Average Orders of Arithmetic Functions

Larry Zhao

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The four functions we will discuss in this talk are

- Divisor function - $d(n)$ or $\sigma_0(n)$
- Totient function - $\phi(n)$
- Sum of divisors function - $\sigma_1(n)$
- Sum of squares function - $r_2(n)$

The Divisor Function

Theorem (Dirichlet)

For all $x \geq 1$,

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$$

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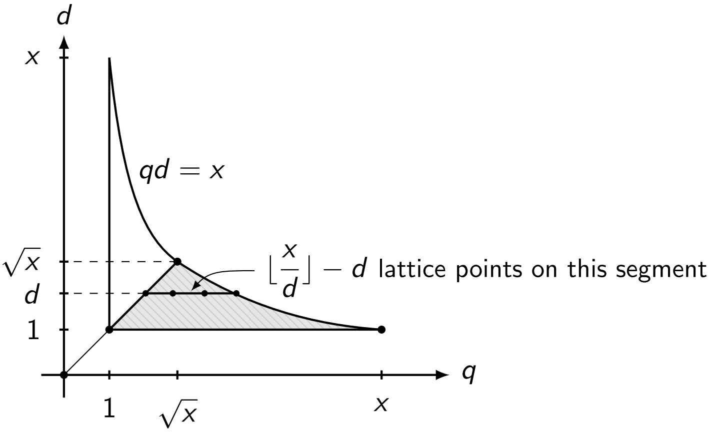
The proof of this uses a very neat geometrical interpretation with lattice points. We can write our partial sums of $d(n)$ as

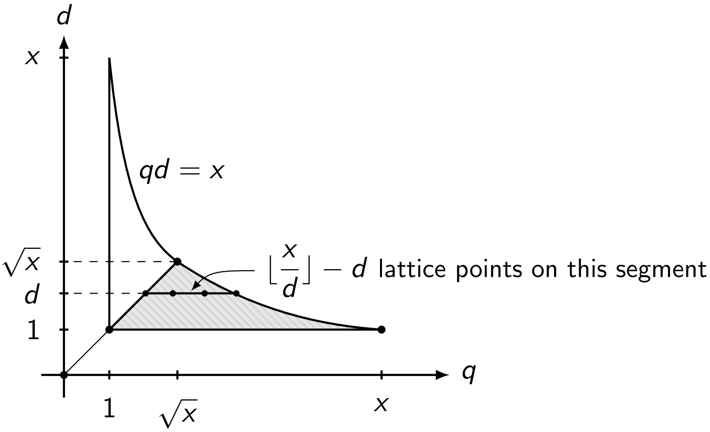
$$\sum_{n \leq x} d(n) = \sum_{\substack{q,d \\ qd \leq x}} 1.$$

Now, our partial sum counts the number of lattice points in the region under the hyperbola $qd = x$.

Dirichlet's Hyperbola Method



Dirichlet's Hyperbola Method



Now, we have

$$\sum_{n \le x} d(n) = 2 \sum_{d \le \sqrt{x}} \left(\lfloor \frac{x}{d} \rfloor - d \right) + \lfloor \sqrt{x} \rfloor.$$

Dirichlet's Hyperbola Method

Before we can finish the proof, we will need some identities from the Euler Summation Formula.

Theorem (Euler's summation formula)

If f has a continuous derivative f' on $[a, b]$, then

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\, dt + \int_a^b f'(t)\cdot\{t\}\, dt + f(a)\cdot\{a\} - f(b)\cdot\{b\}.$$

Corollary

If $x \geq 1$, $\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$

Corollary

If $x \geq 1$, $\sum_{n\leq x} \frac{1}{n^2} = \zeta(2) - \frac{1}{x} + O(\frac{1}{x^2})$

Dirichlet's Hyperbola Method

Now, we just simplify and apply the results from the previous slide.

$$\begin{aligned}
\sum_{n\leq x} d(n) &= 2\sum_{d\leq\sqrt{x}}\left(\frac{x}{d}+O(1)-d\right)+O(\sqrt{x})\\
&= 2x\sum_{d\leq\sqrt{x}}\frac{1}{d}-2\sum_{d\leq\sqrt{x}}d+O(\sqrt{x})\\
&= 2x\left(\log\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)\right)-2\left(\frac{\sqrt{x}^2}{2}+O(\sqrt{x})\right)\\
&= x\log x+(2\gamma-1)x+O(\sqrt{x})
\end{aligned}$$

$\blacksquare$

Sum of Squares Function

Definition

The function $r_2(n)$, or just $r(n)$, counts the number of ways to write n as a sum of two squares.

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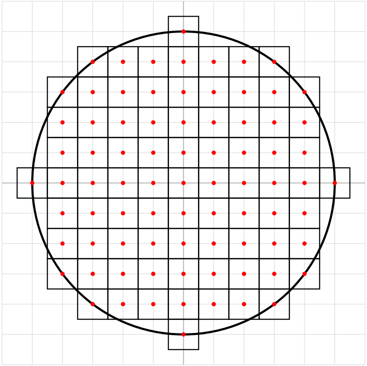
Theorem

The average order of $r(n)$ is π.

