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ON SCHEON-YUA'S PROOF OF THE POSTIVE MASS COMPARISON THEOREMS IN RIEMANNIAN GEOMETRY AND ITS APPLICATIONS TO STRING THEORY.

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ABSTRACT. In this paper, I discribe the proof for the Schoen-Yau positive mass theorem for a three-dimensional Euclidean manifold, as established in Schoen and Yau's original paper [SY79]. The positive mass theorem is, in my opinion, one of the least strange and most intuitive of all the comparison theorems. Its significance is that it shows that an isolated system made of ordinary mass does not possess negative mass on a global scale. This theorem, as it stands, is far more intuitive than any of the volume or triangle angle comparison theorems within Riemannian geometry. However, the mathematics required to prove this intuitive result is somewhat more complex.

1. Introduction

Problem statement/main theorem

In this paper, I discuss the Schoen-Yau positive mass theorem [SY79] 2.1 for a three-dimensional Euclidean manifold, as proven in Schoen and Yau's original paper

1.1. **Background and context.** The link between geometry and gravity is one of the most profound discoveries in modern mathematics and physics [Wal84]. Einstein's theory of general relativity [Wal84] describes gravity as the bending of spacetime, turning geometric questions into physical problems. A notable implication of this link is the Schoen–Yau Positive Mass Theorem, which states that an isolated gravitational system with nonnegative local energy density cannot have negative total mass. In this paper, I discribe the proof of the three-dimensional Schoen–Yau Positive Mass Theorem using Schoen and Yau's original approach. I also describe the necessary geometric tools for the proof, including Riemannian manifolds, metric and curvature tensors, scalar curvature, minimal surfaces [SY79], and ADM mass [ADM62] 2.1. Finally, I highlight the theorem's importance in differential geometry, mathematical relativity, and contemporary theoretical physics.

Comparison theorems [CE08] usually relate to inequalities involving volume, distance, angles, or curvature. Meanwhile, the Positive Mass Theorem [SY79] can be viewed as a form of global comparison theorem. Instead of comparing geometric features such as volumes or geodesic triangles, it compares the local scalar curvature, which reflects local mass density, with the total ADM mass measured at spatial infinity. This establishes a link between local geometric information and global invariants. Such an approach highlights a central concept in comparison geometry: local curvature conditions can heavily influence the manifold's overall structure.

The Schoen-Yau positive mass theorem is a comparison theorem in general relativity. It states that the local mass density in space (a space-like manifold) is non-negative. Then the total mass of the system, that is, the mass seen from spatial infinity, which is called ADM mass, must be positive, unless the system has zero curvature, that is to say, the space (space-like manifold) is the flat Miskosvki space-time. Or, in other words, a manifold with non-negative scalar curvature has non-negative ADM mass.

Schoen-Yau positive mass theorem original paper, Main theorem: The Schoen-Yau positive mass theorem is a comparison theorem in general relativity. It states that the local mass density in

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space (a space-like manifold) is non-negative. Then the total mass of the system, that is, the mass seen from spatial infinity, which is called ADM mass, must be positive unless the system has zero curvature, that is, the space (space-like manifold) is the flat Miskosvki space-time. Or, in other words, a manifold with non-negative scalar curvature has non-negative ADM mass.

1.1.1. *Main contributors.* Richard Schoen [SY79], an emeritus professor at Stanford and a former student of Yao, contributed to the proof of the Schoen-Yau positive-mass theorem for Euclidean manifolds. He used minimizing surfaces to establish the result in dimensions seven and lower.

Shing-Tung Yau [SY79], an eminent mathematician who retired from Harvard University this spring and moved to Tsinghua University in China, has worked on the mathematical analysis of general relativity since the 1970s. In 1979, with the help of one of his former students, Richard Schoen, he proved the Schoen-Yau positive mass theorem, which showed that ADM mass could never take negative values. This is important because, without a lower bound, space and everything within it would become unstable and could take on negative values. In 1982, Yau won the Fields Medal, one of the highest honors in mathematics, in part for his work on the positive mass theorem. Some of Yau's areas of research include differential geometry, partial differential equations, topology, and mathematical physics. He has also written a book with the mathematical journalist Steve Nadis about general relativity and its relation to pure math, titled The Gravity of Math: How Geometry Rules the Universe. His results are essential for defining quasi-local mass and other physical properties.

Edd Witten [Wit81] provided an alternative proof [Wit81] using spinors In 1981, building on the positive energy theorem of Shoen and Yau in the context of supergravity. Witten [Wit81] studied at Brandeis University and Princeton, was a Harvard fellow, served as a junior fellow at the Harvard Society of Fellows from 1977 to 80, and received a MacArthur Foundation fellowship in 1982. He worked at Princeton from 1980 to 87 before joining the Institute for Advanced Study. Witten [Wit81] was awarded the Fields Medal [**?**] at the 1990 International Congress of Mathematicians in Kyoto for his work on superstring theory. He was also recognized for his 1981 discovery of a more straightforward proof for spin manifolds, using the Dirac operator and positive energy principles from physics. The physists who defined ADM mass (known after its authors as "Arnowitt, Deser, Misner mass"): The late Dr. Richard Arnowitt [ADM62] worked in many branches of theoretical physics at Texas A&M University, including general relativity, quantum field theory, high-energy particle phenomenology, supersymmetry and supergravity, and liquid helium theory. Stanley Deser [ADM62], a renowned theoretical physicist, made significant contributions to general relativity, quantum field theory, and high-energy physics. After retiring from Brandeis University in 2005, where he had spent most of his career, he moved to Pasadena to join Caltech as a researcher. At the time of his death, he was a visiting associate in theoretical physics at Caltech and held the title of Ancell Professor of Physics, Emeritus. Charles W. Misner [ADM62] was a distinguished Princeton professor and co-author of the influential textbook Gravitation. Over his career, he mentored 22 Ph.D. students, primarily at Princeton and the University of Maryland. Misner made significant contributions to general relativity, notably developing the ADM formalism with Richard Arnowitt and Stanley Deser. This work earned them the 1994 APS Dannie Heineman Prize for Mathematical Physics and was later honored with the Albert Einstein Society's Einstein Medal in 2015. He was also a Fellow of several prestigious organizations, including the American Academy of Arts and Sciences, the American Physical Society, the AAAS, and the Royal Astronomical Society.

To prove the positive mass theorem, background knowledge of the notation and definitions in general relativity is needed, as I will work backward from it. I will first define manifolds, then tangent spaces, then tensors, then the metric tensor, then the Christoffel symbols, then the Ricci tensor, and finally the Ricci curvature scalar, which I will use repeatedly throughout this paper to prove our theorem. It is also good to know that I shall define what ADM mass means, as it is the total mass of the system seen from spatial infinity.

2. Definitions and preliminaries

2.1. **Terminology.** I included this section to help non-mathematicians and non-physicists understand terminology that may be unfamiliar in this paper. *General relativity*: General relativity is the physical theory of the universe that describes how mass, or more precisely energy, corresponds to the curvature of spacetime and how the geometry of spacetime affects the motion of mass and energy. It is Einstein's general theory of relativity, which marvelously describes gravity and whose accuracy in describing and predicting phenomena in the universe has stood the test of time. It is also the mathematical framework for describing Einstein's general theory of motion and gravity, which relies on differential geometry and tensor calculus, among other things, and is its own field of pure math. Spacial infinity In general, spatial infinity represents a theoretical region that is an infinite distance in space from an isolated system at a constant moment in time. Zero curvature: That is, flat curvature, or, more simply, no curvature. Flat Miskosvki space-time: In Einsteinian relativity, Miskosvki space-time is the space-time manifold physicists use to model 3+1-dimensional flat space-time (flat meaning zero curvature). However, it is not a higher-dimensional Euclidean plane, as was most likely taught in school, but a non-Euclidean one that does not conform to the same notion of distance as the Euclidean plane and that works a bit like the complex plane, hence 3+1 instead of 4-dimensional space-time, due to the use of imaginary numbers. Mass: Mass is often thought of as initial mass, meaning mass or energy that is constrained and therefore allowed to accelerate, unlike massless particles like light, which move at the speed of light and do not accelerate. However, in the context of general relativity, mass can also be thought of as sufficient mass, which includes anything with energy and/or momentum, meaning light would also count. In general relativity, both sufficient and natural mass are equivalent to gravitational mass, which is mass that can be thought of as bending the fabric of space-time. But maybe I should think of it as E/c^2, since it must have units similar to kilograms, such as pounds or the Planck mass. *ADM* [ADM62]: The total mass of the system, that is, the mass measured at spatial infinity, is called the ADM mass. *quasilocal mass* [Bar89] While ADM mass is useful for certain calculations, it doesn't allow physicists to measure the mass contained within a specific finite region. For example, when studying two merging black holes, researchers might want to determine the mass of each black hole before the merger, rather than the total system mass. The mass contained within a particular region, measured from its surface—especially in regions with intense gravity and spacetime curvature—is known as "quasilocal mass." Local mass density Local mass density is the point-by-point energy-momentum of matter and fields within an isolated system. Negative vs non-negative vs positive Non-negative mass is either zero or positive. Positive mass can be thought of as a positive mass charge, and I will prove later that it results in positive or spherical curvature. Negative mass does the opposite; it can be thought of as having negative mass and charge, resulting in negative or hyperbolic curvature.

