Hua's Bound for the Asymptotic Waring Problem

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Euler Circle

Abstract

This paper closely follows T.D Wooley's analysis of Waring's problem and the circle method, which was originally developed by Ramanujan, Hardy and Littlewood. The main result covered is Hua Loo-Keng's initial bound for the Asymptotic Waring problem, $G(k) \leq 2^k + 1$. Although this bound is not optimal in modern literature, Hua's bound used methods from Fourier Analysis and introduced Vinogrodov's method of trigonometric sums, which saw further use in Vinogrodov's Theorem.

1 Introduction

1.1 $g(k)$.

Let $k \in \mathbb{N}$ and s be the least amount of variables such that any natural number can be represented as a sum of powers of k. Let's denote $g(k)$ as the sum of powers of k. Let $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$g(k) = x_1^k + x_2^k + \cdots + x_s^k \tag{1}$$

For $k = 2$, Lagrange proved in 1770 that a minimum of 4 squares was required to represent every natural number.

Then,

$$g(2) = 4$$

The first few values of $g(k)$ are

k	$g(k)$
1	1
2	4
3	9
4	19
$\cdots$	$\cdots$

Let's try to put a bound on $g(k)$, consider

$$n = 2^k \lfloor (3/2)^k \rfloor - 1$$

Since $n < 3^k$, it must be a sum of 2^k and 1. Indeed,

$$n = 2^k + \ldots + 2^k + 1 + \ldots + 1$$

more precisely, there are $(\lfloor (3/2)^k \rfloor - 1)$ 2^k's and $(2^k - 1)$ 1's. This implies that

$$g(k) \geq 2^k \lfloor (3/2)^k \rfloor - 1$$

must be a lower bound for $g(k)$. Actually, this was shown an equality[1] by Niven, Dickson, Pillai, and Rubugunday.

Notice that it is particularly difficult to represent "small" integer values with k-th powers of natural numbers– it's natural to question what would be the case if, for instance, we didn't consider these values.

[1] There is a possible exception of a finite list of k's, as proven by Kurt Mahler in 1957

1.2 $G(k)$.

It's not difficult to see that $G(k) \leq g(k)$; after all, if we remove sufficiently small integer values, we should require less powers of natural numbers to express a sufficiently large number. To do this, we'd like to define the number r, for which $G(k) < g(k)$. Before we define r, lets set a bound on $G(k)$ first.

$$G(k) \leq g(k) = 2^k \lfloor (3/2)^k \rfloor - 1$$

In this paper, we'd like to establish something sharper–that is,

$$G(k) \leq 2^k + 1$$

2 Vinogradov's Method of Trigonometric Sums

The classic setup for the Hardy-Littlewood circle method begins by defining a generating function and then counting the representations of n as a sum of k powers, with a counting function, $R_{s,k}$.

$$\begin{aligned} g_k(z)^s &= \left(\sum_{m_1=1}^{\infty} z^{m_1^k}\right)\left(\sum_{m_2=1}^{\infty} z^{m_2^k}\right)\cdots\left(\sum_{m_s=1}^{\infty} z^{m_s^k}\right) \\ &= \sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty} z^{m_1^k+\ldots+m_s^k} \\ &= \sum_{n=1}^{\infty} R_{s,k}(n) z^n, \end{aligned}$$

where

$$R_{s,k}(n) = \mathrm{card}\{m_1, \ldots, m_s \in \mathbb{N} : m_1^k + \ldots + m_s^k = n\}.$$

But, we can substitute

$$z = re(\theta), e(\theta) = e^{2\pi i\theta}$$

As is usual in analytic number theory. Hence, we can replace $g_k(z)$, with a fourier series. However, this need not be an infinite series since, $n = x_1^k + \ldots + x_s^k, x_i \leq n^{1/k}$. Hence, let $X = \lfloor n^{1/k} \rfloor$, and

$$f_k(\theta) = \sum_{1 \leq x \leq X} e(\theta x^k).$$

Then,

$$\begin{aligned} f_k(\theta)^s &= \left(\sum_{1\leq x_1\leq X} e(\theta x_1^k)\right)\cdots\left(\sum_{1\leq x_s\leq X} e(\theta x_s^k)\right) \\ &= \sum_{1\leq x_1\leq X}\cdots\sum_{1\leq x_s\leq X} e(\theta(x_1^k + \ldots + x_s^k)) \\ &= \sum_{1\leq m\leq sX^k} R^*_{s,k}(m) e(\theta m). \end{aligned}$$

Where,

$$R^*_{s,k}(m) = \mathrm{card}\{m_1, \ldots, m_s \in \mathbb{N} \cap [1, X] : m_1^k + \ldots m_s^k = m\}.$$

We have,

$$\int_0^1 f_k(\theta)^s e(-\theta n)\mathrm{d}\theta = \sum_{1\leq m\leq sX^k} R^*_{s,k}(m)\int_0^1 e(\theta(m-n))\mathrm{d}\theta = R_{s,k}(n).$$

By the orthogonality relation,

$$\int_0^1 e(\theta h)\mathrm{d}\theta = \begin{cases} 0, \text{when } h \in \mathbb{Z}^*, \\ 1, \text{when } h = 0. \end{cases}$$

This gives us something more workable, but we can refine this further. Let us partition the integral into two sets, the major arcs $\mathfrak{M}_\delta$ and the minor arcs, $\mathfrak{m}_\delta$. The motivation for this distinction arises from the integrand's behavior at neighbourhoods of each rational number $\frac{a}{b}$, in which $(a,b)=1$, and $a,b \leq X^{\delta}$. Large contributions to our integrand occur at these neighbourhoods, which motivates us to split integral into two sets. The utility from this, is to bound the integrand over the minor arcs, by $o(n^{s/k-1})$, allowing us to obtain an asymptotic formulae for $R_{s,k}(n)$. Now, for $0<\delta<1$, let

$$\mathfrak{M}_{\delta} = \bigcup_{0 \leq a \leq b \leq X^{\delta}} \bigcup_{(a,b)=1} \mathfrak{M}_{\delta}(b,a),$$

$$\mathfrak{M}_{\delta}(b,a) = \{\theta \in [0,1) : |\theta - a/b| \leq X^{\delta-k}\}$$

Additionally, let

$$\mathfrak{m}_{\delta} = [0,1) - \mathfrak{M}_{\delta}.$$

Before we can go any further, we require some preliminaries.

