The Schoen–Yau Positive Mass Theorem

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Euler Circle

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Goal

- Explain comparison theorems in Riemannian geometry.
- Explain the positive mass theorem.
- Show why it is important in geometry and general relativity.
- Outline Schoen and Yau's proof.
- conclusion
- Questions?
- References

Background

- Gravity is described by geometry.
- Curvature represents gravitational effects.
- Local curvature determines global properties.

Why ADM Mass Matters

- Portrait of mass and energy in space-time.
- Measures total energy of gravity.
- Observed from spatial infinity.
- Rules out isolated systems with negative total mass(shown by the theorem, which will show part of the proof for today).

Definitions

- **Metric Tensor**

$$g = g_{ij}\, dx^i \otimes dx^j,$$

defines lengths, angles, and distances on a manifold.

- **Scalar Curvature**

$$R = g^{ij} R_{ij},$$

the trace of the Ricci tensor, measuring the intrinsic curvature of a manifold at a point.

- **ADM Mass** which measures the total mass (including gravitational energy) of an asymptotically flat spacetime.

Main Theorem

For a complete asymptotically flat manifold of dimension $3 \leq n \leq 7$ with

$$R \geq 0,$$

the ADM mass satisfies

$$m_{\mathrm{ADM}} \geq 0.$$

If the manifold is Euclidean space, the Equality holds.

Comparison Theorems in Riemannian Geometry

Comparison theorems compare the geometry of a manifold with a model space whose curvature is known.

They show that *local curvature conditions determine global geometric properties* such as

- distances,
- angles,
- volume,
- topology,
- or total mass.

The Schoen–Yau Positive Mass Theorem is a comparison theorem because it compares

$$\boxed{\text{Local Scalar Curvature}} \quad \Longrightarrow \quad \boxed{\text{Global ADM Mass.}}$$

Curvature in Riemannian Geometry

The central idea of Riemannian geometry is curvature.

A Riemannian manifold (M, g) is a space equipped with a metric tensor

$$g = g_{ij}\, dx^i \otimes dx^j,$$

which measures lengths, angles, and distances.

Curvature measures how the geometry of a manifold differs from flat Euclidean space.

- Zero curvature: Euclidean space.
- Positive curvature: spherical geometry.
- Negative curvature: hyperbolic geometry.

The scalar curvature

$$R = g^{ij} R_{ij},$$

summarizes the average curvature at each point of the manifold.

Comparison Theorems

Comparison theorems show that **local curvature controls global geometry.**

Rather than computing an entire manifold, they compare it with a model space of constant curvature.

Toponogov's Comparison Theorem
If a complete Riemannian manifold satisfies

$$K_M \geq k,$$

then every geodesic triangle is at least as "fat" as the corresponding triangle in the simply-connected space of constant curvature k.

Schoen–Yau Positive Mass Theorem

If

$$R \geq 0,$$

on a complete asymptotically flat manifold, then

$$m_{\mathrm{ADM}} \geq 0.$$

Instead of comparing triangles, it compares

$$\boxed{\text{local scalar curvature}} \quad \Longrightarrow \quad \boxed{\text{global ADM mass.}}$$

Proof Strategy

1. Assume
$$m_{\mathrm{ADM}} < 0.$$
2. Construct a new asymptotically flat metric.
$$\tilde{g} = u^4 g,$$
where u is chosen so that
$$R_{\tilde{g}} \geq 0.$$
3. Produce a complete stable minimal surface.
$$\Sigma \subset (M, \tilde{g}).$$
4. Use the stability inequality and curvature estimates.
stability relating perimiter to volume
5. by obtaining a contradiction, prove
$$m_{\mathrm{ADM}} \geq 0.$$

Stable Minimal Surfaces

- A minimal surface satisfies
$$H = 0,$$
where H is the mean curvature.
- Stability implies
$$\int_\Sigma \left(|\nabla\phi|^2 - \left(\mathrm{Ric}(\nu,\nu) + |A|^2\right)\phi^2\right) \geq 0.$$
- Gauss equation:
$$2K = R - 2\,\mathrm{Ric}(\nu,\nu) - |A|^2,$$
where K is the Gaussian curvature.
- Gauss–Bonnet:
$$2\pi\chi(\Sigma).$$

Contradiction

- Since
$$R \geq 0,$$
the stability inequality gives
$$\int_\Sigma K\, dA \geq 0.$$
- By Gauss–Bonnet,
$$\int_\Sigma K\, dA = 2\pi\chi(\Sigma),$$
this forces strong topological restrictions on Σ.
- These restrictions contradict the existence of the stable minimizing surface derived from our assumption
$$m_{\mathrm{ADM}} < 0.$$
- Therefore,
$$\boxed{m_{\mathrm{ADM}} \geq 0.}$$

Applications of the Positive Mass Theorem

The Positive Mass Theorem has had a profound impact on mathematical relativity. Two important developments are:

1. **Witten's Spinor Proof (1981)**
   - Gave a shorter proof of the Positive Mass Theorem for spin manifolds.
   - Used spinors and the Dirac operator instead of minimal surface techniques.
   - Bridged supersymmetry and quantum field theory to differential geometry.
2. **Bray's Proof of the Riemannian Penrose Inequality (2001)**
   - Used the Positive Mass Theorem as a key in the proof.
   - Established a conformal flow of metrics.
   - Proved the Riemannian Penrose Inequality
   - Relation between ADM mass and the area of its black hole horizon.

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A}{16\pi}},$$

Conclusion

- The Positive Mass Theorem links local curvature to global mass.
- Fundamental result in differential geometry.
- Foundation for general relativity and mathematical physics.

Questions?

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