2.1.1. **Topological Manifolds:**

A topological manifold is a topological space that locally resembles Euclidean space, meaning it appears flat when zoomed in at any point. Regardless of a shape's global complexity or curvature, zooming in sufficiently on any point reveals a flat structure. Manifolds are common in both mathematics and physics. definition mathematical(got mathmatical defintion: [Lee13]

Definition 1 (Topological Manifold)**.** *A* topological n-manifold *is a topological space M satisfying:*

(1) *M is Hausdorff;*
(2) *M is second-countable;*
(3) *For every point $p \in M$, there exists an open neighborhood $U \subseteq M$ containing p and a homeomorphism $\varphi : U \to V$, where V is an open subset of $\mathbb{R}^n$.*

The pair (U, φ) is called a coordinate chart*, and a collection of charts covering M is called an* atlas*.*

2.1.2. **Smooth Manifold**:

A smooth manifold is a manifold you can do calculus on for example you can take derivatives, define tangent spaces, which I will get to in a minute and integrate functions. [Lee13]

Definition 2 (Smooth Atlas)**.** *A* chart *on M is a pair (U, φ) where $U \subseteq M$ is open and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. If (U, φ) and (V, ψ) are overlapping charts, then the transition maps $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ must be infinitely differentiable (C^∞). I call a collection of charts $(U_\alpha, \varphi_\alpha)$* a smooth atlas *if* $\bigcup_\alpha U_\alpha = M$

[Lee13]

Definition 3 (Smooth Manifold)**.** *An* n-dimensional smooth manifold *is a Hausdorff, second-countable topological space M together with a maximal smooth atlas.*

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2.2. **Tangent Spaces**:

The tangent space consists of all tangent vectors at a specific point on a differentiable manifold, with each tangent vector arising from curves passing through that point. The tangent vectors to the coordinate curves at that point form a natural basis for this space. they also have to be infitly difrentable.

Definition 4 (Tangent Spaces)**.** *[Lee13] Let M be a smooth manifold, and consider a point $p \in M$. The* tangent space *at p, written as T_pM, is the vector space consisting of equivalence classes of smooth curves. an equivalence classes is two curves are the same if there derivitive at a point p are the same.*

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2.3. **Tensors**

A tensor is a geometric object that generalizes scalars, vectors, and matrices. It is a multilinear map whose components transform according to a specific rule as the coordinates change. What makes it different from a matrix? A tensor can be represented as a matrix in a specific coordinate system, but its components follow a specific transformation law whenever the coordinates are changed. [Lee13] **Mathematical definition.**

Let V be a finite-dimensional vector space over $\mathbb{R}$. A tensor of type (r, s) is a multilinear map $T : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \longrightarrow \mathbb{R}$, which is linear in each argument. Equivalently, a tensor of type (r, s) is an element of the tensor product $T \in \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}$. With respect to a basis, a tensor of type (r, s) has components $T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$,

which transform according to the tensor transformation law under a change of coordinates. Consequently, although the components of a tensor depend on the chosen coordinate system, the tensor itself is an intrinsic geometric object.

2.4. **Metric Tensor**:

Measures lengths, angles, distances, and volumes Mathematical definition: The metric tensor is a symmetric (0,2)-tensor. The metric tensor is not just any tensor. It has extra properties: symmetric, nondegenerate, positive definite (in Riemannian geometry). [Lee13]

Definition 5 (metric tensor)**.** *A* Metric Tensor *on a smooth manifold M is a smooth symmetric $(0, 2)$-tensor field $g : TM \times TM \longrightarrow \mathbb{R}$, such that for every point $p \in M$, the bilinear form $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is an inner product. Thus, $g(X, Y) = g(Y, X)$, and $g(X, X) > 0 \quad$ for all $X \neq 0$. In local coordinates, $g = g_{ij}\, dx^i \otimes dx^j$, where g_{ij} are the components of the metric tensor. The inverse metric is denoted by g^{ij}, and satisfies $g^{ik} g_{kj} = \delta^i{}_j$.*

2.5. **Christoffel Symbols.**

2.5.1. **Parallel transport** What is Parallel transport Parallel transport involves moving a vector along a curve while keeping its orientation as parallel as possible, consistent with the manifold's geometry. [Lee13]

Definition 6. *Let $\gamma(t)$ be a smooth curve on M. A vector field $V(t)$ along γ is said to be* parallel *if $\nabla_{\dot\gamma} V = 0$, where ∇ denotes the Levi–Civita connection.*

[Lee13]

Definition 7. *The Christoffel symbols show how the coordinate basis vectors change from point to point. They are determined entirely by the metric. The Levi–Civita connection satisfies $\nabla_{\partial_i}\partial_j = \Gamma^k{}_{ij}\partial_k$, where the Christoffel symbols are given by $\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right)$.*

2.6. **The Riemann Curvature Tensor.** What is the Riemannian tensor

Conceptual Definition. The Riemann curvature tensor captures the intrinsic curvature of a manifold, indicating how parallel transport around a tiny loop varies with the path. Mathematical definition (I know the mathematical definition): How it is defined by Christoffel symbols [Lee13] **Mathematical definition.**

The Riemann curvature tensor is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. In local coordinates,

$R^\ell{}_{ijk} = \partial_j\Gamma^\ell{}_{ik} - \partial_i\Gamma^\ell{}_{jk} + \Gamma^\ell{}_{jm}\Gamma^m{}_{ik} - \Gamma^\ell{}_{im}\Gamma^m{}_{jk}$.

2.7. **The Ricci Tensor.** What are Ricci tenser

Conceptual Definition.

The Ricci tensor is obtained by contracting the Riemann curvature tensor, which involves summing over one upper and one lower index. [Lee13] **Mathematical definition.**

The Ricci tensor is the contraction of the first and third indices of the Riemann curvature tensor, $R_{ij} = R^k{}_{ikj}$.

2.8. **Ricci Curvature. Conceptual Definition.**

Conceptual definition: The Ricci scalar, also called the scalar curvature, is obtained by contracting the Ricci tensor with the inverse metric.

The scalar curvature represents the mean of all sectional curvatures at a point, giving a single value that measures curvature. [Lee13] **Mathematical definition.**

The scalar curvature is the trace of the Ricci tensor with respect to the metric, $R = g^{ij}R_{ij}$.

2.9. **Problem statement/main theorem.** The Schoen-Yau positive mass theorem is a comparison theorem in general relativity. It states that if the local mass density in space (a space-like manifold) is non-negative, then the total mass of the system, that is, the mass seen from spatial infinity, called the ADM mass, must be positive unless the system has zero curvature, that is, the space (space-like manifold) is flat Miskosvki space-time. In other words, a manifold with non-negative scalar curvature has non-negative ADM mass.

Theorem 2.1. *Let (M, g_{ij}) be a complete, asymptotically flat Riemannian manifold of dimension $3 \le n \le 7$, with nonnegative scalar curvature,$R \ge 0$. Then the ADM mass of (M, g_{ij}) is nonnegative: $m_{\mathrm{ADM}} \ge 0$. Moreover, $m_{\mathrm{ADM}} = 0$ if and only if (M, g_{ij}) is isometric to Euclidean space $(\mathbb{R}^n, \delta_{ij})$.*

3. Section 2 Prior results in the literature- what I am using in the next sections to prove the main theorem.

Theorem 3.1. *Let ds^2 be an asymptotically flat metric on an oriented 3-manifold N. If $R \geq 0$ on N, then the total mass of each end is nonnegative.*

Our next result concerns the case when the total mass of one end is zero. In order to prove this, I require the following additional asymptotic decay
condition:

$$|h_{ij}| + r\,|\partial_k h_{ij}| + r^2\,|\partial_k \partial_\ell h_{ij}| \leq C(1+r)^{-q} \tag{3.1}$$

where $q > \frac{1}{2}$ and C is a positive constant.

Theorem 3.2. *Let N be an oriented 3-manifold with an asymptotically flat metric ds^2. Suppose that for some end N_k, the decay condition*
(3.1) holds and the total mass of N_k is zero. If
$R \geq 0$ on N, then ds^2 is flat. In fact, N is isometric to $(\mathbb{R}^3, \delta_{ij})$,
where δ_{ij} is the standard Euclidean metric.

4. Methods/construction/Approach

Explains the machinery of how I will be proving the theorem, (readers should understand how the result is obtained).

Sketch of the proof. The following proof outline follows the original argument of Schoen and Yau [SY79]. Assume, for contradiction, that the ADM mass is negative.

(1) First, modify the metric conformally so that the scalar curvature is strictly positive outside a compact set while preserving the negativity of the ADM mass.
(2) Using the assumption of negative mass, construct a complete area-minimizing surface in the manifold.
(3) Apply the second variation [SY79] formula for area to show that this minimal surface is stable [SY79].
(4) Combine the stability inequality with the Gauss equation [dC92] to obtain an integral inequality relating the scalar curvature, the second fundamental form, and the Gaussian curvature of the surface.
(5) Since the scalar curvature is nonnegative, the inequality forces strong restrictions on the geometry of the minimal surface [SY79]. Using the Gauss–Bonnet theorem together with the asymptotic behavior of the surface leads to a contradiction.
(6) Therefore, the assumption that the ADM mass is negative is false, and hence

$$m_{\mathrm{ADM}} \geq 0.$$

Finally, if

$$m_{\mathrm{ADM}} = 0,$$

rigidity arguments imply that the Ricci curvature vanishes identically. In three dimensions, Ricci-flatness implies that the full Riemann curvature tensor vanishes, so the manifold is isometric to Euclidean space.

■

5. Section 3 Primary results

Main results/Primary results Lemma, proposition, interpretation to prove the main theorem

5.1. **Gauss-bonnet Theorem.** I will be using Gauss-Bonnet theorem [dC92] in this paper to prove Schoen-Yau positive mass theorem. The gap theorem is used in differential geometry and also in topology. It relates the genius of a space to its curvature.

Theorem 5.1 (Gauss-Bonnet Theorem)**.** *Let (M, g) be a compact, oriented, two-dimensional Riemannian manifold with piecewise smooth boundary ∂M. Then $\int_M K\, dA + \int_{\partial M} k_g\, ds + \sum_i \theta_i = 2\pi\, \chi(M)$, where*

- *K is the Gaussian curvature,*
- *k_g is the geodesic curvature of ∂M,*
- *θ_i are the exterior angles at the corners of ∂M,*
- *dA is the Riemannian area element,*
- *ds is the arc-length element along ∂M, and*
- *$\chi(M)$ is the Euler characteristic of M.*

6. Proof of the main theorem

Proof of Theorem 1

Throughout this section I work on a fixed end N_k, and suppose that
x^1, x^2, x^3 are asymptotically flat coordinates on N_k.

Suppose these coordinates describe N_k on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$, where $B_{\sigma_0}(0) = \{\, x \in \mathbb{R}^3 : |x| < \sigma_0 \,\}$, and $r = |x|$ denotes the Euclidean length of $x = (x^1, x^2, x^3)$. I denote the total mass of N_k by M, omitting reference to the index k. I will suppose that $M < 0$ and $R \geq 0$,
in contradiction to Theorem 1.