2.1 Weyl's Inequality and Weyl's Differencing Lemma

Lets consider the pointwise bounds for the sum,

$$f(\alpha) = \sum_{1 \leq x \leq X} e(ax^{k}) \tag{2}$$

With large X, to bound the minor arcs, $\mathfrak{m}_{\delta}$. Let's consider the case for $k=1$.

Lemma 1. *Let $\alpha, X, Y \in \mathbb{R}$, with $Y>1$. Then,*

$$\sum_{X<x\leq X+Y} e(ax) \ll \min\{Y, ||a||^{-1}\}. \tag{3}$$

Proof. One can see that $|e(ax)| \leq 1$, for each x, the left hand side of [3] is at most Y + 1, and this is enough to establish the desired conclusion for $\alpha = 0$. When $\alpha \neq 0$, it is possible to consider the sum as a geometric progression,

$$\begin{aligned}
\sum_{X<x\leq X+Y} e(\alpha x) &= \frac{e(\alpha\lfloor X+Y+1\rfloor) - e(\alpha\lfloor X+1\rfloor)}{e(\alpha)-1} \\
&\ll |e(\alpha/2) - e(-\alpha/2)|^{-1} \\
&\ll |\sin(\pi\alpha)|^{-1}.
\end{aligned}$$

$|\sin(\pi\alpha)|$ is periodic as a function of α, with period 1, so when $0 \leq \alpha \leq 1/2$, $2\alpha \leq \sin(\pi\alpha) \leq \pi\alpha$. Then,

$$\sum_{X<x\leq X+Y} e(\alpha x) \ll ||\alpha||^{-1}.$$

□

[Lemma 1] provides a means to estimate exponential sums for $k=1$. Let us consider the case when, $k>1$. We can use backwards induction to obtain an estimate in terms of linear, exponential sums. Consider a polynomial $\psi(x) \in \mathbb{R}[x]$, and the exponential sum

$$\begin{aligned}
T(X) &= \sum_{1 \leq x \leq X} e(\psi(x)) \\
T(x)^2 &= \sum_{1 \leq y \leq X} \sum_{1 \leq y \leq X} e(\psi(y) - \psi(x)) \\
&= \sum_{|h|<X} \sum_{\substack{1 \leq x \leq X \\ 1 \leq x+h \leq X}} e(\psi(x+h) - \psi(x)),
\end{aligned}$$

in which, we have made a change in variable $y = x + h$. Then, $\psi(x+h) - \psi(x) = h\psi_1(x;h)$. Where $\psi(x;h)$ is a polynomial in x and h. Whose degree is $\deg(\psi) - 1$, with respect to x. Thus,

$$|T(X)|^2 \leq \sum_{|h|<X} \left| \sum_{\substack{1 \leq X \\ 1 \leq x+h \leq X}} e(h\psi_1(x;h)) \right|.$$

The inner sum is one degree lower than $T(X)$, so one can it induct onto itself until [Lemma 1] is applicable.

Let Δ_1 be the forward difference operator defined via

$$\Delta_1(\psi(x); h) = \psi(x+h) - \psi(x).$$

We then define, Δ_i, for $i > 1$ recursively as,

$$\begin{aligned} \Delta_i(\psi(x); \mathbf{h}) =& \Delta_i(\psi(x); h_1, \ldots, h_i) \\ =& \Delta_1(\Delta_{i-1}(\psi(x); h_1, \ldots, h_{i-1}); h_i). \end{aligned}$$

Also, we take $\Delta_0(\psi(x); \mathbf{h}) = \psi(x)$. When $0 \leq i \leq j$, one has,

$$\Delta_i(x^j; \mathbf{h}) = h_1 \ldots h_i p_i(x; h_1 \ldots h_i),$$

where p_i is a polynomial in x of degree $j - i$, with leading coefficient $\binom{k}{j}$. By the linearity of Δ_i, one has,

$$\Delta_i(a_j x^j + \ldots a_1 x; \mathbf{h}) = \sum_{h=1}^{j} a_h \Delta_h(x^h; \mathbf{h}).$$

One can then infer the expressions $\Delta_i(p(x); \mathbf{h})$, for polynomials $p(x)$, by using the special case, $p(x) = x^j$. Note that when $i > \deg(p)$, one has $\Delta_i(p(x); \mathbf{h}) = 0$.

Lemma 2 (Weyl Differencing)**.**

Let $\psi(x)$, be a real valued arithmetic function, and set

$$F(\psi) = \sum_{1 \leq x \leq X} e(\psi(x)).$$

Then for each natural number i, one has

$$|F(\psi)|^{2^j} \leq (2X)^{2^j - j - 1} \sum_{|h_1|<X} \cdots \sum_{h_i < X} \sum_{x \in I_i(h)} e(\Delta_i(\psi(x), h), \tag{4}$$

where $I_j(\mathbf{h})$ denotes the intervals of integers by setting

$$I_i(h_1, \ldots, h_i) = I_{i-1}(h_1, \ldots, h_{i-1}) \cap \{x \in [1, X] : (x + h_i) \in I_{i-1}(h_1, \ldots, h_{i-1})\}.$$

They also satisfy,

$$I_i(h_1, \ldots, h_i) \subseteq I_{i-1}(h_1, \ldots, h_{i-1}) \subseteq I_1(h_1) \subseteq [1, X].$$

If we apply Weyl's differencing lemma $k - 1$ times to our exponential sum[2],

$$f(\alpha) = \sum_{1 \leq x \leq X} e(ax^k),$$

Then we get the bound,

$$|f(\alpha)|^{2^k - 1} \ll X^{2^{k-1} - k} \sum_{h_1, \cdots, h_{k-1}(|h_i|<X)} \sum_{x \in I(h)} e(k!\alpha h_1 \ldots h_{k-1}(x + \frac{1}{2}(h_1 + \ldots + h_{k-1}))).$$

This sum can be applied to [Lemma 1], but we will have to take the average over the products $h_1, \ldots, h_{k-1}$.

