KNOTS, QUIVERS, AND SOME SPECIAL POLYTOPES

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1. Introduction

Higher physics has increasingly been a source of inspiration for mathematicians. *String theory*, in particular, has been a rich source of ideas, providing physical intuition to connect seemingly different mathematical structures. The idea of the *Knot–Quiver Correspondence* was introduced in 2017 by Jakub Jankowski, Piotr Kucharski, Dmitry Noshchenko, and Piotr Sułkowski in *BPS States, Knots and Quivers*.

In 2021, they elaborated on the connection between knots, quivers, and permutahedra in their preprint *Permutahedra for Knots and Quivers*, along with Hélder Larraguível. Larraguível also published his thesis in 2022, *A-polynomials, Symmetries and Permutohedra for Quivers and Knots*, exploring particular connections with 3D Supersymmetric Gauge Theory and Chern–Simons Theory.

Supersymmetry postulates that there is a symmetry between particles of integer spin and particles of half-integer spin: fermions and bosons. Given a system, if some of the supersymmetries are conserved, then it can be shown that the energy of the states is bounded below by the charge associated with the conserved supersymmetry—this is known as the *BPS bound*. States that have the minimum energy allowed by the BPS bound are called *BPS bound states*.

In studying these bound states, it was conjectured that topological string amplitudes encode *HOMFLY knot invariants*. These authors later found that describing these topological string amplitudes required quivers, leading them to the connection between knots and quivers.

In this paper, we aim to provide an elementary introduction to knots and quivers with respect to their intersections with permutohedra and associahedra. We will start by defining knots and quivers and attempting to give a sense of their importance—in the case of quivers, through their applications in representation theory and cluster algebras.
Then, we will move on to defining the invariants of a knot, focusing on the HOMFLY-PT polynomial, taking it to be the invariant of a cabled or satellite knot in particular. We will briefly describe the Knot–Quiver correspondence by comparing the generating function of a quiver and HOMFLY-PT polynomials.
Finally, we will understand how knots and quivers give rise to permutohedra and associahedra in two different interpretations. We will see how graphing degenerate quivers leads to structures consisting of interconnected permutohedra through the local equivalence theorem. Furthermore, we will see how quiver mutations, through the quiver definition of cluster algebras, correspond to and therefore connect with associahedra. We will finally observe some examples of applications of these structures in physics and Lie algebras.

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Contents

Acknowledgements

The author would like to thank Simon Rubinstein-Salzedo and Elsa S. Frankel for their support and valuable discussions. This work was supported by The Euler Circle.

2. Preliminaries

Definition 2.1. A **homomorphism** is a structure-preserving map between two algebraic structures.

Definition 2.2. An **isomorphism** is a structure-preserving mapping or morphism between two structures of the same type that can be reversed by an inverse mapping.

Definition 2.3. The **rank** is the maximum number of linearly independent rows or columns of a matrix.

Definition 2.4. The **kernel** of a linear map, also known as the null space or nullspace, is the part of the domain that is mapped to the zero vector of the codomain.

3. Some Special Polytopes

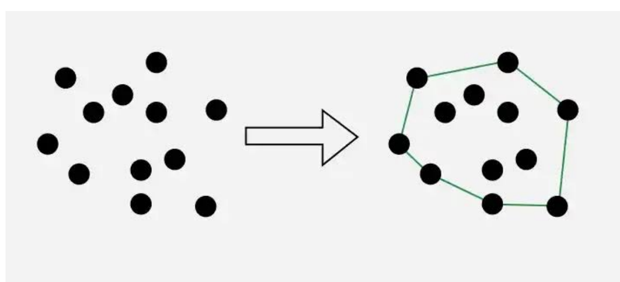


Figure 1. Convex hull

Definition 3.1. The **convex hull** of a finite set S is the smallest possible convex set containing S.

Definition 3.2. A **polytope** is the convex hull of a finite point set in d-dimensional space.

3.1. Permutohedra.

3.1.1. *Defining a Permutohedron.* A **permutohedron** is the convex hull of all the permutations of a finite set. We shall use the notation $P_n(x_1, \ldots, x_n)$ to denote the permutohedron on an n-element set.

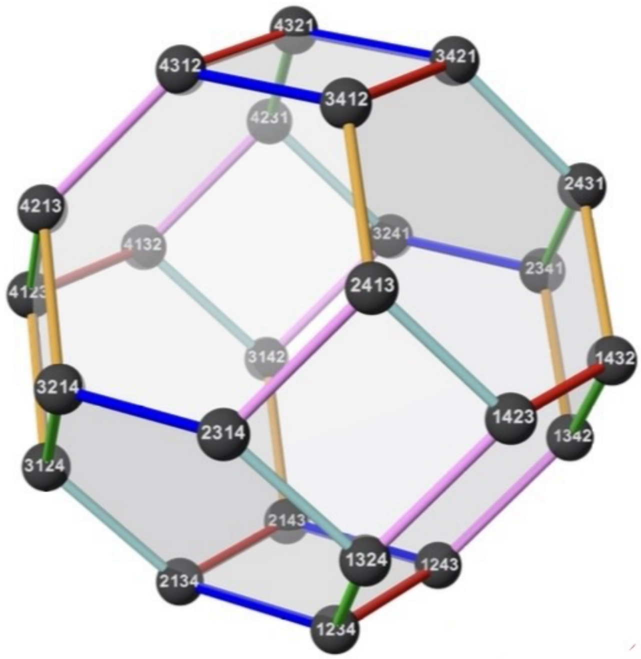


Figure 2. Permutohedron

3.1.2. *Constructing a Permutohedron.* **Vertices:** The coordinates of the vertices of a permutohedron are the permutations of the tuple $(s_1, s_2, \ldots, s_n)$. There are thus a total of $n!$ vertices.

Edges: The edges of a permutohedron correspond to transpositions of adjacent entries in the permutations; e.g., 3412 and 4312 are connected by an edge in the corresponding figure. There are a total of $(n-1)(n!)/2$ edges.