The proof of this statement uses a similar argument to the one we used for $d(n)$. The partial sum $\sum_{n \leq x} r(n)$ counts all the lattice points inside a circle of radius $\sqrt{x}$ centered at the origin.

Geometric Representation of $r(n)$

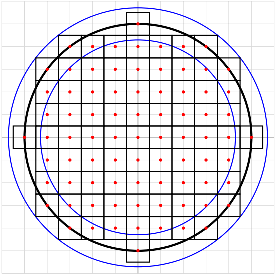
The technique used to count these lattice points involves approximating the area of our circle, πx, with unit squares centered at each point. We now need to take into account the magnitude of error.



$\sum_{n \leq x} r(n) = 81$ for $x = 25$.

Counting Lattice Points in a Circle

Since the distance from the center of a unit square to its furthest point (any of the corners) is $\frac{\sqrt{2}}{2}$, our approximated area cannot exceed the area of a circle of radius $\sqrt{x} + \frac{\sqrt{2}}{2}$. Similarly, we claim that our approximated area is at least that of a circle of radius $\sqrt{x} - \frac{\sqrt{2}}{2}$. To show this, consider an arbitrary point P in our circle of radius $\sqrt{x} - \frac{\sqrt{2}}{2}$. Now, let Q be the nearest lattice point to P. The distance between P and Q is at most $\frac{\sqrt{2}}{2}$. By triangle inequality, this means point Q lies within our original circle of radius $\sqrt{x}$ centered at the origin. Therefore, our approximated area includes the unit square centered at Q, which must therefore include P as desired.



Counting Lattice Points in a Circle

Now, we have

$$\pi\left(\sqrt{x}-\frac{\sqrt{2}}{2}\right)^2 \le \sum_{n\le x} r(n) \le \pi\left(\sqrt{x}+\frac{\sqrt{2}}{2}\right)^2.$$

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After simplifying, we get

$$\sum_{n\leq x} r(n) = \pi x + O(\sqrt{x}).$$

So, the average order of $r(n)$ is π.

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Remark

The generalized sum of squares function $r_k(n)$ also has an average order. To see this, consider the volume of a $k-$dimensional ball.

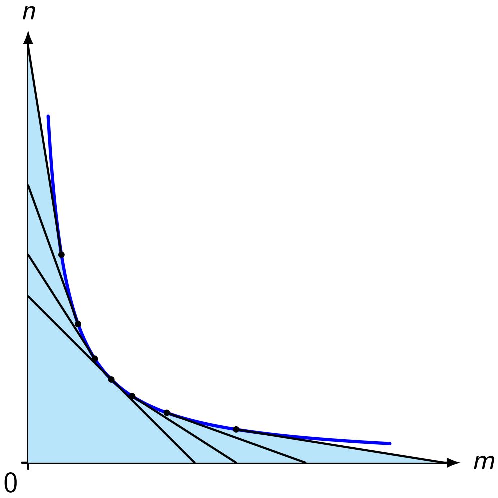
Bounding Error Terms

For both $d(n)$ and $r(n)$, the error term on their respective partial sums was $O(\sqrt{x})$. However, we can do better than this. Dirichlet's Divisor Problem asks what the best error bound is on partial sums for $d(n)$, and the Gauss Circle Problem does the same for $r(n)$. Over the past century, mathematicians have been finding better and better bounds. Coincidentally, error bounds found for one function were exactly the same for the other. This is due to the similar nature of the problems, and also the methods used to determine the error bound. The best error bound $O(x^{\theta})$ is conjectured at $\theta = \frac{1}{4}$, since it was proven by Hardy and Landau that a bound better than that is impossible. The best current bound to date was proven by Martin Huxley in 2003 to be $\theta = \frac{131}{416} \approx 0.31$.

θ	approx.	date
$\frac{1}{2}$	0.50000	
$\frac{1}{3}$	0.33333	1903
$\frac{37}{112}$	0.3306	1925
$\frac{33}{100}$	0.3300	1922
$\frac{27}{82}$	0.3297	1928
$\frac{15}{46}$	0.32609	
$\frac{12}{37}$	0.3242	1963
$\frac{35}{108}$	0.32407	1982
$\frac{139}{429}$	0.32401	
$\frac{17}{53}$	0.32075	1935
$\frac{7}{22}$	0.31818	1988
$\frac{23}{73}$	0.31507	1993
$\frac{131}{416}$	0.31490	2003

Voronoi's 1903 Proof

In my paper, I go over Voronoi's original proof from 1903 for $\theta = \frac{1}{3}$. Here's a diagram that depicts how Voronoi dissected the region underneath the hyperbola.



Euler's Totient Function

Theorem

Euler's totient function $\phi(n)$ has average order $\frac{6n}{\pi^2}$.