[2]see [] for the full proof.

Lemma 3 (Divisor Lemma)**.**

For each $r \in \mathbb{N}$ and $\varepsilon > 0$, one has $\tau_r(n) \ll_{r,\varepsilon} n^{\varepsilon}$, where, for d_i a divisor of n,

$$\tau_r(n) = \sum_{d_1 d_2 \ldots d_r = n} 1.$$

Proof. Consider the prime factorisations of n, and d_i, with $n = d_1, \ldots, d_r$, one obtains the relation,

$$\frac{\tau_r(n)}{n^{\varepsilon}} = \prod_{\substack{p \text{ prime} \\ p^h || n}} \frac{\tau_r(p^h)}{p^{\varepsilon h}} \leq \prod_{p^h || n} \frac{(h+1)^{r-1}}{p^{\varepsilon h}}.$$

We've bounded h, $h = a_1 + \ldots + a_r$ with $0 \leq a_i \leq h$. Since $p^h = p_1^a \ldots p_r^a$. Thus,

$$(h+1)^{r-1} p^{-\varepsilon h} \leq (2^r / p^{\varepsilon})^h,$$

which is atmost 1 for $p \geq 2^{r/\varepsilon}$. We also have, $(h+1)^{r-1}/p^{\varepsilon h}$, which is bounded uniformly in p, in terms of r and ε as h increases. Thus,

$$(h+1)^{r-1} p^{-\varepsilon h} \leq (h+1)^{r-1} 2^{-\varepsilon h} < C(r, \varepsilon).$$

Consequently,

$$\frac{\tau_r(n)}{n^{\varepsilon}} \leq \prod_{p < 2^{r/\varepsilon}} C(r,\varepsilon)^{2^{r/\varepsilon}} \ll_{r,\varepsilon} 1.$$

Then, $\tau_r(n) \ll_{r,\varepsilon} n^{\varepsilon}$, and the proof is complete. □

Theorem 1 (Dirichlet's Approximation Theorem)**.**

For any $n, \alpha \in \mathbb{R}$, $n \geq 1$, there exists p, q such that

$$|q\alpha - p| \geq \frac{1}{\lfloor n \rfloor + 1} < \frac{1}{n}$$

Proof. Let $\alpha \in \mathbb{R}/\mathbb{Q}$, $n \in \mathbb{Z}$. For every k $= 0, 1, 2, \ldots, n$, we can write $k\alpha = m_k + x_k$, for $m_k \in \mathbb{Z}, 0 \leq x_k \leq 1$. Then, the interval $[0, 1)$ can be divided into n intervals with measure $1/n$. Now, there are $n+1$ numbers, $x_0, x_1, \ldots, x_n$ and n intervals. Therefore, by the pidgeonhole principle, at least 2 of them must lie in the same interval. Call them x_i, x_j, $i < j$. Then,

$$|(j-i)\alpha - (m_j - m_i)| = |j\alpha - m_j - (i\alpha - m_j) = |x_j - x_i| < 1/n$$

If we divide by $j - i$,

$$\alpha - \frac{m_j - m_i}{j - i} < \frac{1}{nj - ij} \leq \frac{1}{(j-i)^2}$$

□

Corollary 1.

When $\alpha \in \mathbb{R}/\mathbb{Q}$, there exists infinitely many rational numbers a/b, $(a, b) = 1$ such that $|\alpha - a/b| \leq b^{-2}$.

The next step is to cover Weyl's inequality, which in combination with Hua's lemma, is required to bound the minor arcs by an error term.

Theorem 2 (Weyl's Inequality)**.**

Let $k \geq 2$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that $\alpha \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|a_k - a/q| \leq q^{-2}$. Then,

$$\sum_{1 \leq x \leq X} e(\alpha_1 x + \ldots a_k x^k) \ll X^{1+\varepsilon} (q^{-1} + \frac{1}{X} + \frac{q}{X^k})^{2^{1-k}}. \tag{5}$$

Proof. Write $\psi(x) = a_1 x + \ldots a_k x^k$ and

$$F(\alpha) = \sum_{1 \le x \le X} e(\psi(x)).$$

When $q > X^k$, the estimate is

$$|F(\alpha)| \le \sum_{1 \le x \le X} 1 \le X$$

As desired. So let's assume $q \le X^k$.