Proof. There are a total of $n-1$ transpositions of adjacent elements that can take place for each permutation; i.e., each vertex must be connected to $n-1$ other vertices. Thus, there must be a total of $(n-1)n!$ connections of vertices. However, as each edge connects two vertices, we must divide by two, and thus there are $(n-1)(n!)/2$ edges. ■

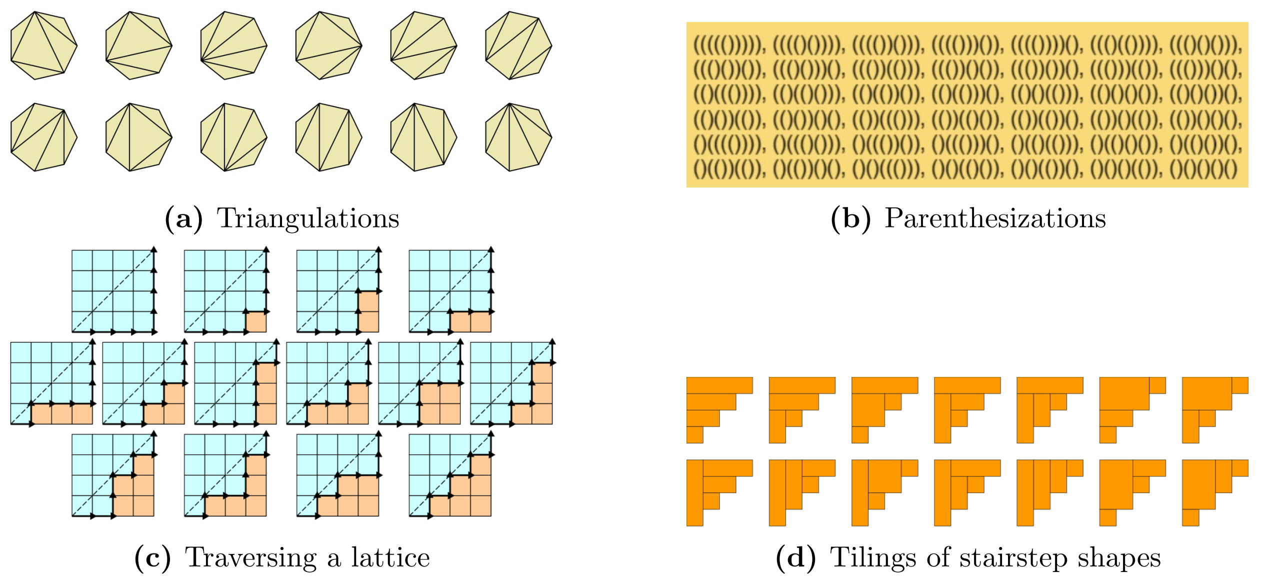
Dimensions: The permutohedron is a hyperplane; i.e., it is an $(n-1)$-dimensional shape that exists in n-dimensional space, where n is the cardinality of the set on which the permutohedron is constructed.

Proof. Consider the permutohedron $P_n(x_1, \ldots, x_n)$ with vertices $t_1, t_2, t_3, \ldots, t_{n!}$, and define a constant $t_1 + t_2 + \cdots + t_n = c$. Note that we have $n!$ vertices, as each vertex represents a permutation of an n-element finite set. We can represent all permutations with a matrix M, which has dimension $n \times n!$. By the Rank–Nullity Theorem, there are a total of $n-1$ linearly independent vectors in M. Hence, these linearly independent vectors add up to give an $(n-1)$-dimensional shape. Essentially, if $s_1 + s_2 + \cdots + s_n = c$, and the values $s_1, \ldots, s_{n-1}$ are chosen, we already know the value of s_n; hence there are only $n-1$ degrees of freedom. ■

3.2. **Associahedra.**

3.2.1. *On the Significance of Catalan Numbers.* The Catalan numbers are a highly important integer sequence that shows up often in counting problems, especially in those that involve recursion. They can be described as

(1) the number of triangulations of a polygon with $n+1$ sides,
(2) the number of parenthesizations possible in a sequence,
(3) the number of ways to traverse a 2D lattice starting from the bottom-left corner, with only rightward and upward moves, without crossing the diagonal,
(4) the number of ways to tile a stairstep shape of height n with n rectangles,
(5) . . . and many more.



(a) Triangulations **(b)** Parenthesizations **(c)** Traversing a lattice **(d)** Tilings of stairstep shapes

Figure 3. Catalan Number Derivations

In fact, Richard P. Stanley, in *Enumerative Combinatorics*, listed 66 ways the Catalan numbers can be derived. The recursive definition of a Catalan number is $C_0 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k\, C_{n-k}, \quad n \geq 0$. The closed-form definition of a Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0$.

3.2.2. *Derivation of Catalan Numbers through Triangulation.*

Proof. As previously stated, a Catalan number C_n can be calculated from the triangulation of an $(n+1)$-sided polygon. We will prove by induction the recursive formula for Catalan numbers through the triangulation of polygons. This is useful both for illustrative purposes and for studying associahedra.

We define the Catalan numbers by

$$C_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ \sum_{k=1}^{n-1} C_k \cdot C_{n-k}, & n \geq 2. \end{cases}$$

Base Case: There is no triangulation of a line. A triangle has exactly one triangulation; i.e., $C_1 = 0$, $C_2 = 1$.

Inductive Step: Assume $\sum_{k=1}^{n-1} C_k \cdot C_{n-k}$ holds for $n > 2$. Then C_{n+1} is derived from the polygon with $n+2$ vertices. Fix an edge E of this polygon. There are n possible triangles including edge E, since there are n possible vertices. There is no overcounting here, as edge E is fixed.

In any triangulation, E belongs to exactly one triangle. The third vertex of this triangle can be chosen among the n remaining vertices. These triangles either partition the $(n+2)$-gon into an $(n+1)$-gon and this triangle itself, or into a $(k+1)$-gon, an $(n+2-k)$-gon, and the triangle itself (with $2 \le k \le n-1$).

