on length l in the same way. Mandelbrot's statement of the Richardson effect is:

$$L(\epsilon) \sim F\epsilon^{1-D}$$

where L represents the length of the coastline, ϵ represents the measurement unit, F is a constant, and D is a parameter that Richardson found depends on the coastline approximated by L. Although Richardson did not expand upon this last parameter, Mandelbrot connected D with a non-integer form of the Hausdorff dimension. A rearranged form of this expression gives us

$$F\epsilon^{-D} \cdot \epsilon$$

where $F\epsilon^{-D}$ is the number of units ϵ needed to obtain L.

7 Conclusion

Irregular natural objects and fractals cannot be measured using classical Euclidean dimension. Therefore, the Hausdorff dimension must be used, allowing us to calculate non-integer dimensionalities. Switching from topological spaces to metric spaces allows for the use of distance as a function, therefore allowing us to define open covers and balls. These ideas and Richardson's study of coastlines led to the Hausdorff measure and the Hausdorff dimension. Now, using these calculations, we can quantify the roughness of fractals, allowing for real world applications in fields like finance and computer graphics.

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