By applying [Lemma 2], with $j = k-1$. Our bound becomes

$$|F(\alpha)|^{2^{k-1}} \ll X^{2^k - k} \sum_{|h_1| < x} \cdots \sum_{|h_{k-1}| < X} \sum_{x \in I_{k-1}(h)} e(\Delta_{k-1}(\psi(x); \mathbf{h})).$$

Where $I_{k-1}(H)$ is a suitable interval of integers in $[1, X]$. Note that

$$\Delta_{k-1}(\psi(x); \mathbf{h}) = k! h_1 \ldots h_k x \alpha_k + \gamma(\alpha, \mathbf{H})$$

Where $\gamma(\alpha, H)$ is independent of x. Thus, applying [Lemma 1] yields,

$$\gamma(\mathbf{h}) \ll \min\{X, ||k! h_1 \ldots h_{k-1} \alpha||^{-1}\}.$$

Thus,

$$\sum_{x \in I_{k-1}(h)} e(\Delta_{k-1}(\psi(x), H)) \ll \min\{X, ||k! h_1 \ldots h_{k-1} \alpha_{k-1}||^{-1}\}.$$

Then, one can assert that,

$$|F(\alpha)|^{2^{k-1}} \ll X^{2^{k-1} - k} \left(X^{k-1} + \sum_{1 \le n \le k! X^{k-1}} \tau_k(n) \min\{X, ||n\alpha_k||^{-1}\} \right).$$

Thus, for $q \le X^k$, we obtain,

$$F(\alpha) \ll X^{1 - 2^{1-k}} + X^{1+\varepsilon} (\frac{1}{q} + \frac{1}{X} + \frac{X}{X^k} + \frac{q}{X^k})^{2^{1-k}}. \tag{6}$$

□

This concludes Weyl's inequality. Now, we can directly apply this to Hua's lemma and use it to directly bound the minor arcs, when $s \ge 2^k + 1$.

3 Minor Arc Analysis

3.1 Hua's Lemma

Lemma 4 (Hua's Lemma)**.**

Suppose that $k \ge 2$ and $1 \le j \le k$. Then, one has

$$\int_0^1 |f(\alpha)|^{2^j} \mathrm{d}\alpha \ll X^{2^j - j + \varepsilon}. \tag{7}$$

Proof. We proceed the proof by induction on j. When $j = 1$, it follows by orthogonality that,

$$\int_0^1 |f(\alpha)|^2 \mathrm{d}\alpha = \int_0^1 f(\alpha) f(-\alpha) \mathrm{d}\alpha = \mathrm{card}\{1 \le x, y \le X : x^k = y^k\} = \lfloor X \rfloor.$$

Now, we have the base case, $j = 1$. We'll apply Weyl's differencing lemma [Lemma 2] to see that,

$$|f(\alpha)|^2 \ll (2X)^{2^J - J - 1} \sum_{|h_1| < X} \cdots \sum_{|h_J < X} \sum_{x \in I_J(h)} e(\alpha \Delta_j(x^k; \mathbf{h})),$$

where $I_j(\mathbf{h})$ is a suitable subinterval of $[1, X]$. It follows that,

$$\int_0^1 |f(\alpha)|^{2^{J+1}} \mathrm{d}\alpha = \int_0^1 (f(\alpha)f(-\alpha))^{2^{J-1}} |f(\alpha)|^{2^J} \mathrm{d}\alpha \ll (2X)^{2^J-J-1} T$$

Where

$$T = \sum_{|h_1|<X} \sum_{|h_J|<X} \sum_{x \in I_J(h)} \int_0^1 (f(\alpha)f(-\alpha))^{2^{J-1}} e(\alpha \Delta_J(x^k; \mathbf{h})) \mathrm{d}\alpha.$$

By orthogonality, T is bounded above by the number of integral solutions to

$$\sum_{i=1}^{2^{J-1}} (u_i^k - v_i^k) = \Delta_J(x^k; \mathbf{h}),$$

with $1 \leq u_i, v_i \leq X$, $(1 \leq i \leq 2^{J-1})$, $1 \leq x \leq X$ and $|h_j| < X$, $(1 \leq j \leq J)$. We've weakened the condition $x \in I_j(\mathbf{h})$ to $1 \leq x \leq X$.

The solutions counted by T are of two types, The first is,

$$\sum_{i=1}^{2^{J-1}} (u_i^k - v_i^k) = 0.$$

We have, $\Delta_J(x^k; \mathbf{h}) = 0$. By orthogonality,

$$\int_0^1 (f(\alpha)f(-\alpha))^{2^{J-1}} \mathrm{d}\alpha = \int_0^1 |f(\alpha)|^{2^J} \mathrm{d}\alpha \ll X^{2^J-J+\varepsilon}.$$

Since $\Delta_J(x^k; \mathbf{h}) = 0$, we have,

$$h_1, \ldots, h_J \left(\frac{k!}{(k-J)!} x^{k-J} + \ldots \right) = 0$$

So either $h_j = 0$ for $1 \leq j \leq J$, or x is a polynomial determined by $h_1, \ldots, h_j$. Then, one has $O(X^J)$ choices for x and $h_1, \ldots, h_j$. Thus, the contribution of this type to T is,

$$\ll X^J \cdot X^{2^J-J+\varepsilon} \ll X^{2^J+\varepsilon}.$$

For the next class of solutions counted by T, one has

$$\sum_{i=1}^{2^J-1} (u_i^k - v_i^k) = N,$$

For some nonzero integer $N = N(\mathbf{u}, \mathbf{v})$, with $|N| \ll X^k$. For each choice of $\mathbf{u}$ and $\mathbf{v}$, we then have

$$h_1, \ldots, h_J \left(\frac{k!}{(k-J)!} x^{k-J} + \ldots \right) = N$$

and thus, we have $O(\tau_{J+1}(N)) \ll X^\varepsilon$ choices for $h_1, \ldots, h_J$ and x. The contribution to T is therefore,

$$\ll \sum_{1 \leq u_i, v_i \leq X | 1 \leq i \leq 2^{J-1}} \tau_{J+1}(N(\mathbf{u}, \mathbf{v})) \ll X^\varepsilon \sum_{1 \leq u_i, v_i \leq X | 1 \leq i \leq 2^{J-1}} 1 \ll X^{2^J+\varepsilon}$$

Thus, one has $T \ll X^{2^J+\varepsilon}$, and then,

$$\int_0^1 |f(\alpha)|^{2^{J+1}} \mathrm{d}\alpha \ll (2)^{2^J-J-1} X^{2^{J+1}-J-1+\varepsilon} \ll X^{2^{J+1}-(J+1)+\varepsilon}.$$