Thus we obtain C_n for the first case and $C_k \times C_{n+1-k}$ for the second case.

Putting everything together, we get $C_{n+1} = C_n + C_n + \sum_{k=2}^{n-1} C_k \times C_{n+1-k}$, which can be rewritten as $C_{n+1} = C_1 \times C_n + C_n \times C_1 + \sum_{k=2}^{n-1} C_k \times C_{n+1-k}$, and hence $C_{n+1} = \sum_{k=1}^{n} C_k \times C_{n+1-k}$. ■

3.2.3. *Defining the Associahedron.*

Definition 3.3. The **associahedron** is a shape whose number of vertices is a Catalan number, and where each vertex corresponds to a specific triangulation of a polygon.

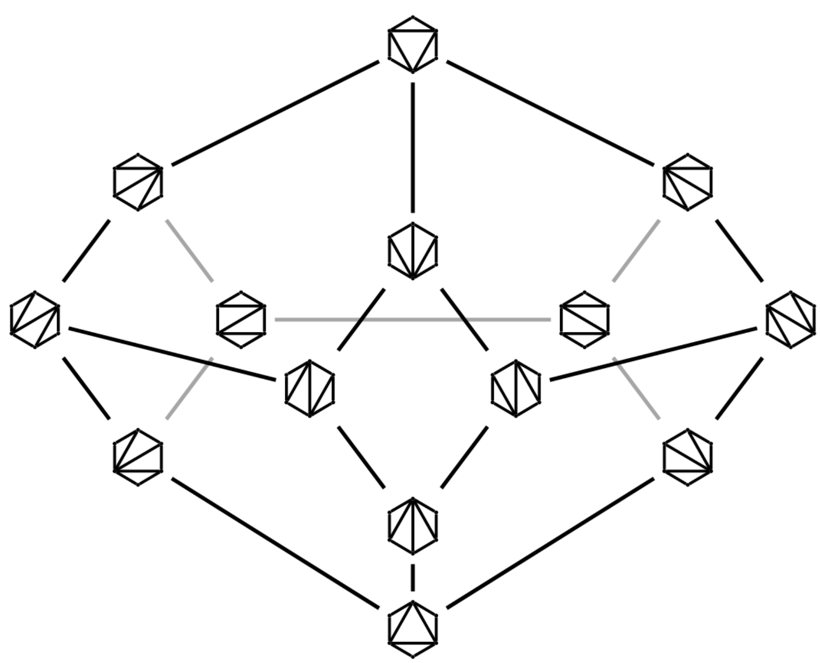


Figure 4. An associahedron

3.2.4. *Construction of Associahedra.* Though we have established their connections to Catalan numbers, it can still be a little counterintuitive to visualize them. We aim to give a clearer idea of this in this section. **Vertices**
An associahedron's vertices are calculated as follows through the **secondary polytope** method.

1 For the associahedron A_n, a single triangulation of an $(n+1)$-gon is taken.
2 For any vertex v_i of the polygon, s_i is the sum of the areas of the triangles that contain v_i.
3 The vector $[s_1, s_2, \ldots, s_i, \ldots, s_n]$ obtained is the coordinate of one vertex of the associahedron.
4 The process is repeated for all possible triangulations and the corresponding vertices to obtain the full associahedron.

Edges
Two vertices of the associahedron are connected by an edge if the triangulations they correspond to differ only by "flipping" a diagonal. (By a flip, we mean choosing an arbitrary quadrilateral in the polygon in which one of the diagonals is drawn, and then drawing the opposite diagonal instead.)

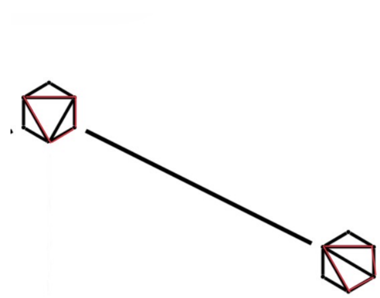


Figure 5. A flip in an associahedron

4. Knots

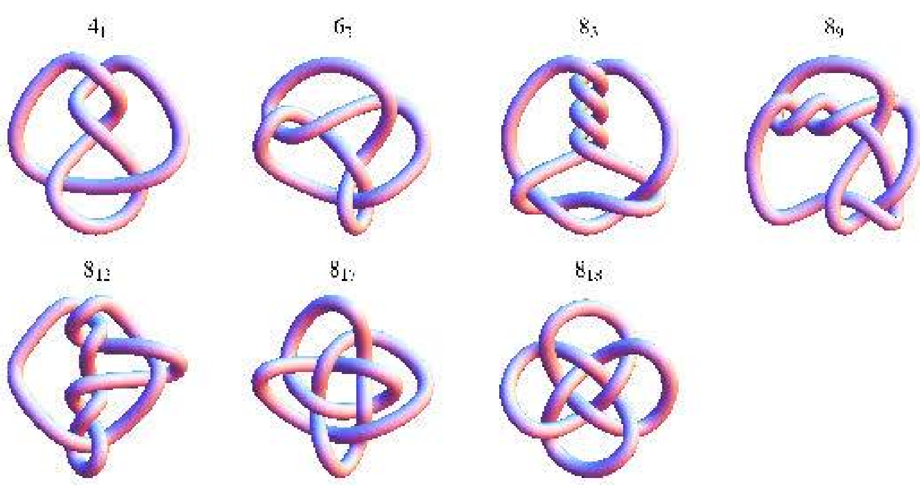


Figure 6. Knots

Knots are fairly intuitive to visualize—mathematical knots are merely refined versions of the typical knots that show up in everyday life, in shoelaces, headphone cords, etc.

Definition 4.1. A **knot** is a smooth embedding of a circle S^1 into three-dimensional space $\mathbb{R}^3$.

Think of a circle in two-dimensional space, with points (x, y). We can "add an extra z coordinate" and **embed** our circle in three-dimensional space, giving us more "flexibility" to twist things around. A good visualization is a hair tie, a loop that can be bent and twisted in 3D space.