The proof involves three steps. The first allows us to assume that
$R > 0$ outside a compact subset of $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$.
The second uses the assumption $M < 0$ to prove the existence of a complete
area-minimizing surface. The third uses second variation [SY79] arguments to show
that this is impossible if $R \geq 0$.

Step 1.
If ds^2 is an asymptotically flat metric on N_k with
$R \geq 0$, and with the total mass of N_k negative, then there exists
an asymptotically flat metric
$d\tilde{s}^2$,
conformally equivalent to ds^2, having scalar curvature
$\tilde{R} \geq 0$ on N_k,
$\tilde{R} > 0$
outside a compact subset of N_k, and having negative total mass.

Proof. Let $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$ represent N_k as described above. Let Δ denote the Laplace operator on functions. For a function ϕ on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$, $\Delta\phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial \phi}{\partial x^j} \right)$, where, as usual, $g = \det(g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. Using the asymptotic expansion (1.1), I calculate the expansion of Δr on
$\mathbb{R}^3 \setminus B_{\sigma_0}(0)$ to obtain

$$\Delta r = \frac{2}{r} - \frac{2M}{r^2} + O\left(\frac{1}{r^3}\right). \tag{2.1}$$

It follows that there exists a number
$\sigma > \sigma_0$
such that
$\Delta\left(\frac{1}{r}\right) < 0, \qquad r > \sigma.$

Choose $t_0 \gg 0$, and let $\eta(t)$ be a C^5 function which satisfies

$$\eta(t) = 0 \quad \text{for } t > 2t_0, \qquad \eta'(t) > 0, \qquad \eta(t) < 0 \quad \text{for } t \in (0,\infty). \tag{2.2}$$

Define a C^5 function $\phi : N_k \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} 1 + \eta(3t_0) & \text{on } N \setminus N_k, \\ 1 + \eta(-M)\, r^{-4} & \text{on } \mathbb{R}^3 \setminus B_{\sigma_0}(0) = N_k. \end{cases}$$

From (2.1) and (2.2) I see that

$$\Delta\phi \le 0 \quad \text{on } N, \qquad \Delta\phi < 0 \quad \text{for } r > 2\sigma_0. \tag{2.3}$$

I now define a new metric $d\tilde{s}^{\,2} = \phi^4\, ds^2$. The metric $d\tilde{s}^{\,2}$ is asymptotically flat since on all ends other than N_k it is a constant multiple of ds^2, and on N_k I have $\tilde{g}_{ij} = \left(1 - \frac{M}{r}\right)^4 \left(\delta_{ij} + \frac{M}{r} a_{ij} + O\left(\frac{1}{r^2}\right)\right) = 1 + \frac{M}{r} a_{ij} + O\left(\frac{1}{r^2}\right)$. Thus the new mass of N_k is $\tilde{M} = \frac{M}{2} < 0$. ■

The well-known formula for the scalar curvature $\tilde{R}$ under a conformal change is $\tilde{R} = \phi^{-5}\left(-8\Delta\phi + R\phi\right)$. I replace our original metric ds^2 by $d\tilde{s}^{\,2}$ but maintain the notation ds^2, so that I are assuming $R > 0$ on N, $R > 0$ outside a compact subset of N_k, and $M < 0$.

Step 2. There exists a complete area minimizing (relative to ds^2) surface S properly imbedded in N so that $S \cap (N \setminus N_k)$ is compact, and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by x^1, x^2, x^3.

Proof. Let $\sigma > 2z_0$, and let C_σ be the circle of Euclidean radius σ centered at 0 in the x^1x^2-plane. Let S_σ be the smooth imbedded oriented surface of least ds^2-area among all competing surfaces regardless of topological type having boundary curve C_σ. A discussion of this known existence result is given in the Appendix to this paper. I wish to extract a sequence $\sigma_i \to \infty$ so that S_{σ_i} converges to the required surface S.

I first show that there is a compact subset $K_0 \subset N$ so that I have

$$S_\sigma \cap (N \setminus N_k) \subset K_0 \quad \text{for every } \sigma > 2z_0. \tag{2.4}$$

That is, I show that the S_σ cannot run to infinity in an end other than N_k. To see this, let $N_{k'}$ be another end, with asymptotically flat coordinate system y^1, y^2, y^3 associating $N_{k'}$ with $\mathbb{R}^3 \setminus B_{z_0}(0)$ where $B_{z_0}(0) = \{y : |y| < z_0\}$. In this coordinate system, the metric ds^2 has the form $ds^2 = \sum_{i,j=1}^3 g'_{ij}\, dy^i\, dy^j$, with g'_{ij} satisfying (1.1).

I calculate the covariant Hessian of the function $|y|^2$, that is, $D_{ij}|y|^2 = \frac{\partial^2 |y|^2}{\partial y^i \partial y^j} - D_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^j}\right)(|y|^2)$.

By (1.1) I see $D_{ij}|y|^2 = 2\delta_{ij} + O\left(\frac{1}{|y|}\right)$ as $|y| \to \infty$, where δ_{ij} is the Kronecker delta.

In particular, I see that there exists $z_1 > z_0$ so that the function $|y|^2$ is a convex function for $|y| > z_1$. Since $\partial S_\sigma = C_\sigma$, which lies in N_k, I may apply the maximum principle to conclude that $S_\sigma \cap N_{k'} \subset B_{z_1}(0)$.

Since $N_{k'}$ was any end of N other than N_k, I have established (2.4).

I now analyze the behavior of $S_\sigma \cap N_k$. In fact, I bound the height of $S_\sigma \cap N_k$ in the x^3 direction. For any $h > 0$, I let $E_h = \{x \in \mathbb{R}^3 : |x^3| \le h\}$.

I show that there exists a number $h > z_0$ so that

$$N_k \cap S_\sigma \subset E_h \quad \text{for all } \sigma > 2z_0. \tag{2.5}$$

To accomplish this, I again use a maximum principle, this time for the function x^3 restricted to $S_\sigma \cap N_k$. I must first compute the asymptotic behavior of the covariant Hessian of x^3 on N_k. If D is the Riemannian connection for ds^2, define Γ_{ij}^ℓ by $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{\ell=1}^3 \Gamma_{ij}^\ell \frac{\partial}{\partial x^\ell}$.

Then Γ^ℓ_{ij} has the following expression in terms of ds^2: $\Gamma^\ell_{ij} = \frac{1}{2} g^{\ell m} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right)$.

The Hessian of any function φ is given by $D_{ij}\varphi = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \Gamma^\ell_{ij} \frac{\partial \varphi}{\partial x^\ell}$.

By direct calculation using (1.1) and (2.6) I have

$$D_{ij} x^3 = -\Gamma^3_{ij} = \frac{M}{r^3} \left(x^i \delta_{j3} + x^j \delta_{i3} - x^3 \delta_{ij} \right) + O\left(\frac{1}{r^3} \right). \tag{2.7}$$

Let h be the maximum for x^3 on $S_\sigma \cap N_k$, and suppose this maximum occurs at the point $x_0 \in S_\sigma$. If $h < z_0$, I have established (2.5). So suppose $h > z_0$. The tangent space to S_σ at x_0 is then spanned by $\frac{\partial}{\partial x^1}(x_0), \quad \frac{\partial}{\partial x^2}(x_0)$.

Let v_1, v_2 be tangent vector fields to S_σ defined in a neighborhood of x_0 and satisfying $v_i(x_0) = \frac{\partial}{\partial x^i}(x_0), \quad i = 1, 2$.

Let $(q_{ij})_{1 \le i,j \le 2}$ be the restriction of ds^2 to S_σ in terms of the basis field v_1, v_2. Let V be the induced connection on S_σ, and note $V_i V_j x^3 = V_{v_i} V_{v_j}(x^3) = v_i v_j x^3 - D_{v_i} v_j (x^3) + \langle D_{v_i} v_j, v \rangle \, v(x^3)$, where v is the unit normal field of S_σ. Evaluating at the point x_0 I have $V_{ij} x^3 = D_{ij} x^3 + h_{ij} v(x^3)$, where $h_{ij} = \langle D_{v_i} v_j, v \rangle (x_0)$ is the second fundamental form.

Contracting with respect to (q^{ij}) I have $\sum_{i,j=1}^2 q^{ij} V_{ij} x^3 = \sum_{i,j=1}^2 q^{ij} D_{ij} x^+ \sum_{i,j=1}^2 q^{ij} h_{ij} \, v(x^3)$.

Since S_σ is minimal I have $\sum_{i,j=1}^2 q^{ij} h_{ij} = 0$, so applying (2.7) I see $\sum_{i,j=1}^2 q^{ij} V_{ij} x^3 = -\frac{2Mh}{r^3} + O\left(\frac{1}{r^3}\right)$.

Since $M < 0$, I see that h sufficiently large implies that $\sum_{i,j=1}^2 q^{ij} V_{ij} x^3 > 0$ at x_0, contradicting the fact that x^3 attains a maximum there. A similar argument gives a lower bound on $x^3|_{S_\sigma \cap N_k}$, and I have established (2.5).

Now, let $\rho > 2z_0$ and define the set $A^\rho = (N \setminus N_k) \cup \left\{ x : |x| > \sigma_0, \, (x^1)^2 (x^2)^2 \le \rho^2 \right\}$.

For any $\sigma > \rho$, (2.4) and (2.5) imply

$$S_\sigma \cap A^\rho \subset (K_0 \cup E_h) \cap A^\rho, \tag{2.8}$$

which is a compact subset of N.

I now quote a local interior regularity estimate for area minimizing surfaces which is discussed in the Appendix.

(2.1) Regularity Estimate. Let $U_r(x)$ denote the geodesic ball of radius r about $x \in N$. There exists a number $r_0 > 0$ so that for any point $x_0 \in S_\sigma$ with $U_{r_0}(x_0) \cap \partial S_\sigma = \varnothing$, it is true that $S_\sigma \cap U_{r_0}(x_0)$ can be written as the graph of a C^3 function f_σ over the tangent plane to S_σ in a normal coordinate system on $U_{r_0}(x_0)$. Moreover, there is a constant c_1 depending only on (N, ds^2) which bounds all derivatives of f_σ up to order three in $U_{r_0}(x_0)$. (Note that both r_0 and c_1 are independent of σ.)