This confirms induction with $j = J + 1$ and the lemma concludes. □

In addition, for $s \geq 2^k + 1$, we have the collorary

Corollary 2.

$$\int_{\mathfrak{m}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha \ll X^{s-k-\delta 2^{-k}}.$$

Proof. By using Weyl's inequality and Hua's lemma, one has,

$$\begin{aligned}\left|\int_{\mathfrak{m}_\delta} f(\alpha)^s e(-n\alpha)\right| \leq &(\sup|f(\alpha)|)^{s-2^k}\int_0^1 |f(\alpha)|^{2^k}\mathrm{d}\alpha \\ \ll&(X^{1-\delta 2^{1-k}+\varepsilon})^{s-2^k}X^{2k-k+\varepsilon} \\ \ll&X^{s-k-(s-2^k)\delta 2^{1-k}+s\varepsilon} = o(n^{s/k-1}).\end{aligned}$$

□

This concludes our treatment of the minor arcs, with our bound for the integral over the minor arcs,

$$\int_{\mathfrak{m}_\delta} f(\alpha)^s e(-na)\mathrm{d}\alpha = o(n^{s/k-1})$$

Recall that,

$$R_{s,k}(n) = \int_{\mathfrak{m}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha + \int_{\mathfrak{M}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha$$

Now that we have a bounded expression for the minor arcs,

$$R_{s,k}(n) = \int_{\mathfrak{M}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha + o(n^{s/k-1}).$$

Thus, we can justifiably turn our attention to the major arcs.

4 Major Arc Analysis

Lets try to asymptotically evaluate $f(\alpha)$ for $\alpha \in \mathfrak{M}_\delta$. We can use the mean value theorem with $\beta = \alpha - a/b$, so that $|\beta| \leq X^{\delta-k}$. Then, we have,

$$\begin{aligned}\sum_{1\leq x\leq X} e(\alpha x^k) =&\sum_{r=1}^{b}\sum_{b\leq y\leq (X-r)/b} e((\beta + a/b)(yb-r)^k) \\ =&\sum_{r=1}^{b} e(ar^k/b)\sum_{(1-r)/b\leq y\leq(X-r)/b} e(\beta(yb+r^k)\end{aligned}$$

Then, since β is small, we can approximate the inner sum by a smooth function with control of the error terms. We do this with the mean value theorem. Let, $F(z)$ be differentiable on $[a,b]$ with $a<b$. Then,

$$F(a) - F(b) = (a-b)F'(c)$$

For some $c \in (a,b)$. Trivially,

$$e(F(z)) = \int_{-1/2}^{1/2} e(F(z))\mathrm{d}u.$$

Hence,

$$|e(F(z)) - \int_{-1/2}^{1/2} e(F(z+u))\mathrm{d}u| \leq \underset{\sup}{|u| \leq} 1/2|e(F(z+u) - e(F(z))) \ll \underset{\sup}{|u| \leq} 1/2|F'(z+u)|$$

Thus, we can infer that

$$\sum_{(1-r)b\leq y\leq(X-r)/b} e(\beta(yb+r)^k) - \int_{-r/b}^{(X-r)/b} e(\beta(zb+r)^k)\mathrm{d}z \ll 1 + X^k|\beta|.$$

Finally, we have

$$\int_{-r/b}^{(X-r)/b} e(\beta(zb+r)^k)\mathrm{d}z = \frac{1}{b}\int_0^X e(\beta\gamma^k)\mathrm{d}\gamma.$$

After this, it can be inferred from the previous equations that,

$$f(\alpha) - \frac{1}{b}S(b,a)v(\alpha - a/b) \ll b + X^k|b\alpha - a|.$$

Where

$$S(b,a) = \sum_{r=1}^{b} e(ar^k/b)$$

and,

$$v(\beta) = \int_0^X e(\beta\gamma^k)\mathrm{d}\gamma.$$

Let's turn our attention toward the formula for the major arc contribution. We have,

$$\int_{\mathfrak{M}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha = \sum_{1\le b\le X^\delta}\sum_{a=1}^{b}\int_{-X^{\delta-k}}^{X^{\delta-k}} f(\beta + a/b)^s e(-n(\beta + a/b))\mathrm{d}\beta.$$

Whenever $\delta < 1/5$, this is

$$\sum_{1\le b\le X^\delta}\sum_{a=1}^{b}(\frac{1}{b}S(b,a))^s e(-na/b)\int_{-X^{\delta-k}}^{X^{\delta-k}} v(\beta)e(-\beta n)\mathrm{d}\beta.$$

The next lemma will be our asymptotic formula for the major arcs. We will need more notation for this. Let the truncated singular series, be

$$\mathfrak{S}_{s,k}(n;B) = \sum_{1\le b\le B}\sum_{a=1}^{b}(\frac{1}{b}S(b,a))^s e(-na/b).$$

And the truncated singular integral be,

$$J_{s,k}(n;B) = \int_{-BX^{-k}}^{BX^{-k}} v(\beta)^s e(-\beta n)\mathrm{d}\beta.$$

Lemma 5.

The work we've done earlier can be summarized as follows, When $0 < \delta < 1$, one has,

$$\int_{\mathfrak{M}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha = J_{s,k}(n;X^\delta)\mathfrak{S}_{s,k}(n;X^\delta) + O(X^{s-k+(5\delta-1)}).$$

Corollary 3.