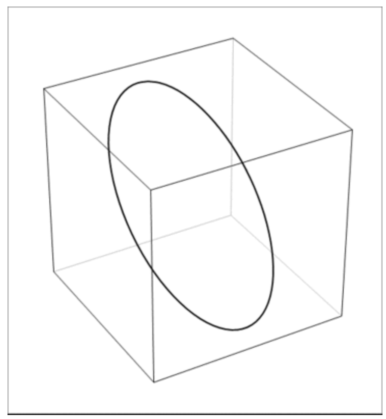


Figure 7. Embedding of a circle

4.1. **Knot Equivalence.** The central concern of knot theory is the knot equivalence problem; i.e., telling which knots are the same and which knots are different. This may seem like a very trivial problem. However, a look at the figure below might help illustrate this.

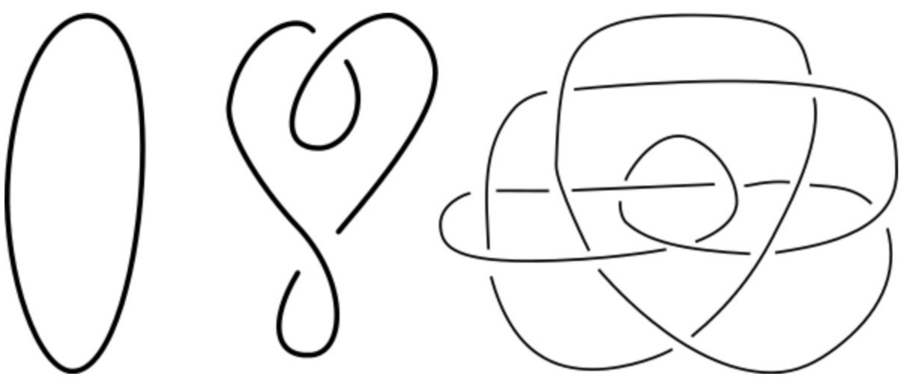


Figure 8. Knot equivalence

As you can see in the figure, it is very easy to see that we can "untwist" the second knot to get the first one. Thus, the first and second knots are the same. However, we can also similarly "untwist" the third knot to get the first knot; however, this is much harder to visualize. Thus, knot equivalence is in fact a suitably difficult problem.

4.2. **Reidemeister Moves.** The difficulty is caused by the fact that you may "untwist" or "twist" a knot—deform it—without fundamentally changing it. Reidemeister reduced all such deformations that do not change a knot to three moves:

- (I) the twist move,
- (II) the poke move,
- (III) the slide move.

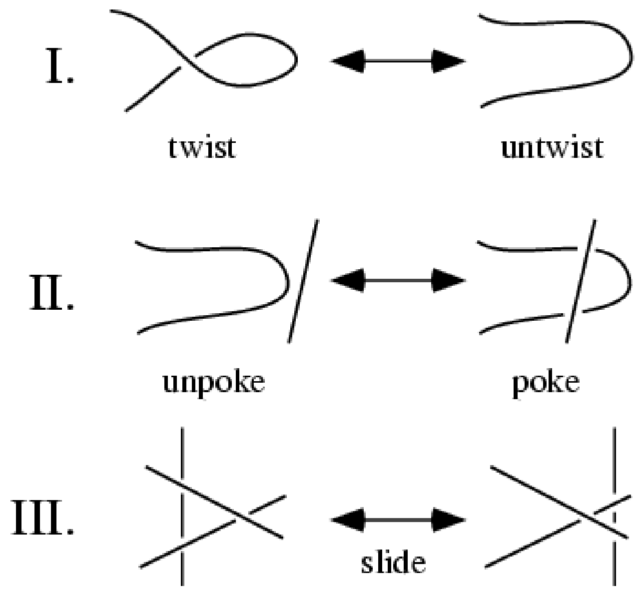


Figure 9. Reidemeister Moves

Theorem 4.2. *Two knots are equivalent if and only if they result from each other by a finite number of Reidemeister moves.*

4.3. **Invariants.** Knot invariants are properties of a knot that can help us with the knot equivalence problem. Knot invariants are properties that remain constant however the knot is deformed, provided that the knot is not fundamentally changed. Essentially, a knot invariant is a property that does not change under Reidemeister moves.

4.4. **HOMFLY–PT Polynomial.** The HOMFLY–PT polynomial is an important invariant of oriented knots and links. It generalizes both the Alexander polynomial and the Jones polynomial. More generally, one can define the *colored HOMFLY–PT polynomial*, which assigns representations (or "colors") to the components of a knot or link.

A convenient generating function for the colored HOMFLY–PT polynomials is given by

Definition 4.3 (Colored HOMFLY–PT Generating Function)**.**

$$P_K(x, a, q) = \sum_{r=0}^{\infty} \frac{x^r}{(q^2; q^2)_r} P_r(a, q),$$

where

$$(q^2; q^2)_r = \prod_{k=1}^{r} (1 - q^{2k})$$

is the q-Pochhammer symbol, and $P_r(a,q)$ denotes the HOMFLY–PT polynomial colored by the r-th symmetric representation.

4.4.1. *Calculating HOMFLY–PT Polynomials: A Brief Overview.* The HOMFLY–PT polynomial is computed recursively using the *skein relation*. Rather than evaluating the entire knot at once, crossings are successively replaced according to a local rule until the diagram is reduced to simpler links whose polynomials are already known. Although the full derivation is beyond the scope of this report, the resulting polynomial is a powerful knot invariant: if two knots have different HOMFLY–PT polynomials, then the knots are necessarily distinct.