It then follows from (2.8) and the Regularity Estimate that I can choose a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \cap A^\rho$ converges in C^2 topology. Since this can be done for any $\rho > 2z_0$, I can take a sequence $\rho_j \to \infty$ and by extracting a diagonal sequence I find a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \to S$, an imbedded C^2-surface, uniformly in C^2 norm on compact subsets of N.

The surface S is properly imbedded by (2.8), and is clearly area minimizing on any compact subset of N. From (2.4) I have $S \cap (N \setminus N_k) \subset K_0$, and hence $S \cap (N \setminus N_k)$ is compact. From (2.5) I have $S \cap N_k \subset E_h$, which is the region between two parallel 2-planes in $\mathbb{R}^3$. This completes the proof of Step 2. ∎

Step 3. The surface S constructed in Step 2 cannot exist.

Proof. For any $\sigma > z_0$, let $S(\sigma)$ be the set $S(\sigma) = \left[S \cap (N \setminus N_k) \right] \cup \left[S \cap B_\sigma(0) \right]$.

The sets $S(\sigma)$ form an exhaustion of S, and I can see

$$\operatorname{Area}(S(\sigma)) < C_2 \sigma^2. \tag{2.9}$$

for a constant C_2 independent of $\sigma \ge z_0$.

To prove (2.9), I note that if S has transverse intersection with $\partial B_\sigma(0)$ then this intersection is a union of oriented C^2 Jordan curves on $\partial B_\sigma(0)$ which bound $S(\sigma)$. It follows that these curves bound a domain $O \subset \partial B_\sigma(0)$. Thus I have $\partial S(\sigma) = \partial O$, so I can apply the area minimizing property of S to conclude that $\mathrm{Area}(S(\sigma)) < \mathrm{Area}(O) < \mathrm{Area}(\partial B_\sigma(0))$.

Since (1.1) implies that ds^2 is uniformly equivalent to the Euclidean metric on $\mathbb{R}^3 \setminus B_{z_0}(0)$, (2.9) follows for those $\sigma > z_0$ for which $S \cap \partial B_\sigma(0)$ is transverse. Since this is true except for σ in a set of measure zero, (2.9) follows for any $\sigma > z_0$ by approximation.

I can use (2.9) to bound the integrals of certain functions on S. For $\alpha > 0$ I have $\int_{S(\sigma)} \frac{1}{1+\sigma^{-\alpha} r^\alpha}\, d\mu < \mathrm{Area}(S(\sigma)) + \int_\sigma^\infty \frac{\alpha t^{\alpha-1}}{(t^\alpha+\sigma^\alpha)^2} \mathrm{Area}(S(t))\, dt$.

If $\alpha > 2$, I can integrate by parts and apply (2.9) to get $\int_{S(\sigma)} \frac{1}{1+\sigma^{-\alpha} r^\alpha}\, d\mu < C_2\sigma^2 + C_2\alpha \int_\sigma^\infty \frac{t^{\alpha+1}}{(t^\alpha+\sigma^\alpha)^2}\, dt$.

$$\leq C_2\sigma^2 + C_2\alpha \int_\sigma^\infty \frac{t^{\alpha+1}}{(t^\alpha+\sigma^\alpha)^2}\, dt.$$

It follows that I have $\int_{S(\sigma)} \frac{1}{1+\sigma^{-\alpha} r^\alpha}\, d\mu < C(\alpha) \qquad$ whenever $\alpha > 2$.(2.10)

Applying similar reasoning, I have for $\sigma_0 < \sigma_1 < \sigma_2$,

$$\int_{S(\sigma_2)\setminus S(\sigma_1)} \frac{1}{r^2}\, d\mu < 2C_2 \log \frac{\sigma_2}{\sigma_1}. \tag{2.11}$$

I now introduce the second variation [SY79] inequality for S. This inequality expresses the fact that up to second order S has smallest area in a one-parameter compactly supported deformation of S.

Let e_1, e_2, e_3 be orthonormal (with respect to ds^2) vector fields defined locally on N. I use the notation $K_{ij} =$ sectional curvature of the section $\{e_i, e_j\}$.

The Ricci tensor can then be written $\mathrm{Ric}(e_i) = \sum_{j=1}^3 K_{ij}$, where I let $K_{ii} = 0$. The scalar curvature R is then given by $R = K_{12} + K_{13} + K_{23}$.

Let v be the unit normal vector field of S, and choose a frame $e_1, e_2, e_3 = v$ adapted to S.

Let A denote the second fundamental form of S, i.e. the matrix in terms of e_1, e_2 is $h_{ij} = (D_{e_i} v, e_j)$.

It is well-known that A is a symmetric quadratic tensor on S. I let $\|A\|^2$ denote the length of A with respect to ds^2, i.e. $\|A\|^2 = \sum_{i,j=1}^2 h_{ij}^2$.

The condition that S is a minimal surface [SY79] is $\mathrm{Trace}(A) = h_{11} + h_{22} = 0$.

The second variation [SY79] inequality (see [10]) for S is $\int_S \left(|\nabla f|^2 - (\mathrm{Ric}(v) + \|A\|^2) f^2\right) \geq 0$ for any C^2 function f with compact support on S.

After integration by parts I see $\int_S (\mathrm{Ric}(v) + \|A\|^2) f^2 \leq \int_S |\nabla f|^2$ for any C^2 function f with compact support on S. By approximation I see easily that (2.13) holds for any Lipschitz function f with compact support on S.

The Gauss curvature equation expresses the Gauss curvature K of S as $K = K_{12} + h_{11}h_{22} - h_{12}^2$.(2.14)

Applying (2.12) and the symmetry of A gives $\frac{1}{2}\|A\|^2 = K_{12} - K$.

Putting this into (2.13) gives $\int_S \left(\mathrm{Ric}(v) + K_{12} - K + \frac{1}{2}\|A\|^2\right) f^2 \leq \int_S |\nabla f|^2$.

That is, I have $\int_S \left(R - K + \frac{1}{2}\|A\|^2\right) f^2 \leq \int_S |\nabla f|^2$.

I now choose a suitable cutoff function for f in our inequalities. For $\sigma > \sigma_0$ define a function φ by $\varphi = \begin{cases} 1, & \text{on } S(\sigma), \\ \dfrac{\log(\sigma^2/r)}{\log \sigma}, & \text{on } S(\sigma^2) \setminus S(\sigma), \\ 0, & \text{outside } S(\sigma^2). \end{cases}$

Let g be a Lipschitz function on S satisfying $|g| \leq 1$ and $g = 1$ outside a compact subset of S. Setting $f = \varphi g$ in (2.13) and applying the Schwarz inequality gives $\int_S (\mathrm{Ric}(v) + \|A\|^2)\varphi^2 g^2 \leq 2\int_S |\nabla\varphi|^2 + 2\int_S \varphi^2|\nabla g|^2$.

Since $|\nabla\varphi| = \frac{1}{r\log\sigma}$ on $S(\sigma^2) \setminus S(\sigma)$, this becomes $\leq \frac{2}{(\log\sigma)^2}\int_{S(\sigma^2)\setminus S(\sigma)} \frac{1}{r^2} + 2\int_S |\nabla g|^2$.

Because of (1.1), there is a constant C_3 with $|\mathrm{Ric}(v)| \leq \frac{C_3}{r^3}$.

Thus our inequality implies, by rearranging and using the definition of φ and g, $\int_{S(\sigma)} \|A\|^2 g^2 \leq \frac{2C_3}{(\log\sigma)^2}\int_{S(\sigma^2)\setminus S(\sigma)} \frac{1}{r^2} + 2\int_S |\nabla g|^2 + \int_S |\mathrm{Ric}(v)| g^2$.

Applying (2.11) I have $\int_{S(\sigma)} \|A\|^2 g^2 \leq 2C_2C_3(\log\sigma)^{-1} + 2\int_S |\nabla g|^2 + \int_S |\mathrm{Ric}(v)| g^2$.

Letting $\sigma \to \infty$ I conclude

$$\int_S \|A\|^2 g^2 \leq 2\int_S |\nabla g|^2 + \int_S |\mathrm{Ric}(v)| g^2. \tag{2.16}$$

for any Lipschitz function g with $|g| \leq 1$ and $g = 1$ outside a compact subset of S.

By (1.1), I have $\mathrm{Ric}(v) = O\left(\frac{1}{r^3}\right)$, so choosing $g \equiv 1$ on S and applying (2.10) with $\alpha = 3$ I conclude from (2.16) that $\int_S \|A\|^2 < \infty$.

(The formula (2.16) with $g \neq 1$ will be used later.)

Formula (2.14) implies that $|K| \leq |K_{12}| + \|A\|^2$.

By (1.1) I have $K_{12} = O\left(\frac{1}{r^3}\right)$, so (2.10) implies $\int_S |K_{12}| < \infty$.

Thus I have

$$\int_S |K| < \infty. \tag{2.17}$$

I now use the function $f = \varphi$ in inequality (2.15) and let $\sigma \to \infty$ as above to conclude $\int_S \left(R - K + \frac{1}{2}\|A\|^2\right) \leq 0$.

Since $R > 0$, and $R > 0$ outside a compact subset of S, I conclude $\int_S K > 0$. (2.18) ■

[2.1] The Cohn–Vossen inequality says that $\int_S K \leq 2\pi\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. Combining this with (2.18) I see immediately that S is homeomorphic to $\mathbb{R}^2$.

In light of (2.18), the proof of Step 3 will be finished if I can show $\int_S K \leq 0$.

Since this is a very important part of our proof of Theorem 1, I give two proofs of this inequality. The first proof is conceptually very clear, and has the advantage of being more general than the second. The first proof, however, uses a deep theorem of R. Finn [**?**] and A. Huber [**?**] concerning the Gauss–Bonnet theorem [dC92] on open Riemann surfaces, while the second uses no outside results and is special to our situation.

Claim. $\int_S K \leq 0$.

First Proof. I first note that inequality (2.17) and Remark 2.1 imply, by a result of A. Huber [**?**], that S is conformally equivalent to the complex plane. Thus there is a conformal diffeomorphism [SY79] $F : \mathbb{C} \longrightarrow S$.