Here is a very brief overview of the proof. The proof uses the following relation.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

where μ is the Möbius function. Then, the proof considers series of the form $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$. From here, one can show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

using products of Dirichlet series. After some simplifying and Corollaries of the Euler Summation formula, the proof is complete.

Applications of the Totient Function

While the proof for the average order of the Totient function didn't use lattice points, there is a very neat application involving lattice points.

Remark

When finding the average order of $\phi(n)$, the following result is obtained, and it is what we will be using later.

$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x)$$

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$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x)$$

Definition

A lattice point P is said to be visible from the origin if the line segment joining $(0,0)$ to P does not pass through any other lattice points.

Density of Visible Lattice Points

Theorem

A non-zero lattice point (a, b) *is visible from the origin iff* $\gcd(a, b) = 1$.

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The probability that a randomly selected lattice point is visible from the origin is $\frac{6}{\pi^2}$.

Note that this is equivalent to saying that the density of visible lattice points from the origin is $\frac{6}{\pi^2} \approx 61\%$.

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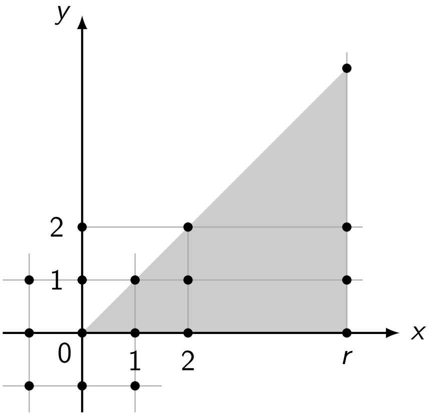
To prove this, consider squares of side length $2r$ centered at the origin, where r is an integer. We want to find the ratio of visible points to total points contained in this square as $r \to \infty$. To count the number of points visible from the origin, we use symmetry by dividing the coordinate plane into 8 sections using the lines $y = x$, $y = -x$, $x = 0$, and $y = 0$.

Density of Visible Lattice Points



$$\sum_{0 \le x \le r} \sum_{\substack{0 \le y < x \\ \gcd(x,y)=1}} 1 = \sum_{0 \le x \le r} \phi(x)$$

$$= \frac{3r^2}{\pi^2} + O(r \log r)$$

Density of Visible Lattice Points

We now need to multiply our expression $\frac{3r^2}{\pi^2} + O(r \log r)$ by 8, and divide by the total number of points in our square, $(2r+1)^2 = 4r^2 + O(r)$.

$$\frac{8\left(\frac{3r^2}{\pi^2} + O(r \log r)\right)}{4r^2 + O(r)} = \frac{\frac{6}{\pi^2} + O\left(\frac{\log r}{r}\right)}{1 + O\left(\frac{1}{r}\right)}$$

which approaches $\frac{6}{\pi^2}$ as $r \to \infty$. $\blacksquare$

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which approaches $\frac{6}{\pi^2}$ as $r \to \infty$. ■

From here, we directly obtain the following result.

Corollary

The probability two random integers are relatively prime is $\frac{6}{\pi^2}$.

Sum of Divisors Function

Theorem

The average order of $\sigma(n)$ *is* $\frac{n\pi^2}{6}$.

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Proof.

Letting $q = \frac{n}{d}$, we write our partial sum as a double sum.

$$
\begin{aligned}
\sum_{n\le x} \sigma(n) = \sum_{n\le x} \sum_{d|n} d &= \sum_{d\le x} \sum_{q\le \frac{x}{d}} q \\
&= \sum_{d\le x} \left(\frac{1}{2} \left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right) \right) \\
&= \frac{x^2}{2} \sum_{d\le x} \frac{1}{d^2} + O\left(x \sum_{d\le x} \frac{1}{d} \right) \\
&= \frac{x^2}{2} \left(\zeta(2) - \frac{1}{x} + O\left(\frac{1}{x^2}\right) \right) + O(x\log x) \\
&= \frac{\zeta(2)x^2}{2} + O(x\log x)
\end{aligned}
$$

Sum of Divisors Function

Thus we have shown that

$$\sum_{n\leq x}\sigma(n)\sim\frac{\pi^2x^2}{12}\sim\sum_{n\leq x}\frac{\pi^2n}{6}$$

as desired. $\blacksquare$

Sum of Divisors Function

Thus we have shown that

$$\sum_{n \leq x} \sigma(n) \sim \frac{\pi^2 x^2}{12} \sim \sum_{n \leq x} \frac{\pi^2 n}{6}$$

as desired. ∎

Like with $r(n)$, $\sigma(n)$ is a special case of a more general function $\sigma_k(n)$, which sums the divisors of n raised to the kth power. We can also find the average order of this generalized function, though we won't prove it here.

Theorem

For $x, k > 0$,

$$\sum_{n \leq x} \sigma_k(n) \sim \frac{\zeta(k+1)}{k+1} x^{k+1}.$$

Thank you

Thank you for your attention!