(1) The HOMFLY–PT polynomial of the unknot is normalized to be

$$P_{\bigcirc}(a,z) = 1.$$

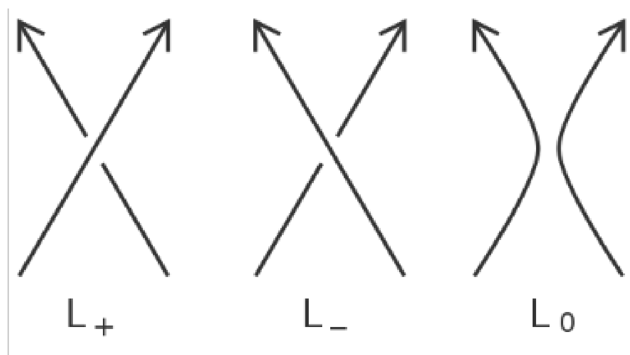


Figure 10. The skein relation relating the links L_+, L_-, and L_0.

(2) Suppose three oriented links L_+, L_-, and L_0 differ only at a single crossing, as shown in Figure 10. Their HOMFLY–PT polynomials satisfy the skein relation

$$aP_{L_+}(a,z) - a^{-1}P_{L_-}(a,z) = zP_{L_0}(a,z).$$

(3) If an oriented link L is the disjoint union of two links L_1 and L_2, then

$$P_L(a,z) = \frac{a - a^{-1}}{z}\, P_{L_1}(a,z)\, P_{L_2}(a,z).$$

(4) If L is the connected sum of two links L_1 and L_2, then

$$P_L(a,z) = P_{L_1}(a,z)\, P_{L_2}(a,z).$$

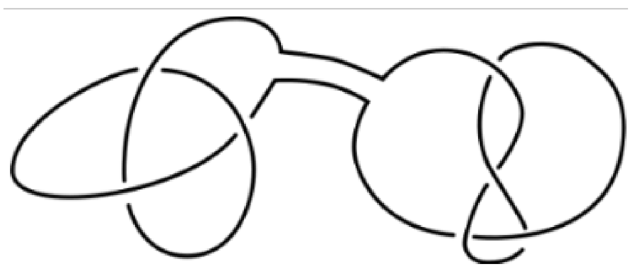


Figure 11. Connect Loop

However, we are considering the colored HOMFLY polynomial, as in the knot invariant of a *colored* knot. (To clarify, coloring here does not mean coloring in the sense of the four-color theorem—it is a concept from representation theory, which we will not get into here. Instead, we will use cabled knots as a substitute to visualize this.)

4.5. **Cabled Knots.** A cabled or satellite knot is a knot that is made up of another knot; i.e., the cross-section of the knot's "string" is made up of a knot itself. Think of a knot being made of yarn, where the yarn itself is made of string knotted together. Cabled knots are an easier way to visualize colored HOMFLY polynomials.

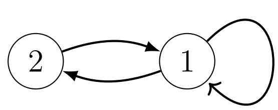
(a) Knot **(b)** Cross-section

Figure 12. Cabled knot

5. Quivers

Definition 5.1. A **quiver** is a directed graph $Q = (V, E)$.

For any quiver whose vertices are labeled $1, \dots, n$, define the matrix C to be the adjacency matrix of the underlying (undirected) graph G. This is the matrix with entries C_{ij}, where C_{ij} is the number of edges between vertices i and j.



$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

5.1. **Quivers and Representation Theory.** The main idea of representation theory is to study structures like groups or algebras through their representations. A representation of a group G could be a homomorphism from the group to a group of isomorphisms from a vector space V to itself. Thus, each element of G would be represented by isomorphisms of a vector space, allowing us to study a group with tools from linear algebra.
A representation of a given quiver assigns to each vertex i a vector space $V(i)$, and to each arrow $\alpha : i \to j$ a linear map $V(\alpha) : V(i) \to V(j)$.

5.2. **Quiver Generating Function.**

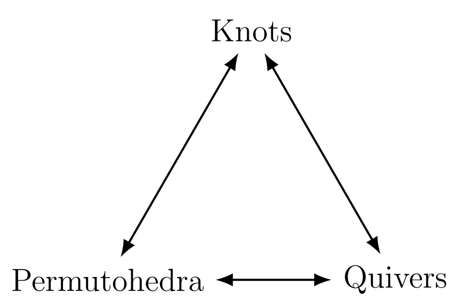
$$
\begin{aligned}
P_Q(x,q) &= \sum_{d\geq 0} \frac{x^d}{(q^2;q^2)_d}\, C_d(q) \\
&= \sum_{d_1,\ldots,d_m\geq 0} \frac{x_1^{d_1}\cdots x_m^{d_m}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}}\, q^{\sum_{i,j=1}^m C_{ij}d_id_j}
\end{aligned}
$$

Definition 5.2. A **quiver series** is an expansion of the quiver generating function.

- $x_1, x_2, \ldots, x_m$ are parameters corresponding to the m nodes of the quiver,
- $d_1, d_2, \ldots, d_m$ are the dimensions of the vector spaces linked to each node in the representation-theoretic interpretation of quivers,
- q is a deformation parameter,
- the denominator is the q-Pochhammer product, which shows up in many generating functions and prevents overcounting of the quivers.

6. Knots, Quivers, and Permutohedra

This section will piece together three of the components that we have discussed so far in the paper: knots, quivers, and permutohedra.
We will first discuss how knots and quivers correspond, then study how quivers can be plotted to generate permutohedra, thus tying them all together.



6.1. **Knot–Quiver Correspondence.**

Piotr Kucharski, Markus Reineke, Marko Stosic, and Piotr Sułkowski introduced the knot–quiver correspondence in the 2017 paper titled *BPS states, knots and quivers.*

$$
\sum_{r=0}^{\infty} \frac{x^r}{(q^2;q^2)_r}\, P_r(a,q) = \sum_{d_1,\ldots,d_m\geq 0} \frac{x_1^{d_1}\cdots x_m^{d_m}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}}\, q^{\sum_{i,j=1}^m C_{ij}d_id_j}
$$

Knots and quivers correspond as follows. However, it is very important to note that this is not a one-to-one correspondence; many quivers may correspond to one knot.