Let D_i denote the disk of radius i in $\mathbb{C}$, and let C_i be the circle of radius i. For $i = 1, 2, \ldots$, let $L_i = \mathrm{Length}(F(C_i)),_i = \mathrm{Area}(F(D_i))$.

The simply connected case of the theorem of R. Finn [**?**] and A. Huber [**?**] says that

$$= \int_S K = 2\pi - \lim_{i\to\infty} \frac{L_i^2}{2A_i}. \text{ (2.19)}$$

=

Thus to show $\int_S K \leq 0$, it suffices to show $\lim_{i\to\infty} \frac{L_i^2}{4\pi A_i} \geq 1$.

Since S is properly imbedded in N with $S \cap (N \setminus N_k)$ compact, I see that $F(C_i)$ lies outside any compact subset of N_k for i sufficiently large. Thus, for large i, let ℓ_i be the Euclidean length of $F(C_i)$. Inequality (1.1) implies $\ell_i \leq (1 + o(1))L_i \qquad$ as $i \to \infty$.

For the immersed disk Σ_i of least Euclidean area with boundary curve $F(C_i)$, I have the well-known inequality whose proof can be found in [?], $A(\Sigma_i) \leq \frac{\ell_i^2}{4\pi}$, where $A(\cdot)$ denotes Euclidean area.

Let $\widetilde{\Sigma}_i$ be an oriented surface of least Euclidean area among all surfaces with boundary $F(C_i)$ regardless of topological type (see examples). Since $A(\widetilde{\Sigma}_i)(\Sigma_i)$, I have $A(\widetilde{\Sigma}_i) \leq \frac{\ell_i^2}{4\pi}$.

Because $F(C_i)$ lies outside any compact set for i sufficiently large, I can find a sequence $a_i \to \infty$ with $\widetilde{\Sigma}_i \cap B_{a_i}(0) \subset F(C_i)$.

Since from Step 2 I know that $F(C_i) \subset E_h = \{x \in \mathbb{R}^3 : |x^3| < h\}$, it follows that $\widetilde{\Sigma}_i \subset E_h$ by the convex hull property of minimal surfaces.

Since $\widetilde{\Sigma}_i$ does not retract onto its boundary circle, there is a point $x_0 \in \widetilde{\Sigma}_i \cap \{(0,0,x^3) : x^3 \in \mathbb{R}\}$.

A well-known inequality (see [?]) implies $A(\widetilde{\Sigma}_i \cap B_r(x_0)) \geq \pi r^2$.

Thus I clearly have

$$A(\widetilde{\Sigma}_i \cap B_{a_i}(0)) \geq (1+o(1))\pi a_i^2. \tag{2.23}$$

I wish to compare the ds^2-area of $\widetilde{\Sigma}_i$ with the Euclidean area, but I cannot do it directly since $\widetilde{\Sigma}_i \cap B_{z_0}(0)$ may be nonempty, and ds^2 is not defined on this part of $\widetilde{\Sigma}_i$.

I get around this problem by modifying $\widetilde{\Sigma}_i$ near 0. Let $a \in [z_0, z_0+1]$ be such that $\partial B_a(0)$ and $\widetilde{\Sigma}_i$ have transverse (or empty) intersection. I can then find a domain $\Omega_i \subset \partial B_a(0)$ so that $\partial\Omega_i = \widetilde{\Sigma}_i \cap \partial B_a(0)$.

I then define a new surface $\widehat{\Sigma}_i = (\widetilde{\Sigma}_i \setminus B_a(0)) \cup \Omega_i$.

Now (2.23) implies $A(\widetilde{\Sigma}_i) \to \infty$, so I let $\widehat{A}_i = A(\widehat{\Sigma}_i)$, and conclude $\widehat{A}_i \leq (1+o(1))A(\widetilde{\Sigma}_i)$, which combines with (2.22) to give

$$\widehat{A}_i \leq (1+o(1))\frac{\ell_i^2}{4\pi}. \tag{2.24}$$

By the area minimizing property of $\widetilde{\Sigma}_i$, I also have $A(\widetilde{\Sigma}_i) \leq \widehat{A}_i$, so $\widehat{A}_i \to \infty$.

By choosing a_i smaller if necessary, I may take

$$A(\widehat{\Sigma}_i \cap B_{a_i}(0)) \leq \sqrt{\widehat{A}_i}, \tag{2.24a}$$

and $A(\widehat{\Sigma}_i \setminus B_{a_i}(0)) \to \infty$.

This can be done because of (2.23) and (1.1). If the a_i remained bounded, say $a_i \leq C$ for all i, then by comparison as in the proof of (2.9) I would have $A(\widehat{\Sigma}_i \cap B_C(0)) \leq C'$, which would imply that $A(\widehat{\Sigma}_i \cap B_{a_i}(0))$ is a bounded sequence. Thus I must have $a_i \to \infty$.

It then follows by asymptotic flatness that $A(\widetilde{\Sigma}_i \setminus B_{a_i}(0)) \leq (1+o(1))\widehat{A}_i$.

Combining this with (2.24a) I have

$$A(\widetilde{\Sigma}_i) \leq (1+o(1))\widehat{A}_i \qquad \text{as } i \to \infty. \tag{2.25}$$

Using the area minimizing property of S and inequalities (2.21), (2.24), and (2.25) I have $A_i \leq A(\widetilde{\Sigma}_i) \leq (1+o(1))\widehat{A}_i \leq (1+o(1))\frac{\ell_i^2}{4\pi} \leq (1+o(1))\frac{L_i^2}{4\pi}$ as $i \to \infty$.

I thus conclude $\liminf_{i\to\infty} \frac{L_i^2}{4\pi A_i} \geq 1$, establishing (2.20). This completes the first proof of our claim. ■

Second Proof. I now give a proof of the claim in which I directly apply the Gauss–Bonnet theorem [dC92] with boundary, and estimate the boundary terms.

For any $x \in \mathbb{R}^3$, let $x' = (x^1, x^2, 0)$, $\qquad r' = |x'| = \sqrt{(x^1)^2 + (x^2)^2}$.

Consider the cylinder $P_\sigma = \{x \in \mathbb{R}^3 : r' < \sigma\}$.

For any $\sigma > z_0$ for which $\partial P_\sigma \cap S$ is transverse, I have by Remark 2.1 at least one circle in this intersection which is not homologous to zero in $\mathbb{R}^3 \setminus P_\sigma$. Choose one of these circles, and let D_σ be the connected component of this circle in $S \cap \big[(N \setminus N_k) \cup P_\sigma\big]$.

I claim that for σ sufficiently large, D_σ is a disk.

To see this, recall from the first proof that S is conformally equivalent to $\mathbb{C}$, so I have a conformal diffeomorphism [SY79] $F : \mathbb{C} \to S$.

Now $F^{-1}(D_\sigma)$ is a bounded, connected region in $\mathbb{C}$. By transversality, the function r' changes sign across each boundary component of $F^{-1}(D_\sigma)$. If $F^{-1}(D_\sigma)$ is not simply connected, then there is a bounded domain $\mathcal{O} \subset \mathbb{C} \setminus F^{-1}(D_\sigma)$.

Thus on $\partial F(\mathcal{O})$ I have $r' = \sigma$, and inside $F(\mathcal{O})$ at some points I have $r' > \sigma$.

Thus r' takes a maximum at some point of $F(\mathcal{O})$.

I claim that $(r')^2$ is a subharmonic function on S for r' sufficiently large, which will give a contradiction.

I calculate $\Delta x^i = \sum_{j=1}^2 e_j e_j x^i - \nabla_{e_j} e_j x^i$.

Now $e_j x^i = \left(e_j, \frac{\partial}{\partial x^i}\right) + O\left(\frac{1}{r}\right)$, so $\Delta x^i = \sum_{j=1}^2 \Big[g(e_j, e_j^T - x^i) + h_{jj}(v, \partial_{x^i}) - g(e_j, \nabla_{e_j} \partial_{x^i})\Big] + O\left(\frac{1}{r^2}\right) = O\left(\frac{1}{r^2}\right)$.

since $\sum_{j=1}^2 h_{jj} = 0$.

Thus I have $\Delta (r')^2 = 2 \sum_{i,j=1}^2 \left(e_i, \frac{\partial}{\partial x^j}\right)^2 + O\left(\frac{1}{r}\right)$.

Since both $\{e_1, e_2\}$ and $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ span 2-dimensional subspaces of $\mathbb{R}^3$, they must intersect in at least a line, so the norm of their projection is asymptotically bounded below, i.e. $\sum_{i,j=1}^2 \left(e_i, \frac{\partial}{\partial x^j}\right)^2 \geq 1 - O\left(\frac{1}{r}\right)$.

So I have $\Delta(r')^2 \geq 2 - O\left(\frac{1}{r}\right) \qquad$ as $r \to \infty$.

Thus for r sufficiently large, in particular for r' sufficiently large, I have $\Delta (r')^2 > 0$.

Thus it follows that D_σ is a disk for σ large.

I may choose the D_σ to be increasing, so that $D_\sigma \subset D_{\hat\sigma} \qquad$ when $\hat\sigma > \sigma$.

Since S is connected, the D_σ form an exhaustion of S, and I apply the Gauss–Bonnet theorem [dC92] on D_σ, so that $\int_{D_\sigma} K + \int_{\partial D_\sigma} k = 2\pi$, where k is the geodesic curvature of ∂D_σ relative to the inner normal.

Thus our proof will be complete if I can find a sequence $\sigma_i \to \infty$ so that $\int_{\partial D_{\sigma_i}} k \geq 2\pi - o(1) \qquad$ as $i \to \infty$.

In a neighborhood of ∂D_σ, I choose a frame e_1, e_2, e_3, where e_1 is the positively oriented unit tangent vector of ∂D_σ, e_2 is the inner normal to D_σ, and $e_3 = v$ is the unit normal of S in $\mathbb{R}^3$ relative to ds^2.

The geodesic curvature k is given by $k = (D_{e_1} e_1, e_2)$.

Since $r' = \sigma$ on ∂D_σ, I have $(e_1, \nabla r') = 0 \qquad$ on ∂D_σ.