When $s \ge 2^k + 1$ and $0 < \delta < 1/5$, one has

$$R_{s,k}(n) = J_{s,k}(n;X^\delta)\mathfrak{S}_{s,k}(n;X^\delta) + o(X^{s-k}) \tag{8}$$

Proof. Since $[0,1)$ is the disjoint union of $\mathfrak{m}_\delta$ and $\mathfrak{M}_\delta$, one has,

$$R_{s,k}(n) = \int_{\mathfrak{M}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha + \int_{\mathfrak{m}_\delta} f(\alpha)^s e(-n\alpha)\mathrm{d}\alpha$$

The conclusion follows from, [Lemma 5] and [Corollary 2] □

5 The Singular Integral $J_{s,k}$.

We first consider the truncated singular integral, $J_{s,k}(n;B)$. Our first step is to complete the integral to obtain the complete singular integral,

$$J_{s,k}(n) = \int_{-\infty}^{\infty} v(\beta)e(-n\beta)\mathrm{d}\beta$$

We will consider the domain of integration. Recall that,

$$v(\beta) = \int_0^x e(\beta\gamma^k)\mathrm{d}\gamma$$

Lemma 6.

$$v(\beta) \ll X(1+X^k|\beta|)^{-1/k}.$$

The estimate $|v(\beta) \leq X$ is trivial. Since $|v(\beta)| = |v(-\beta)|$, we can assume that $\beta > X^{-k}$. Next, set $u = \beta\gamma^k$ and see that when $\beta > 0$, one has,

$$\begin{aligned} v(\beta) =& \frac{1}{k\beta^{1/k}} \int_0^{\beta X^k} u^{-1+1/k} e(u)\mathrm{d}u, \\ |v(\beta)| \leq& \frac{1}{k\beta^{1/k}} \left| \int_0^{\beta X^k} u^{-1+1/k} e(u)\mathrm{d}u \right| \end{aligned}$$

Because $u^{-1+1/k}$ decreases monotonically to 0 as $u \to \infty$, the integral is uniformly bounded by Dirichlet's convergence test.

$$\left| \int_0^{\beta X^k} u^{-1+1/k} e(u)\mathrm{d}u \right| \leq \sup_{Y \geq 0} \left| \int_0^{Y} u^{-1+1/k} e(u)\mathrm{d}u \right| < \infty$$

Hence, when $|\beta| > X^{-k}$, one has,

$$v(\beta) \ll |\beta|^{-1/k} \ll X(1+X^k|\beta|)^{-1/k}.$$

Then, our conclusion follows from $|v(\beta)| \leq X$, applied when $|\beta| \leq X^{-k}$.

Corollary 4.

Suppose $s \geq k+1$, then the singular series $J_{s,k}(n)$ converges absolutely, and moreover, we have

$$J_{s,k}(n;Q) - J_{s,k}(n) \ll X^{s-k}Q^{-1/k}.$$

Proof. By applying [Lemma 6], one sees that,

$$J_{s,k}(n) \ll \int_{-\infty}^{\infty} \frac{X^s}{(1+X^k|\beta|)^{s/k}}\mathrm{d}\beta \ll X^s \int_0^{\infty} \frac{X^s}{(1+X^k|\beta|)^{s/k}}\mathrm{d}\beta \ll X^{s-k}$$

Thus, the integral defining $J_{s,k}(n)$ is indeed absolutely convergent. Moreover,

$$J_{s,k}(n;Q) - J_{s,k}(n) \ll \int_{QX^{-k}}^{\infty} \frac{X^s}{(1+X^k\beta)^{1+1/k}}\mathrm{d}\beta \ll X^{s-k}Q^{-1/k}.$$

□

Theorem 3.

When $s \geq k+1$, one has

$$J_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1},$$

Where,

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t \text{ (Re(z) ¿ 0).}$$

Proof. We begin by observing that,

$$\begin{aligned} J_{s,k}(n) &= \lim_{K\to\infty}\int_{-K}^{K} v(\beta)^s e(-\beta n)\mathrm{d}\beta \\ &= \lim_{B\to\infty}\int_{[0,X]^s}\int_{-K}^{K} e(\beta(\gamma_1^k+\ldots+\gamma_s^k-n))\mathrm{d}\beta\ \mathrm{d}\gamma \end{aligned}$$

Notice that, for $\phi\neq 0$, one has,

$$\int_{-K}^{K} e(\beta\phi)\ \mathrm{d}\beta = \frac{\sin(2\pi B\phi)}{\pi\phi}.$$

When $\phi = 0$, we take the right hand side to be $2B$. We also obtain,

$$J_{s,k}(n) = \lim_{B\to\infty}\int_{[0,X]^s}\frac{\sin(2\pi B(\gamma_1^k+\ldots+\gamma_s^k-n))}{\pi\gamma_1^k+\ldots\pi\gamma_s^k-\pi n}\mathrm{d}\gamma$$

We'd like to turn this into a linear problem, so let's substitute $u_i=\gamma_i^k(1\leq i\leq s)$, and recall that, $n = X^k$. Thus,

$$J_{s,k}(n) = k^{-s}\lim_{B\to\infty} I(B)$$

Where,

$$I(B) = \int_{[0,n]^s}\frac{\sin(2\pi B(u_1+\ldots+u_s-n))}{\pi u_1+\ldots+\pi u_s - n}(u_1\ldots u_s)^{-1+1/k}\mathrm{d}\mathbf{u}$$

We can substitute further with $v = u_1+\ldots+u_s$ and make the change of variable $(u_1,\ldots,u_s)\to(u_1,\ldots,u_{s-1},v)$, obtaining,

$$I(K) = \int_0^{sn}\Psi(v)\frac{\sin(2\pi B(v-n))}{\pi(v-n)}\mathrm{d}v,$$

Where,

$$\Psi(v) = \int_{\mathfrak{B}(n)}(u_1\ldots u_{s-1})^{-1+1/k}(n-u_1-\ldots-u_{s-1})^{-1+1/k}\mathrm{d}u.$$

Note that,

$$\mathfrak{B}(v) = \{(u_1,\ldots,u_{s-1})\in[0,n]^{s-1} : 0\leq u_1+\ldots+u_{s-1}\leq n\}$$

