The Probabilistic Method

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Euler Circle

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Finding Things...Non-Constructively?

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Suppose we want show that some combinatorial object exists. A lot of the time, it is *really* hard to find a construction for our desired object. Can we show that this object exists without necessarily building it?

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Suppose we want show that some combinatorial object exists. A lot of the time, it is *really* hard to find a construction for our desired object. Can we show that this object exists without necessarily building it?

Answer: Yes! We can use the following:

- Create a random object according to some procedure
- If the random construction has a positive probability of being the object we want, then....

Our object must necessarily exist! This approach to proving existence is known as **The Probabilistic Method.**

Applying the Probabilistic Method

Let's consider a simple example:

Definition

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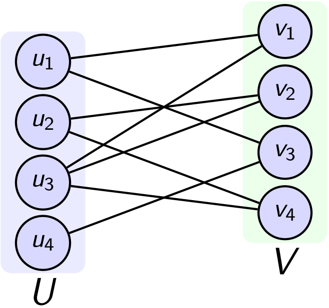
For instance, take the following graph, where we may partition the vertices into sets U and V.



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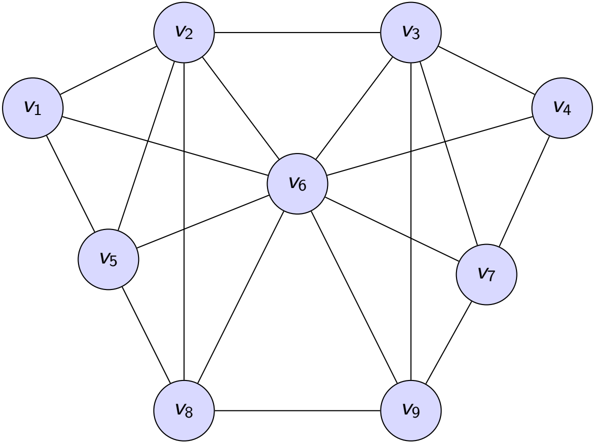
We build a random construction:

- We color every vertex black or white independently.
- Then, we keep only the edges joining different colors.

Claim: This resulting graph is bipartite. It remains to calculate the number of edges in the resulting graph.

Applying The Probabilistic Method



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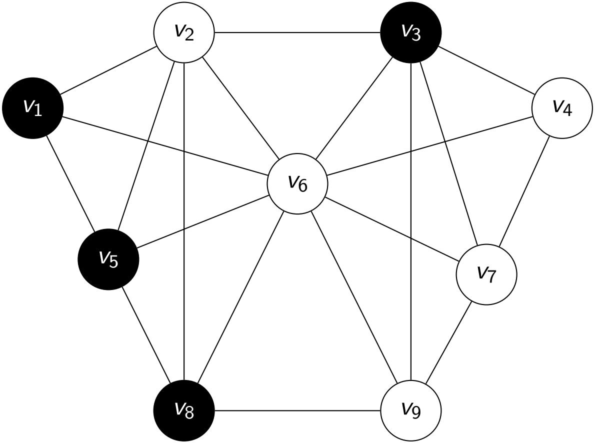

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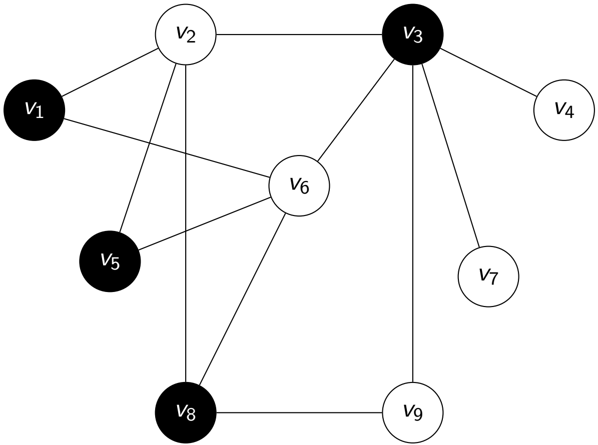

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Although applications become increasingly complex, they generally retain this flavor. We:

1. Formulate a random construction
2. Make particular observations about the construction
3. Leverage some probabilistic tool to show it is our desired object with positive probability.

Stronger Techniques in the Probabilistic Method

We can improve upon basic probabilistic arguments using a number of techniques. Among them,

- **The First Moment Method**
- **Alterations**
- **The Lovasz Local Lemma**
- The Second Moment Method
- Containers
- Martingale Concentration Inequalities
- The Rodl Nibble
- Dependent Random Choice

The First Moment Method

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Proposition

For a discrete non-negative random variable X, if $\mathbb{E}[X] < 1$, then $X = 0$ with positive probability.

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Proposition

For a discrete non-negative random variable X, if $\mathbb{E}[X] < 1$, then $X = 0$ with positive probability.

In practice, we apply the First Moment Method to perform an argument analogous to complementary counting. That is, we observe the number of "bad" characteristics of an object and let X be the number of bad characteristics.

Alteration

On their own, basic probabilistic arguments are usually not that strong; for discrete random X, $\mathbb{E}[X] < 1$ is effectively analogous to saying that the average case has very few (often zero) bad characteristics.