Differentiating this with respect to e_1 gives $(D_{e_1} e_1, \nabla r') + (e_1, D_{e_1} \nabla r') = 0$.

Now (1.1) implies that $\nabla r' = \frac{x'}{\sigma} + O\left(\frac{1}{\sigma}\right)$, and $D_{e_1} \nabla r' = \frac{e_1}{\sigma} + O\left(\frac{1}{\sigma^2}\right)$, so I have $(D_{e_1} e_1, \frac{x'}{\sigma}) + \frac{1}{\sigma} - \frac{1}{\sigma}\left(e_1, \frac{\partial}{\partial x^3}\right)^2 = O\left(\frac{1}{\sigma}\right) \|D_{e_1} e_1\| + O\left(\frac{1}{\sigma^2}\right)$.

Since $D_{e_1} e_1 = k e_2 - h_{11} v$, this gives

$$k\left(e_2, \frac{x'}{\sigma}\right) + \frac{1}{\sigma} - \frac{h_{11}}{\sigma}(v, x') = O\left(\frac{1}{\sigma}\right) \|D_{e_1} e_1\| + O\left(\frac{1}{\sigma^2}\right). \tag{2.27}$$

Suppose $\delta \in [\sigma, 2\sigma]$.

I now apply the divergence theorem for the vector field $\frac{\partial}{\partial x^3}$ on the volume enclosed by $D_{2\sigma} \setminus D_\sigma, \quad \partial P_{2\sigma}, \quad \partial P_\sigma$, and the plane annulus $Q_\sigma = \{x : x^3 = -h,\ \sigma \leq r' \leq 2\sigma\}$, where h is a bound on $|x^3|$ for $x \in S \cap N_k$ (see Step 2).

By (1.1), $\operatorname{div}\left(\frac{\partial}{\partial x^3}\right) = O\left(\frac{1}{r^2}\right)$, so I have $\int_{D_{2\sigma}\setminus D_\sigma}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right) = O(\sigma)$, where I have used the fact that $\left(\frac{\partial}{\partial x^3}, n\right) = O\left(\frac{1}{\sigma}\right)$ where n is the unit normal of $\partial P_{2\sigma}$ and ∂P_σ.

Applying the area minimizing property of S on $D_{2\sigma}\setminus D_\sigma$ as compared with the union of Q_σ and the part of $\partial P_{2\sigma}\cup\partial P_\sigma$ between S and Q_σ, I have $A(D_{2\sigma}\setminus D_\sigma) \le A(Q_\sigma) + O(\sigma)$.(2.28)

Combined with the above inequality this gives $\int_{D_{2\sigma}\setminus D_\sigma}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right) = O(\sigma)$.

The coarea formula (see [16, p. 258]) gives $\int_\sigma^{2\sigma}\left(\int_{D_{2\sigma}\cap\partial P_t}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right)ds\right)dt = \int_{D_{2\sigma}\cap(P_{2\sigma}\setminus P_\sigma)}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right)\|\nabla r$

Since $D_{2\sigma}\cap(P_{2\sigma}\setminus P_\sigma)\subset D_{2\sigma}\setminus D_\sigma$, I can combine these inequalities to obtain

$$\int_\sigma^{2\sigma}\int_{D_{2\sigma}\cap\partial P_t}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right)ds\,dt \le O(\sigma). \tag{2.29}$$

Again using the coarea formula I have $\int_\sigma^{2\sigma} L(D_{2\sigma}\cap\partial P_t)\,dt = \int_{D_{2\sigma}\cap(P_{2\sigma}\setminus P_\sigma)}\|\nabla r'\|$.

Combined with (2.28) this gives $\int_\sigma^{2\sigma} L(D_{2\sigma}\cap\partial P_t)\,dt = O(\sigma^2)$.

I must now bound the second fundamental form of S on ∂D_σ.

To do this, I apply inequality (2.16) with the following choice of g:

$$g(x) = \begin{cases} 0, & x\in S\cap\left[(N\setminus N_k)\cup P_{\sqrt{\sigma}}\right],\\ \dfrac{\log(r'/\sqrt{\sigma})}{\log(1/\sqrt{\sigma})}, & x\in S\cap(P_\sigma\setminus P_{\sqrt{\sigma}}),\\ 1, & x\in S\cap\left[(N\setminus N_k)\cup P_\sigma\right].\end{cases}$$

This implies $\int_{S\setminus[(N\setminus N_k)\cup P_\sigma]}\|A\|^2 \le \frac{2}{(\log\sqrt{\sigma})^2}\int_{S\cap(P_\sigma\setminus P_{\sqrt{\sigma}})}\frac{\|\nabla r'\|^2}{(r')^2} + \int_{S\setminus[(N\setminus N_k)\cup P_{\sqrt{\sigma}}]}|\operatorname{Ric}(v)|$.

This implies by (1.1) $\int_{S\setminus[(N\setminus N_k)\cup P_\sigma]}\|A\|^2 \le \frac{2C_1}{(\log\sqrt{\sigma})^2}\int_{S\cap(P_\sigma\setminus P_{\sqrt{\sigma}})}\frac{1}{(r')^2} + \int_{S\setminus[(N\setminus N_k)\cup P_{\sqrt{\sigma}}]}O\left(\frac{1}{r^3}\right)$.

Similar reasoning as that used in deriving (2.10) and (2.11), using r' in place of r and the fact that x^3 is bounded on $S\cap N_k$, implies $\int_{S\cap(P_\sigma\setminus P_{\sqrt{\sigma}})}\frac{1}{(r')^2} = O(\log\sqrt{\sigma})$, and $\int_{S\setminus[(N\setminus N_k)\cup P_{\sqrt{\sigma}}]}O\left(\frac{1}{r^3}\right) = O\left(\frac{1}{\sqrt{\sigma}}\right)$.

It therefore follows that $\int_{D_{2\sigma}\cap(P_{2\sigma}\setminus P_\sigma)}\|A\|^2 = o(1)$.

The coarea formula gives $\int_\sigma^{2\sigma}\left(\int_{D_{2\sigma}\cap\partial P_t}\|A\|^2\right)dt = \int_{D_{2\sigma}\cap(P_{2\sigma}\setminus P_\sigma)}\|A\|^2\|\nabla r'\|$.

This then implies $\int_\sigma^{2\sigma}\left(\int_{D_{2\sigma}\cap\partial P_t}\|A\|^2\right)dt = o(1)$.

Now (2.29), (2.30), and (2.31) imply that there exists $\hat\sigma\in[\sigma,2\sigma]$ satisfying $\int_{D_{2\sigma}\cap\partial P_{\hat\sigma}}\left(1-\left(v,\frac{\partial}{\partial x^3}\right)\right) = O(1)$, $L(D_{2\sigma}\cap\partial P_{\hat\sigma}) = O(\sigma)$, and $\int_{D_{2\sigma}\cap\partial P_{\hat\sigma}}\|A\|^2 = o\left(\frac{1}{\sigma}\right)$ as $\sigma\to\infty$.

Applying the Schwarz inequality and the condition on boundary length I have $\left(\int_{D_{2\sigma}\cap\partial P_{\hat\sigma}}\|A\|\right)^2 \le L(D_{2\sigma}\cap\partial P_{\hat\sigma})\,o\left(\frac{1}{\sigma}\right) = o(1)$.

Since $\partial D_{\hat\sigma}$ is one component of $D_{2\sigma}\cap\partial P_{\hat\sigma}$, I have shown $L(\partial D_{\hat\sigma}) = O(\sigma)$, and $\int_{\partial D_{\hat\sigma}}\|A\| = o(1)$.

I use (2.32) to estimate the terms in (2.27). Note that (2.27) and (2.32) imply $k\left(e_2,\frac{\partial x^3}{\partial x}\right) = O(1) + O\left(\frac{1}{\sigma}\right)\int_{\partial D_\sigma}\|D_{e_1}e_1\| + \cdots$

To bound the last term, I note that by (1.1) I have $(e_1,x')^2+(e_2,x')^2 = 1-(v,x')^2+O\left(\frac{1}{\sigma}\right)$, which implies by (2.32) that $\int_{\partial D_\sigma}\left(e_1,x'\right)^2+\left(e_2,x'\right)^2 = O(1)$.

I now give a pointwise lower bound on (e_2,x'). In fact I show that $\sup_{\partial D_\sigma}\left(1+(x',e_2)\right) = O(1)$.

I first note that by (1.1) $1-(v,e_2) = (v,\partial_{x^3}) + O\left(\frac{1}{\sigma}\right) = \frac{1}{2}\left(1-(v,x')\right) + O\left(\frac{1}{\sigma}\right)$.

Then applying (2.32) gives $\int_{\partial D_\sigma}\left(1-(x',e_2)\right) = O(1)$.

Now since e_2 is the inner normal to D_σ, and $\nabla r' = -\frac{x'}{\sigma}+O\left(\frac{1}{\sigma}\right)$ is the outer normal, I have $|(x',e_2)|\le O\left(\frac{1}{\sigma}\right)$, so $1-(v,e_2)^2 = \left(1-(v,e_2)\right)\left(1+(v,e_2)\right) \ge (1-O(1/\sigma))\left(1+(v,e_2)\right)+O(1/\sigma)$.

Combining with (2.36) then gives $1 + (x', e_2) = O(1)$.

Since ∂D_σ is not homologous to zero in $\mathbb{R}^2 \setminus P_\sigma$, its projection onto the x^1x^2-plane must be the circle of radius σ centered at the origin. I therefore have $L(\partial D_\sigma) \geq 2\pi\sigma - O(1)$.

Now (2.37) and (2.38) together imply that there is a point $x_0 \in \partial D_\sigma$ with $1+(x', e_2)(x_0) = O\big(\frac{1}{\sigma}\big)$.

Differentiating $1 + (x', e_2)$ along ∂D_σ gives $e_1\big(1 + (x', e_2)\big) = (D_{e_1}e_2, x') + (e_2, e_1) + O\left(\frac{1}{\sigma}^2\right)$ $= -k(e_1, x') + h_{12}(v, x') + O\left(\frac{1}{\sigma}^2\right) \leq \frac{1}{\sigma}\|D_{e_1}e_1\| + \|A\| + O\left(\frac{1}{\sigma}^2\right)$.