The condition on $u_1,\ldots,u_{s-1}$ in the definition of $\mathfrak{B}(v)$ may be rephrased as $v-n\leq u_1+\ldots+u_{s-1}\leq v$. Since $\Psi(v)$ is a function of bounded variation, it follows from Fourier's integral theorem that, since $n\in(0,sn)$, one has,

$$\lim_{B\to\infty} I(B) = \Psi(n) = \int_{\mathfrak{B}(n)}(u_1\ldots u_{s-1})^{-1+1/k}(n-u_1-\ldots u_{s-1})^{-1+1/k}\mathrm{d}\mathbf{u}.$$

Thus,

$$J_{s,k}(n) = k^{-s}\Psi(n) = k^{-s}\int_0^n\cdots\int_0^n(u_1\ldots u_{s-1})^{-1+1/k}(n-u_1-\ldots-u_{s-1})^{-1+1/k}\ \mathrm{d}\mathbf{u}.$$

Finally, we apply induction to show that,

$$J_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}n^{s/k-1}$$

When $s=2$,

$$\begin{aligned} J_{2,k}(n) =& \frac{1}{k^2}\int_0^n u_1^{-1+1/k}(n-u_1)^{-1+1/k}\mathrm{d}u_1 \\ =& \frac{n^{-1+1/k}}{k^2}\int_0^1 v^{-1+1/k}(1-v)^{-1+1/k}\mathrm{d}v. \end{aligned}$$

Thus, on recalling the Beta function, one has

$$J_{s,k}(n) = \frac{n^{-1+2/k}}{k^2}B(1/k,1/k) = \frac{n^{-1+2/k}}{k^2}\frac{\Gamma(1/k)^2}{\Gamma(2/k)} = \frac{\Gamma(1+1/k)^2}{\Gamma(2/k)}n^{-1+2/k}.$$

Induction holds for $s = 2$.

Suppose now that the inductive hypothesis holds for $s = t$, then,

$$J_{t+1,k}(n) = \frac{1}{k}\int_0^n u_t^{-1+1/k} J_{t,k}(n-u_t)\mathrm{d}u_t = \frac{\Gamma(1+1/k)^t}{k\Gamma(t/k)}\int_0^n u_t^{-1+1/k}(n-u_t)^{-1+t/k}\mathrm{d}u_t.$$

Recall the classical Beta function,

$$\begin{aligned}J_{t+1,k}(n) =& \frac{\Gamma(1+1/k)^t}{\Gamma(t/k)} n^{-1+(t+1)/k}\mathrm{B}(1/k, t/k)\\ =& \frac{n^{-1+(t+1)/k}\Gamma(1+1/k)^t\Gamma(1/k)\Gamma(t/k)}{k\Gamma(t/k)\Gamma((t+1)/k)}\\ =& \frac{n^{s/k-1}\Gamma(1+1/k)^s}{\Gamma((t+1)/k)}.\end{aligned}$$

This confirms the inductive hypothesis, with t replaced by $t+1$. We have therefore shown that, for $s \geq k+1$, one has

$$J_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1}.$$

□

Corollary 5.

Suppose that $s \geq k+1$. Then, As $Q \to \infty$,

$$J_{s,k}(n;Q) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1} + O(n^{s/k-1}Q^{-1/k})$$

Proof. This follows by the previous corollary and lemma. It's noteworthy to point out that $X = n^{1/k}$. □

Now, we will shift our attention to the truncated singular series. Recall that,

$$\mathfrak{S}_{s,k}(n) = \sum_{b=1}^{\infty}\sum_{a=1}^{b}(b^{-1}S(b,a))^s e(-na/b)$$

We do this by considering the tail of the infinite sum.

Lemma 7.

For $a, b \in \mathbb{Z}$ with $(a,b) = 1$, one has $S(b,a) \ll b^{1-2^{1-k}+\varepsilon}$.

Proof. We'll apply Weyl's inequality with $\alpha_k = a/b$ and $X = b$ to obtain,

$$\sum_{r=1}^{b} e(ar^k/b) \ll b^{1+\varepsilon}(\frac{2b^{k-1}+b}{b^k})^{2^{1-k}}.$$

By applying [Lemma 7] to estimate the tail of the truncated singular series, we have

$$\sum_{b>B}\sum_{\substack{a=1\\(a,b)=1}}^{b}\left|\frac{S(b,a)^s e(-na/b)}{b}\right| \ll \sum_{b>B}\phi(b)\left(b^{\varepsilon-2^{1-k}}\right)^s$$

When $s \geq 2^k+1$,

$$\sum_{b>B}\sum_{\substack{a=1\\(a,b)=1}}^{b}|\frac{S(b,a)^s e(-na/b)}{b} \ll \sum_{b>B} b^{\varepsilon-1-2^{1-k}} \ll B^{-2^{-k}}.$$

Additionally, $\mathfrak{S}_{s,k}(n)$ is absolutely convergent under these conditions and we have

$$\mathfrak{S}_{s,k}(n) - \mathfrak{S}_{s,k}(n;B) \ll B^{-2^{-k}}.$$

These results will be summarised in the following lemma.

Lemma 8.

Suppose $s \geq 2^k + 1$, then $\mathfrak{S}_{s,k}(n)$ is absolutely convergent and

$$\mathfrak{S}_{s,k}(n) - \mathfrak{S}_{s,k}(n; B) \ll B^{-2^{-k}}.$$

□

We need to define a new function, $A(b, n)$. It is a multiplicative function in b, and it is related to the singular series. Let

$$A(b, n) = \sum_{a=1}^{b} (b^{-1} S(b, a))^s e(-na/b).$$

Lemma 9.

$A(b, n)$ is a multiplicative function of b.