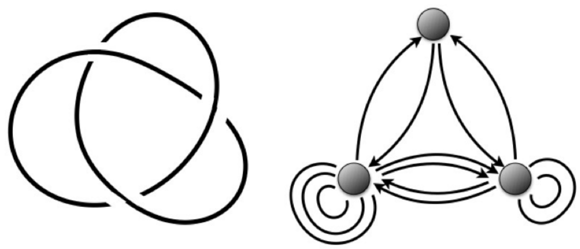


Figure 13. Knot–quiver correspondence

6.2. **Local Equivalence Theorem.**

Theorem 6.1. *The Local Equivalence Theorem states that for a given quiver of size* m *(equal to the number of HOMFLY-PT generators of the corresponding knot), encoded in a symmetric matrix* C*, there exist equivalent quivers such that their matrices differ from* C *only by a transposition of two non-diagonal elements* C_{ab} *and* C_{cd}*, as long as the values of these two elements differ by* 1 *and certain extra conditions are met.*

6.3. **Plotting the Quivers.** The Local Equivalence Theorem states that, under certain conditions, an equivalent quiver can be obtained by exchanging the values assigned to two nodes. This connects naturally to the concept of transpositions in permutohedra, where two vertices are joined by an edge if one can be obtained from the other by swapping adjacent elements.

This idea provides the intuition for constructing the permutohedron graph. Using the Local Equivalence Theorem, we identify all quivers equivalent to a given knot—in this case, the 9_1 trefoil knot—and represent them graphically. Each vertex corresponds to a quiver matrix, and edges connect quivers that differ by a transposition. The resulting structure is a network of interconnected permutohedra of varying sizes, where the diversity in size arises from different interpretations of the quiver derived from the generating function.