Applying (2.32) and (2.34) I see $e_1\big(1 + (x', e_2)\big) = O(1)$.

Now I write for any $x \in \partial D_\sigma$, $1 + (x', e_2)(x) = 1 + (x', e_2)(x_0) + \int_{x_0}^{x} e_1\big(1 + (x', e_2)\big)$.

Thus combining this with (2.39) and (2.40) I have established (2.35).

Now (2.33), (2.34), and (2.35) together imply $\int_{\partial D_\sigma} k = O(1) + O\big(\frac{1}{\sigma}\big) \int_{\partial D_\sigma} \|D_{e_1}e_1\|$.

Since $D_{e_1}e_1 = ke_2 - h_{11}v$, I have $\|D_{e_1}e_1\| \leq |k| + \|A\|$, so integrating and applying (2.32) I have $\int_{\partial D_\sigma} \|D_{e_1}e_1\| \leq \int_{\partial D_\sigma} |k| + O(1)$.

Combining this with (2.41) gives $\int_{\partial D_\sigma} k = O(1)$.

I may now rewrite (2.27) using (2.32), (2.34), and (2.42): $k \geq \frac{1}{\sigma}\big(1 + (x', e_2)\big) + \frac{1}{\sigma}L(\partial D_\sigma) - o(1)$.

Applying (2.35), (2.38), and (2.42) then gives $\int_{\partial D_\sigma} k \geq 2\pi - o(\sigma)$.

Since this holds for σ arbitrarily large, I can choose a sequence $\sigma_i \to \infty$ for which (2.26) holds. This completes the second proof of our claim.

This finishes the proof of Theorem 1.

■

Proof. In this section I prove Theorem 2, which states that if for some end N_k condition (1.2) holds, the total mass of M_k is zero, and $R \geq 0$, then ds^2 is flat. I first note that by removing the other ends outside a convex ball, I may assume that N has only one end, so that $N \setminus N_k$ is compact.

I will need the following Sobolev inequality for functions with compact support on N.

Lemma 3.1. There is a constant $c_1 > 0$ depending on N and the constants k_1, k_2, k_3 of (1.1) such that for any function ϕ with compact support on N, I have the inequality $\left(\int_N |\phi|^6\right)^{1/3} \leq c_1 \int_N |\nabla\phi|^2$. Note that I do not require $\phi = 0$ on ∂N.

Proof. I prove the inequality by contradiction. If it were not true, I could find a sequence of functions f_i with compact support such that $\int_N f_i^6 = 1, \qquad \int_N |\nabla f_i|^2 < \frac{1}{i}$.

Since N_k is identified with $\mathbb{R}^3 \setminus B_{r_0}(0)$ and ds^2 is uniformly equivalent to the Euclidean metric, I obtain the Euclidean Sobolev inequality.

Thus by (3.1) I have $f_i \to 0$ in L^6 on N_k. If I choose a precompact coordinate neighborhood $\Omega \subset N$, for any C^1 function g defined on Ω I have the Euclidean inequality of the same form: $\int_\Omega |g\phi|^6 \leq C \left(\int_\Omega |\nabla(g\phi)|^2\right)^3$.

Applying this inequality and (3.1) to the functions f_ig, I obtain a sequence f_i such that $f_i \to 0$ in L^6 locally. Since $\int_N f_i^6 = 1$, this yields a contradiction. This proves Lemma 3.1.

I now study equations of the form $\Delta v - fv = h$ on N, where f, h satisfy $|f| \leq k_7(1 + r^5)^{-1}$, $|\nabla f| \leq k_8(1 + r^5)^{-1}$, $|h| \leq k_7(1 + r^5)^{-1}$, $|\nabla h| \leq k_8(1 + r^5)^{-1}$.

Let f_+, f_- be the positive and negative parts of f, so that $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Lemma 3.2. Suppose (1.1) holds and N_k has zero total mass. There exists a number $\varepsilon_0 > 0$ depending only on N and k_1, k_2, k_3 such that if $\left(\int_N (f_-)^{3/2}\right)^{2/3} \leq \varepsilon_0$, then (3.2) has a unique solution v defined on N satisfying $v = O(1/r)$ as $r \to \infty$ and $\frac{\partial v}{\partial n} = 0$ on ∂N, where n is the outward unit normal vector to ∂N. Moreover, the solution satisfies $v = \frac{A}{r} + O(r^{-2})$, $|\nabla v| \leq k_9(1 + r^2)^{-1}$, $|\nabla^2 v| \leq k_{10}(1 + r^3)^{-1}$, on N_k, where $A = \frac{1}{4\pi}\int_N (fv + h)$, and the constants depend only on k_5, k_6, k_7.

Proof. Throughout the proof I use $c_1, c_2, c_3, \ldots$ to denote constants depending only on k_1, k_2, k_3.

To prove existence of v, I solve the boundary value problem on exhaustion domains: $\Delta v_a - f v_a = h \quad \text{on } N \cap B_a(0),\ v_a = 0 \quad \text{on } \partial B_a(0), \qquad \frac{\partial v_a}{\partial n} = 0 \quad \text{on } \partial N.$

If v_a satisfies (3.4), I multiply by v_a and integrate by parts to obtain $\int |\nabla v_a|^2 = -\int f v_a^2 - \int h v_a$.

Using Hölder's inequality, $\int |\nabla v_a|^2 \leq \left(\int |f|^{3/2}\right)^{2/3} \left(\int v_a^6\right)^{1/3} + \left(\int |h|^{6/5}\right)^{5/6} \left(\int v_a^6\right)^{1/6}$.

If $h = 0$, Lemma 3.1 gives control of v_a. Choosing ε_0 sufficiently small ensures coercivity, so standard elliptic theory yields a unique smooth solution v_a.

Applying (3.3), Lemma 3.1, and the smallness assumption, I obtain a uniform bound $\int |v_a|^6 \leq c_3$.

Standard elliptic estimates imply that $\{v_a\}$ is equicontinuous in C^2 on compact subsets of N. Hence a subsequence converges to a solution v of (3.2) satisfying $\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial N, \quad v = O(1/r)$, and $\int_N |v|^6 \leq C_3, \qquad \|v\|_{L^\infty} \leq C_4$.

This completes the proof of Lemma 3.2.

To analyze the asymptotic behavior of v, I derive a potential theoretic expression for v. For $x, y \in N_k$, let $\Phi_x(y) = \frac{(y^i - x^i)(y^i - x^i)}{|y-x|}$.

By direct calculation I have $\Delta_y \Phi_x(y) = -4\pi\delta_x(y) + \psi_x(y)$, where $\delta_x(y)$ is a point mass at x, and by (1.1) $\psi_x(y)$ satisfies $|\psi_x(y)| \leq c_5 |x-y|^{-2}|x|^{-3} \quad \text{for } y \in B_1(x)$.

Again by (1.1) I see that for $y \in B_1(x)$ I have

$$|\Phi_x(y)| \leq c_6 \left((1+|y|)|x-y|^{-3} + |x|^2|x-y|^{-3} + (1+|y|)^3|x-y|^{-2}\right). \tag{6.1}$$

Also I have

$$c_7^{-1}|x-y| \leq G_x(y) \leq c_7|x-y|, \quad c_8^{-1} \leq |\nabla_y G_x(y)| \leq c_8, \quad \lim_{|x|\to\infty} |x|\,|G_x(y)|^{-1} = 1. \tag{6.2}$$

Multiplying (3.2) by $[G_x(y)]^{-1}$, integrating by parts twice on the set $D_\delta(x) = \{y \in N_k : G_x(y) < \delta\}$, where $\delta \in (a/2, a)$, and applying (3.6), I obtain

$$\begin{aligned} 4\pi v(x) = {} & \int_{D_\delta(x)} G_x(y) v(y)\, d\mu(y) - \int_{D_\delta(x)} (fv+h)(y)\,[G_x(y)]^{-1}\, d\mu(y) \\ & + \int_{\partial D_\delta(x)} v\,\partial_n [G_x(y)]^{-1}\, dA(y) - \int_{\partial D_\delta(x)} \partial_n v\,[G_x(y)]^{-1}\, dA(y). \end{aligned} \tag{6.3}$$

Applying Stokes' theorem and using (3.2), I rewrite boundary terms as $\int_{\partial D_\delta(x)} \partial_n v + \int_{D_\delta(x)} Av = \int_{\partial D_\delta(x)} \partial_n v + \int_{D_\delta(x)} (fv+h)$.

From (3.3) and (3.5) I obtain $\int_{\partial D_\delta(x)} (fv+h)\,[G_x(y)]^{-1}\, dA(y) \leq c_9$.

Since $\text{Area}\{\Phi_x(y) = \delta\} \leq C_{12}\delta^2$, I may choose $\delta \in (a/2, a)$ so that

$$\int_{\{\Phi_x(y)=\delta\}} v \leq C_{15}\, a^{-1}. \tag{6.4}$$

Thus I obtain $\int_{\{\Phi_x(y)=\delta\}} v\,[G_x(y)]^{-1} \leq c_6$.

Letting $a \to \infty$ and applying (3.10)–(3.11) in (3.9), I obtain

$$\begin{aligned} 4\pi v(x) = {} & \int_{N_k} \Phi_x(y) v(y)\, d\mu(y) - \int_{N_k} (fv+h)(y)[G_x(y)]^{-1}\, d\mu(y) \\ & - \int_{\partial N_k} \partial_n v\,[G_x(y)]^{-1}\, dA(y) + \int_{\partial N_k} v\,\partial_n [G_x(y)]^{-1}\, dA(y). \end{aligned} \tag{6.5}$$

I may write
$v(x) = q(x) + \mathcal{F}(x)$,

where

$$4\pi q(x) = \int_{N_k} \Phi_x(y)\, v(y)\, d\mu(y) - \int_{N_k} (fv+h)(y)\big[G_x(y)^{-1} - |x|^{-1}\big]\, d\mu(y)$$

$$- \int_{\partial B_\rho(0)} v(y)\, \partial_n G_x(y)^{-1}\, dA(y) + \int_{\partial B_\rho(0)} v(y)\, \partial_n G_x(y)\, dA(y). \tag{3.17}$$

I see directly that

$\big|\Delta_y G_x(y)^{-1} - |x|^{-1}\big| \le C_{26}\, |x|^{-1}|x-y|^{-1}$, which combined with (3.17), (3.3), and (3.8) shows that

$$|\mathcal{F}(x)| \le C_{27}(1+|x|)^{-1}, \quad x \in N_k. \tag{3.18}$$

To estimate the derivatives of q, I record the following Schauder estimate.