Proof. Suppose $(b, r) = 1$. Then by the Chinese Remainder Theorem, there exists a bijection between the residue classes a modulo br, with $(a, br) = 1$, and the ordered pairs, (c, d) with c modulo b and d modulo r satisfying, $(c, b) = (d, r) = 1$, via the relation $a \equiv cr + dq \pmod{br}$. Thus,

$$\begin{aligned} A(br, n) = &\sum_{\substack{a=1 \\ (a,br)=1}}^{br} ((br)^{-1} S(br, a))^s e(-na/br) \\ = &\sum_{\substack{c=1 \\ (c,b)}}^{b} \sum_{\substack{d=1 \\ d,r}}^{r} ((br)^{-1} S(br, cr + bc))^s e(\frac{-cr + bd}{br} n) \end{aligned}$$

Thus,

$$A(br, n) = A(b, n)A(r, n)$$

□

We have,

$$\mathfrak{S}_{s,k}(n) = \sum_{b=1}^{\infty} A(b, n)$$

The multiplicity of $A(b, n)$ should then suggest that, $\mathfrak{S}_{s,k}(n)$ should factor as a product,

$$\sigma(p) = \sum_{h=0}^{\infty} A(p^h, n)$$

Theorem 4.

Suppose that $s \geq 2^k + 1$. Then the following holds,

1. The series $\sigma(p)$ is converges absolutely, and one has
$$|\sigma(p) - 1| \ll p^{-1-2^k}.$$
2. the infinite product,
$$\prod_{p \text{ prime}} \sigma(p)$$
converges absolutely,
3. one has $\mathfrak{S}_{s,k}(n) = \prod_p \sigma(p)$
4. There exists a number $C = C(k)$ such that,
$$1/2 < \prod_{p \geq C(k)} \sigma(p) < 3/2$$

Proof. We will start with the first claim of [Theorem 4] From [Lemma 7], whenever $(a,p) = 1$, we have

$$S(p^h, a) \ll p^{h(1-2^{1-k}+\varepsilon)}$$

Then, whenever $s \geq 2^k + 1$, one finds that,

$$A(p^h, n) = \sum_{\substack{a=1 \\ (a,p)=1}}^{p^h} (p^{-h} S(p^h, a))^s e(-na/p^h) \ll p^{h(1-s2^{1-k})+\varepsilon} \ll p^{-h(1+2^{-k})}$$

Hence,

$$\sigma(p) - 1 = \sum_{h=1}^{\infty} A(p^h, n) \ll \sum_{h=1}^{\infty} p^{-h(1+2^{-k})} \ll p^{-1-2^{-k}}.$$

Thus, $\sigma(p)$ converges absolutely, and one has, $|\sigma(p) - 1| \ll p^{-1-2^{-k}}$.

Now, we'll try to prove the second claim. From the conclusion of the first claim, there exists a positve number $T = T(k)$ with the property $|\sigma(p) - 1| \leq Tp^{-1-2^{-k}}$. Hence, when p is sufficiently large, we have,

$$\log(1 + |\sigma(p) - 1) \leq \log(1 + Tp^{-1-2^{-k}}) \leq Tp^{-1-2^{-k-1}},$$

Whence,

$$\sum_{p \text{ prime}} \log(1 + |\sigma(p) - 1|) \ll T \sum_p p^{-1-2^{-k-1}} \ll 1.$$

Thus, the infinite product $\prod_p \sigma(p)$ is absolutely convergent. We have to reuse the multiplicative property of $A(b, n)$ for the proof of the third claim.

$$\mathfrak{S}_{s,k}(n) = \sum_{b=1}^{\infty} A(b, n) = \sum_{b=1}^{\infty} \prod_{p^h || b} A(p^h, n).$$

Since we've shown that $\prod_p \sigma(p)$ converges absolutely, and $\sum_{q=1}^{\infty} A(b, n)$ converges absolutely too, we can rearrange to obtain,

$$\mathfrak{S}_{s,k}(n) = \prod_p \sum_{h=0}^{\infty} A(b, n) = \sum_{b=1}^{\infty} \prod_{p^h || b} A(p^h, n).$$

Now, we would like to establish our final claim. Whenever p is large in terms of k, one has,

$$1 - p^{-1-2^{-k}} \leq \sigma(p) \leq 1 + p^{-1-2^{-k}}.$$

If we have $C = C(k)$, such that it is sufficiently large, one finds that

$$\left| \prod_{p \geq C(k)} \sigma(p) - 1 \right| \leq \sum_{n \geq C(k)} n^{-1-2^{-k}} \ll C(k)^{-2^{-k}}.$$

If $C(k)$ is large in terms of k, we can infer that,

$$\left| \prod_{p \geq C(k)} \sigma(p) - 1 \right| < 1/2$$

and thus, we conclude that,

$$1/2 < \prod_{p \geq C(k)} \sigma(p) < 3/2$$

The proof of [Theorem 4] is complete. $\square$

We can now show that $G(k) \leq 2^k + 1$.

Proof. In [Corollary 3], when $s \geq 2^k + 1$ and $0 < \delta < 1/5$, one has

$$R_{s,k}(n) = J_{s,k}(n; X^\delta) \mathfrak{S}_{s,k}(n; X^\delta) + o(X^{s-k})$$

Also [Corollary 5] shows that,

$$J_{s,k}(n; X^\delta) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} n^{s/k-1} + O(n^{s/k-1-\delta/k^2})$$

and [Lemma 8] and [Theorem 4] show that,

$$\mathfrak{S}_{s,k}(n; X^\delta) = \mathfrak{S}_{s,k}(n) + O(n^{-\delta 2^{-k}/k})$$

Where $\mathfrak{S}_{s,k}(n) = 1$. Thus, when $s > 2^k + 1$,

$$R_{s,k} = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} n^{s/k-1} \mathfrak{S}_{s,k}(n) + o(n^{s/k-1}) \gg n^{s/k-1}. \tag{9}$$

Then, as $R_{s,k} \to \infty$, $n \to \infty$, $G(k) \leq 2^k + 1$. $\square$

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