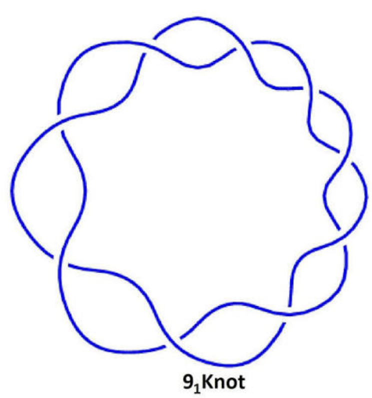


Figure 14. 9_1 torus knot

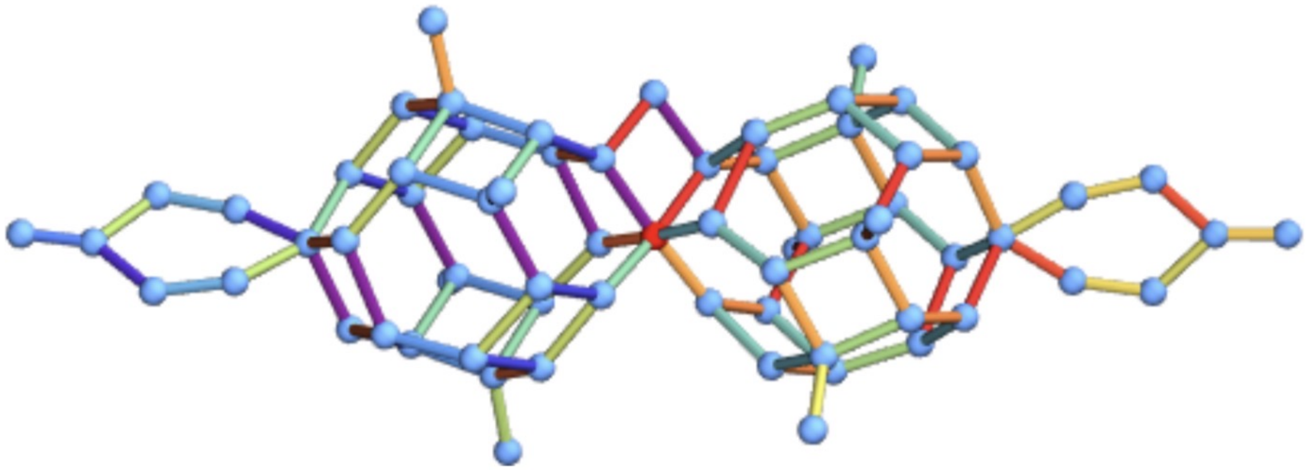


Figure 15. Permutohedra graph of the 9_1 torus knot

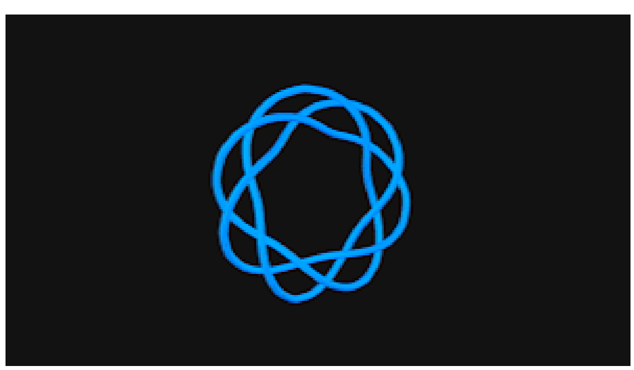


Figure 16. 11_1 torus knot

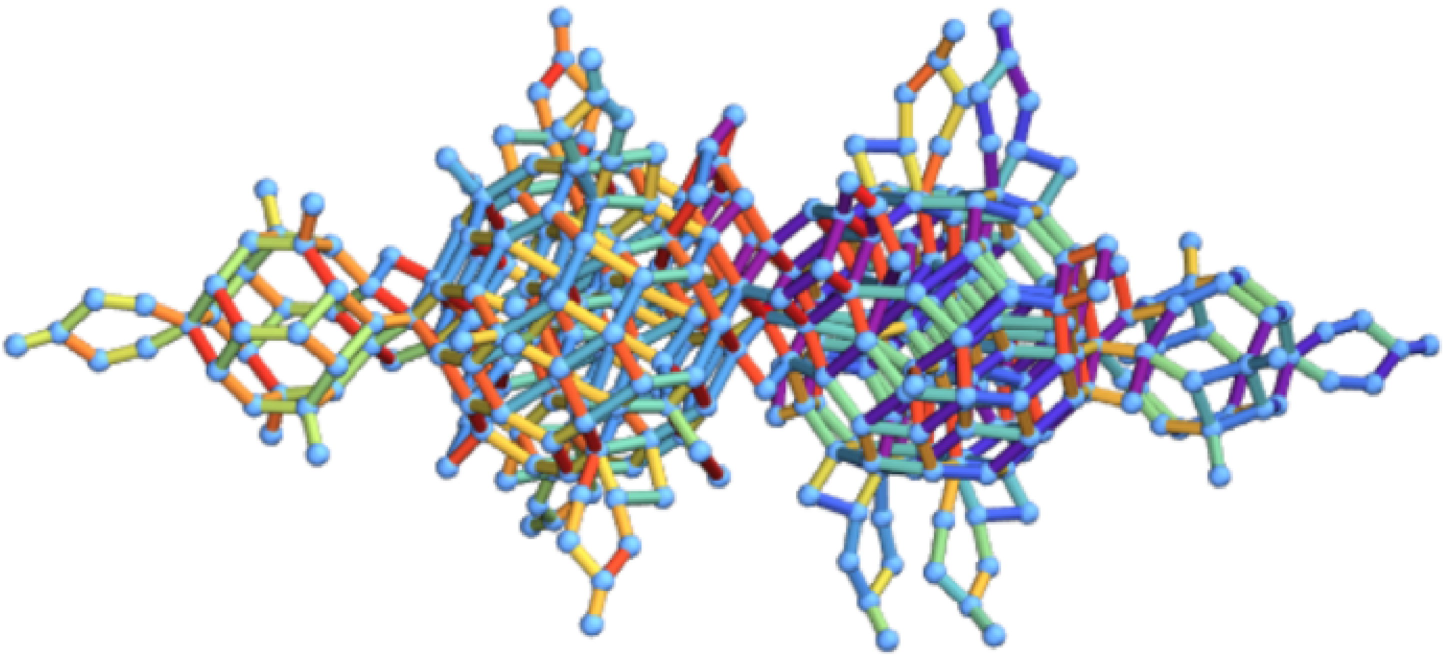


Figure 17. Permutohedra graph of the 9_1 torus knot

7. Quiver Mutations and Associahedra

7.1. Cluster Algebras.

7.2. Redefining the Quiver.

We redefine what a quiver is in the context of cluster algebras. A quiver is a finite oriented graph; we disallow loops (i.e., an arrow may not connect a vertex to itself) and oriented 2-cycles (i.e., no arrows of opposite orientation may connect the same pair of vertices).

We designate some vertices as frozen. The remaining vertices are called mutable. We will always assume that there are no arrows between pairs of frozen vertices. (Such arrows would make no difference in the future construction of a cluster algebra associated with a quiver.)

7.3. Quiver Mutations.

Definition 7.1. Let k be a mutable vertex in a quiver Q. The quiver mutation μ_k transforms Q into a new quiver $Q' = \mu_k(Q)$ via a sequence of three steps: (1) For each oriented two-arrow path $i \to k \to j$, add a new arrow $i \to j$ (unless both i and j are frozen, in which case do nothing). (2) Reverse the direction of all arrows incident to the vertex k. (3) Repeatedly remove oriented 2-cycles until unable to do so.

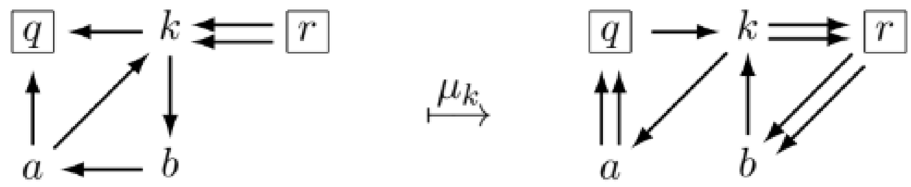


Figure 18. A quiver mutation

7.4. **Quivers and Triangulations.** We now associate a quiver to each triangulation of a convex m-gon P_m and explain how flips of such triangulations correspond to quiver mutations. Let T be a triangulation of the polygon P_m by pairwise noncrossing diagonals. The quiver $Q(T)$ associated to T is defined as follows. The frozen vertices of $Q(T)$ are labeled by the sides of P_m, and the mutable vertices of $Q(T)$ are labeled by the diagonals of T. If two diagonals, or a diagonal and a boundary segment, belong to the same triangle, we connect the corresponding vertices in $Q(T)$ by an arrow whose orientation is determined by the clockwise orientation of the boundary of the triangle.

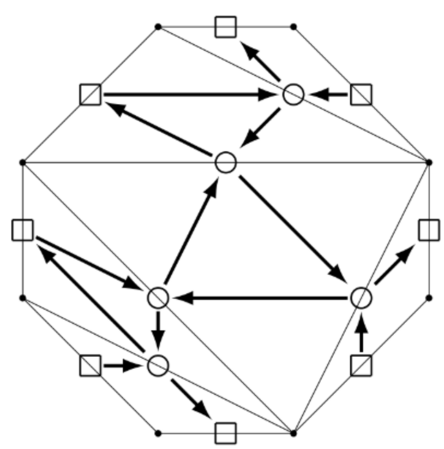


Figure 19. Associating a quiver with triangulations

7.5. **Connecting to the Associahedron.** We have previously established that we can associate a triangulation of a polygon with a vertex of an associahedron, and that two triangulations connected by a flip are joined by an edge.
Here, we have associated triangulations with quivers, and flips with mutations.
Thus, the vertices of an associahedron represent quivers and the edges of an associahedron represent quiver mutations.