Let L be an elliptic operator on the unit ball in $\mathbb{R}^3$ of the form $Lu = \sum_{i,j=1}^3 a_{ij}(\xi)\, \partial_{ij} u + \sum_{j=1}^3 b_j(\xi)\, \partial_j u + c(\xi) u$, where $\xi = (\xi_1, \xi_2, \xi_3)$ are Cartesian coordinates. For a function φ on an open set Ω and $0 < \alpha < 1$, define the norms $\|\varphi\|_{0,\Omega} = \sup_{\xi\in\Omega} |\varphi(\xi)|$, $[\varphi]_{\alpha,\Omega} = \sup_{\xi\neq\eta} \frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|^\alpha}$, $\|\varphi\|_{1,\alpha,\Omega} = \|\varphi\|_{0,\Omega} + \|\nabla\varphi\|_{0,\Omega} + [\nabla\varphi]_{\alpha,\Omega}$.

Let $B_r = \{\xi : |\xi| < r\}$. Suppose there exists $A > 0$ such that

$$\|a_{ij}\|_{0,\alpha,B_1} + \|b_j\|_{0,\alpha,B_1} + \|c\|_{0,\alpha,B_1} \le A, \tag{3.19}$$

and $A^{-1}|\zeta|^2 \le \sum_{i,j=1}^3 a_{ij}(\xi)\zeta_i\zeta_j, \quad \forall \zeta \in \mathbb{R}^3,\ \xi \in B_1$.

Then for any $C^{2,\alpha}$ function u on B_1, $\|u\|_{2,\alpha,B_{1/2}} \le C\big(\|Lu\|_{0,\alpha,B_1} + \|u\|_{0,B_1}\big)$, where C depends only on α and A.

Now fix a point $x \in N_k$, and assume that $|x| = \frac{1}{2}|x_0| > \rho_0$. I set $\xi = \frac{1}{|x|}(y-x)$, where y is the asymptotic coordinate on N_k. Define $u(\xi) = q(x), \quad a_{ij}(\xi) = g_{ij}(y), \quad b_j(\xi) = g_{ij}(y)\Gamma^j_{ij}(y), \quad c(\xi) = -|x|^2 f(y)$.

Then $Lu(\xi) = |x|^2\big(\Delta q(x) - fq(x)\big)$.

From (3.2) and the definition of q, $\Delta q - fq = f|x|^{-1} + h - \Delta|x|^{-1}$.

Using (1.1), I obtain $|Lu(\xi)| \le C_{30}\big(|x|^{-3} + |x|^{-2}\big)$.

Thus (3.19) is satisfied uniformly in x, and hence $\|u\|_{2,\alpha,B_{1/2}} \le C_{31}\big(|x|^{-3} + |x|^{-2} + \|u\|_{0,B_1}\big)$.

By (3.18), this yields for any $0 < \alpha < 1$, $|\nabla q(x)| \le C_{32}|x|^{-3}, \qquad |D^2 q(x)| \le C_{32}|x|^{-4}$.

This establishes the required decay estimates for q. The expression for the constant A follows by integrating (3.2) over $(N \setminus N_k) \cup (N_k \cap B_\rho(0))$ and using the boundary condition $\frac{\partial v}{\partial n} = 0 \quad$ on ∂N, and letting $\rho \to \infty$.

To prove uniqueness, suppose g is another solution of (3.2) satisfying $g = O(1/r)$ and $\partial_n g = 0$ on ∂N. Let $u = v - g$, so that $\Delta u - fu = 0, \quad u = O(1/r), \quad \partial_n u = 0$ on ∂N.(3.21)

I show $u \equiv 0$. Let $\delta > 0$ and define $E_\delta = \{x \in N : u(x) > \delta\}$. Since $u \to 0$ at infinity, E_δ is compact. Multiplying (3.21) by u and integrating over E_δ, $\int_{E_\delta} u\Delta u = \int_{E_\delta} fu^2$.

Integrating by parts and applying (3.21), $\int_{E_\delta} |\nabla u|^2 \le \int_{E_\delta} fu^2 \le \Big(\int_{E_\delta} f^{3/2}\Big)^{2/3} \Big(\int_{E_\delta} u^6\Big)^{1/3}$.

Applying Lemma 3.1 and letting $\delta \to 0$, I conclude $u \equiv 0$.

To prove that φ is everywhere positive on N, I let $E = \{x \in N : \varphi(x) < 0\}$. Since φ is asymptotic to one, I see that E is compact. If E is nonempty, I multiply (3.22) by φ and integrate by parts on E, using $\frac{\partial\varphi}{\partial n} = 0$ on ∂N, to obtain $\int_E |D\varphi|^2 = -\int_E \frac{1}{6} R\varphi^2$.

Applying Lemma 3.1 to this inequality gives $\big(\int_E \varphi^6\big)^{1/3} \le C_1 \big(\int_E |D\varphi|^2\big)^{1/3}$, which is a contradiction since $e_0 < \frac{1}{3}C_1^{-1}$. Hence $\varphi > 0$ on N. The Hopf maximum principle then implies $\varphi > 0$ on ∂N as well.

To compute the mean curvature of ∂N under the conformal metric $\tilde{g} = \varphi^4 g$, I use the transformation law $\tilde{H} = \varphi^{-2}\left(H + 4\frac{\partial\varphi}{\partial n}\right)$, which implies $\tilde{H} > 0$ since $\frac{\partial\varphi}{\partial n} = 0$ and $\varphi > 0$.

Thus $\tilde{g}$ is asymptotically flat, scalar flat, and the mass formula follows from Lemma 3.2.

A special case yields:

If $M = 0$, $R \geq 0$, and $R \not\equiv 0$, then there exists a metric conformally equivalent to g which is asymptotically flat, scalar flat, and for which N_k has negative total mass.

Using Theorem 1 and this corollary, I conclude that if $M = 0$ and $R \geq 0$, then $R = 0$ on N.

I define a one-parameter family of metrics $g_{ij}(t) = g_{ij} + tS_{ij}$, where S_{ij} is the Ricci tensor. Then $g(0) = g$, and for small t, the metric remains asymptotically flat with ∂N having positive mean curvature.

Let $R(t)$ be the scalar curvature of $g(t)$. Then $\frac{d}{dt}R(t)\bigg|_{t=0} = -\Delta R + 6\,\mathrm{Ric} - |\mathrm{Ric}|^2$. Since $R = 0$, this simplifies to $R'(0) = -|\mathrm{Ric}|^2$.

Using Lemma 3.3, I obtain a conformal factor φ_t such that $\varphi_t^4 g(t)$ is scalar flat. Its mass is $M(t) = -\frac{1}{32\pi}\int_N R(t)\varphi_t\,d\mu_t$.

I show that $M'(0)$ exists and can be computed by differentiating under the integral sign. Standard elliptic estimates give compactness of φ_t, and hence $\varphi_t \to \varphi_0$ in $C^{2,\alpha}$ on compact sets.

Therefore, $M'(0) = -\frac{1}{32\pi}\int_N |\mathrm{Ric}|^2\,d\mu$.

Since $M'(0) \geq 0$, if $\mathrm{Ric} \not\equiv 0$, then for small negative t I would obtain a metric with negative mass, contradicting Theorem 1. Hence $\mathrm{Ric} \equiv 0$, and in dimension three this implies that g is flat.

This completes the proof of Theorem 2.

■

7. Discussion/interpretation and Applications to theoretical physics, i.e., String theory/black holes

the positive mass theorem shows that space and everything within it would become unstable without a lower bound and would take on negative values. The results from the positive mass theorem are essential for defining quasi-local mass and other physical properties. Surprising implications What does the theorem mean What changed limitations

8. Further

8.1. Future work Questions. The Schoen–Yau Positive Mass Theorem demonstrates that nonnegative scalar curvature imposes strong global restrictions on asymptotically flat manifolds, ensuring that the ADM mass is nonnegative and identifying Euclidean space as the only zero-mass example. This theorem underscores a profound rigidity linking local curvature conditions to the overall geometry. Future research may explore the extent to which these rigidity properties extend beyond the Riemannian framework. Specifically, the Lorentzian variant of the positive mass theorem and its relationship with the dominant energy condition remain key topics in mathematical relativity. Other important avenues include examining cases where the scalar curvature is nonnegative but the ADM mass is small yet nonzero. Quantitative forms of the theorem that estimate a manifold's proximity to Euclidean space based on its mass are actively studied. Extensions to higher dimensions and to metrics with lower regularity also pose challenges, especially since Schoen and Yau's proof covers dimensions $3 \leq n \leq 7$. Moving beyond this range requires overcoming analytical difficulties linked to minimal hypersurfaces and regularity theory. Moreover, the proof techniques hint at deeper links between geometric analysis and general relativity. Methods involving minimal surfaces [SY79], curvature comparison, and spinorial approaches (as in Witten's proof) [Wit81] continue to shape modern research, with applications to black hole geometry and quasi-local mass. Overall, the connection between curvature, topology, and global mass remains a vibrant and active field of study.

8.2. **Future Questions.** The positive mass theorem is now a fundamental link between differential geometry and general relativity. While it confirms that physically realistic isolated gravitational systems have nonnegative total mass, many key questions still exist. Research is ongoing into extending the theorem to more general geometric contexts, such as manifolds with lower regularity, singularities, or different asymptotic behaviors. Additionally, researchers are exploring the theorem's stability: when the ADM mass is nearly zero, how closely must the manifold resemble Euclidean space? Recent efforts in quantitative and geometric stability aim to answer this.

Despite decades of progress, many problems remain open. Extending positive mass results to increasingly general settings, understanding the geometry of near-equality cases, and exploring connections with quantum gravity continue to be active areas of research. These developments illustrate how a theorem originally motivated by Einstein's theory of general relativity has become a powerful bridge between geometry, analysis, and modern theoretical physics.

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