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Rather than pursuing strictly random construction, it is natural to apply the **Alteration Method**. Alteration (or Deletion) involves the construction of a random object which is then manually "fixed" to avoid any undesirable elements. We typically:

- Devise an optimistic random construction
- Go in and fix the imperfections

Let's examine an example where we apply alteration.

Definition

Let A be a matrix. Let S_x be a subset of column indices, and S_y be a subset of the row indices. The **Submatrix** B consists of all entries $a_{i,j}$, where $i \in S_y$, and $j \in S_x$

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Using alteration, we can show that:

Theorem

For every positive integers $n \geq k \geq 2$, there exists an $n \times n$ matrix with $\{0, 1\}$ entries, with at least $n^{\frac{2k}{k+1}}\left(1 - \frac{1}{k!^2}\right)$ 1's, such that there is no $k \times k$ submatrix consisting of all $1's$

Applying Alterations

Proof.

Let $p \in [0, 1]$ to be decided later. Define Q as a random n by n matrix with $\{0, 1\}$ entries formed by setting $a_{i,j} = 1$ with probability p independently for all $1 \leq i, j \leq n$.

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Consider the following procedure: Create a new matrix Q' by "deleting" (changing to 0) one entry in every $k \times k$ sub-matrix of Q that contains all ones. Then Q' has no $k \times k$ sub-matrices with all ones.

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$$\begin{aligned} \mathbb{E}[\text{1's in } Q\text{'}] &= \mathbb{E}[1's \text{ in } Q] - \mathbb{E}[k \times k \text{ submatrix with all 1's}] \\ &= n^2 p - \binom{n}{k}^2 p^{-k^2} \end{aligned}$$

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Optimizing p and simplifying gives $\mathbb{E}[\text{1's in } Q'] = n^{\frac{2k}{k+1}}\left(1 - \frac{1}{k!^2}\right)$. ■

The Lovasz Local Lemma

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Lemma

Lovasz Local Lemma; Symmetric: *Let $A_1, ...A_n$ be events such that no event has probability larger than p of occurring. Suppose that A_i is mutually independent from a set of size at least $(n-d) < n$ events for all i. Then*

$$e \cdot p \cdot d \leq 1$$

implies that with positive probability, no events A_i occur.

Applying the LLL

The Symmetric LLL produces the following result:

Theorem

Let $G = (V, E)$ be a graph with maximum degree Δ and let $V = V_1 \cup V_2 ... \cup V_r$ be a partition of the vertices, where each $|V_i| = \lceil 2e\Delta \rceil$. Then there exists a set of vertices M containing one vertex in each V_i, such that there are no edges in M connecting two vertices in the set.

Applying the LLL

Randomly select a vertex $v_i \in V_i$ for each i. Let A_i be the probability that edge $i \in E$ has both endpoints in $M = \{v_1, ...v_r\}$. We may show

$$\mathbb{P}(A_i) \leq (\frac{r}{r \times 2e\Delta})^2 = \frac{1}{(2e\Delta)^2}$$

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Each A_i is dependent on some other A_j if there exists a V_k such that i and j are incident to(connect) vertices in V_k, So $d = 2(2e\Delta)(\Delta)$. Hence

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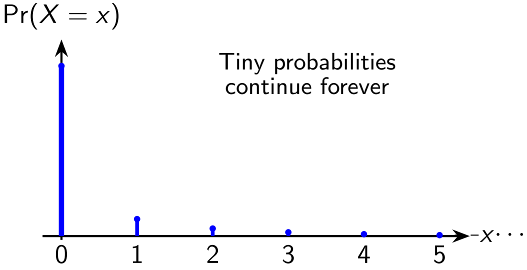
We finish by the Symmetric LLL.

The Second Moment Method

However, expected value alone is often not sufficient to make strong conclusions about a bound. Specifically, considering expected value alone reveals nothing about the *structure* of the distribution for a variable X.
If we wish to show $X = 0$ almost never, $\mathbb{E}[X]$ is not sufficient.



The Second Moment Method

Therefore, we want to bring in techniques from probability theory that describe the concentration and density of a variable X around the mean.

Definition

The *variance* of random variable X is

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

The *covariance* of random variables X, Y is

$$\mathrm{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Showing $X = O(\mathbb{E}X)$ whp

Using Variance, we can deduce the following:

Lemma

Let $X = X_1 + X_2 .... + X_n$ be the sum of the indicator variables for events $A_1, A_2, ...A_n$. Let Δ be defined as follows:

$$\Delta = \max_i \sum_{j:j \not\perp i} \mathbb{P}(A_j | A_i)$$

That is, Δ is the maximum possible value across all events A_i of the sum of the expression $\mathbb{P}(A_j|A_i)$ for all events A_j dependent on A_i. Then if

$$\Delta = o(\mathbb{E}[X])$$

then $X > 0$ and X is asymptotically equivalent to $\mathbb{E}[X]$ with high probability.

Conclusions

The probabilistic method is a fundamentally simple tool that leverages randomness to non-constructively prove existence. Through refinements via a number of increasingly complex techniques, we may apply probabilistic arguments to produce sophisticated results across Combinatorics.

Questions?
Thank you for your time!