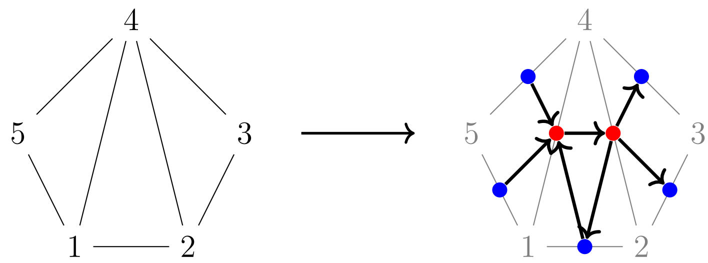


Figure 20. A flip and its corresponding quiver mutation

8. Conclusions and Further Scope

The Knot–Quiver Correspondence is deeply intertwined with higher physics, and several of the sources cited here are in fact physics papers. Furthermore, these ideas are important tools for investigating Lie theory. We attempt to provide some non-technical illustrations of some of these connections.

8.1. **Quantum Field Theory.**

Quantum Field Theory is one of the most successful theories in any field to date. We explore within Quantum Field Theory **3D Supersymmetric Gauge Theory**.

8.1.1. *Supersymmetric Gauge Theory.*

> *"... a gauge is a "coordinate system" that varies depending on one's "location" with respect to some "base space" or "parameter space", a gauge transform is a change of coordinates applied to each such location, and a gauge theory is a model for some physical or mathematical system to which gauge transforms can be applied."*

Supersymmetry is the idea that there exists a corresponding boson (an integer-spin particle) for every fermion (a half-integer-spin particle). The motivation for a supersymmetric version of gauge theory can be the fact that gauge invariance is consistent with supersymmetry. The Lagrangian partition function of Supersymmetric Gauge Theory reduces to a quiver series.

$$Z = \int \mathcal{D}\Phi \, e^{-S[\Phi]}$$

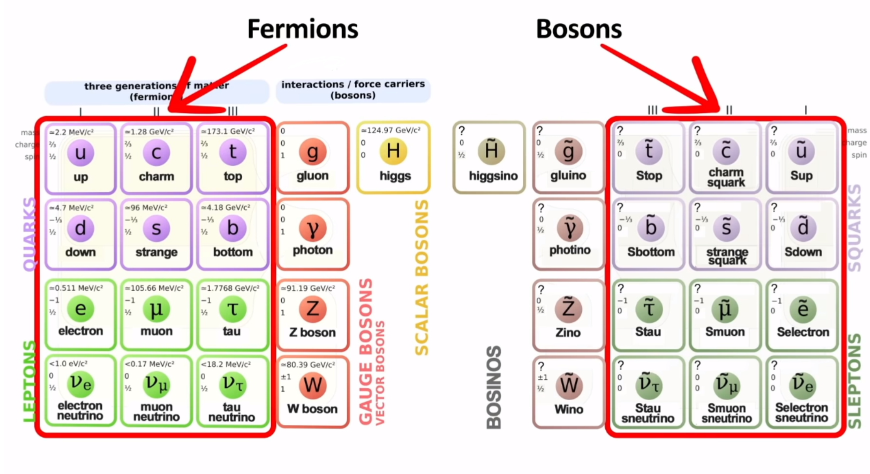


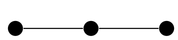
Figure 21. A depiction of supersymmetry

8.2. **Quiver Theory and Lie Algebras.** A **Lie group** is a group that is also a differentiable manifold, such that group multiplication and taking inverses are both differentiable. A **Lie algebra** is the tangent space at the identity point of the Lie group. Quivers have many intersections with Lie theory and are an increasingly important tool for exploring Lie algebras. To illustrate this connection, we take a brief look at **Gabriel's Theorem**.

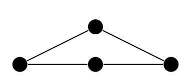
For a vector space E, a set of vectors S is a root system if:

- The set of vectors S spans the root system.
- The only scalar multiple of a vector v in S that is also in S is v itself.
- The set of vectors is perpendicular to the hyperplane (a hyperplane is an $(n-1)$-dimensional structure existing in an n-dimensional space).
- The projection of any vector v_1 in S onto any other v_2 in S must be an integer or half-integer.

Gabriel's Theorem states: A (connected) quiver is of finite type if and only if its underlying *undirected* graph is one of the ADE Dynkin diagrams, i.e., specific graphical representations of Lie algebras.



Finite



Affine (infinite)

If you take *mutations* on these quivers, finite means that you will eventually get back to the data you started with by taking the *same mutation* over and over.

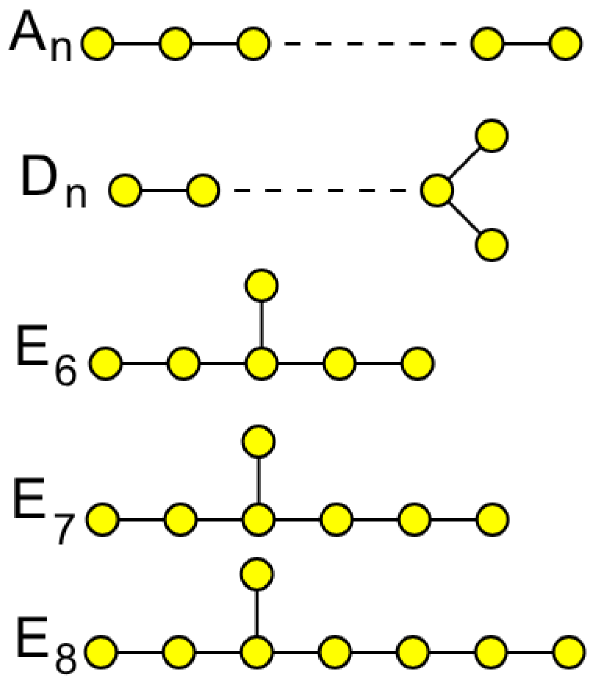


Figure 22. Dynkin diagrams

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[3] Piotr Kucharski, Markus Reineke, Marko Stosic, and Piotr Sułkowski. Knots-quivers correspondence. *arXiv preprint arXiv:1707.04017*, 2017.

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