ON AVERAGE ORDERS OF ARITHMETIC FUNCTIONS

LARRY ZHAO

Abstract. This expository paper discusses average orders of arithmetic functions and their error terms. Geometric methods and interpretations are used whenever possible, as this creates interesting connections between analytic number theory and lattice-point geometry. We will present the average orders of four common arithmetic functions. Then, we will discuss an application of the asymptotics for the Euler's totient function in counting visible lattice points. Finally, we will discuss Voronoï's contribution to the Dirichlet Divisor Problem, leaving out some of the more technical details. Such presentations of Voronoï's original geometric proof are hard to find in English, as he wrote his 1903 paper in French. The aim is to provide students with a strong foundation in analytic number theory, while also illustrating the essence of geometric methods in a field that seems purely analytical.

1. Introduction

When studying arithmetic functions, it is often more useful to consider their average orders, because the functions themselves can often fluctuate, e.g. divisor function on primes. Dirichlet is often cited as the first mathematician to study these average orders.

Definition 1.1. The *average order* of an arithmetic function f is g if

$$\sum_{n\leq x} f(n) \sim \sum_{n\leq x} g(n).$$

The four functions we will discuss in this paper are the following.

- Divisor function - $d(n)$ or $\sigma_0(n)$
- Totient function - $\phi(n)$
- Sum of divisors function - $\sigma_1(n)$
- Sum of squares function - $r_2(n)$

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2. Background

Theorem 2.1 (Euler's summation formula)**.** *If f has a continuous derivative f' on $[a, b]$, then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\ dt + \int_a^b f'(t)(t - \lfloor t\rfloor)\ dt + f(a)(a - \lfloor a\rfloor) - f(b)(b - \lfloor b\rfloor).$$

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Proof. We have

$$\begin{aligned}
\sum_{a<n\le b} f(n) &= \int_a^b f(t)\ d\lfloor t\rfloor \\
&= \int_a^b f(t)\ d(t-(t-\lfloor t\rfloor)) \\
&= \int_a^b f(t)\ dt - \int_a^b f(t)\ d(t-\lfloor t\rfloor) \\
&= \int_a^b f(t)\ dt - \left([f(t)(t-\lfloor t\rfloor)]_a^b - \int_a^b f'(t)(t-\lfloor t\rfloor)\ dt\right) \\
&= \int_a^b f(t)\ dt + \int_a^b f'(t)(t-\lfloor t\rfloor)\ dt + f(a)(a-\lfloor a\rfloor) - f(b)\cdot(b-\lfloor b\rfloor)
\end{aligned}$$

■

The proofs of the following two corollaries are not particularly interesting so we won't discuss them here.

Corollary 2.2. *If* $x \ge 1$, $\sum_{n\le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$

Corollary 2.3. *If* $x \ge 1$, $\sum_{n\le x} \frac{1}{n^2} = \zeta(2) - \frac{1}{x} + O(\frac{1}{x^2})$

3. Average Order of $d(n)$

Definition 3.1. The function $d(n)$ counts the number of positive divisors of n. Equivalently,

$$d(n) = \sum_{d\mid n} 1.$$

We will prove a weaker version of Dirichlet's asymptotic formula for partial sums of $d(n)$, though we will still find the average order of $d(n)$.

Theorem 3.2. *The function* $f(x) = \frac{1}{x}\sum_{n\le x} d(n)$ *is asymptotic to* $\log x$*. More precisely, for* $x \ge 1$,

$$\sum_{n\le x} d(n) = x\log x + O(x).$$

Proof. We can rewrite the LHS as

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d\mid n} 1.$$

Now, to swap the double sum, let $q = \frac{n}{d}$.

$$\begin{aligned}
\sum_{n\le x} d(n) &= \sum_{d\le x}\sum_{q\le\frac{x}{d}} 1 \\
&= \sum_{d\le x}\lfloor\frac{x}{d}\rfloor \\
&= \sum_{d\le x}\left(\frac{x}{d} + O(1)\right)
\end{aligned}$$

Using Corollary 2.2, we have

$$\begin{aligned}\sum_{n\le x} d(n) &= x\sum_{d\le x}\frac{1}{d} + O(x)\\ &= x\left(\log x + \gamma + O\left(\frac{1}{x}\right)\right) + O(x)\\ &= x\log x + O(x)\end{aligned}$$

Therefore, the average

$$f(x) = \frac{1}{x}\sum_{n\le x} d(n) = \log x + \frac{O(x)}{x},$$

so $f(x) \sim \log x$ as $x \to \infty$, i.e. $\lim_{x\to\infty} \frac{f(x)}{\log(x)} = 1$. ∎

The above proof gives an error bound of $O(x)$. We will now prove Dirichlet's error bound of $O\left(x^{\frac{1}{2}}\right)$, using Dirichlet's hyperbola method.

Theorem 3.3. *For all $x \ge 1$,*

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

The following proof is adapted from [Apo76, Chapter 3].

Proof. Since

$$\sum_{n\le x} d(n) = \sum_{\substack{q,d\\ qd\le x}} 1,$$

we are trying to count lattice points (q, d) under the hyperbola in the shaded region depicted in Figure 1. Using symmetry across the line $q = d$, we can count lattice points our half-section row-by-row. We first account for the diagonal, which contains $\lfloor\sqrt{x}\rfloor$ lattice points. Now, if we sum over $1 \le d \le \sqrt{x}$, for each row, we start counting at $(d, d+1)$ and end on $(d, \lfloor\frac{x}{d}\rfloor)$, N.B. we aren't counting (d, d) since that was already accounted for on the diagonal. So, each row will have $\lfloor\frac{x}{d}\rfloor - d$ lattice points. Thus, we have

$$\sum_{n\le x} d(n) = 2\sum_{d\le\sqrt{x}}\left(\lfloor\frac{x}{d}\rfloor - d\right) + \lfloor\sqrt{x}\rfloor.$$

Now, we can do some rearranging and apply Corollary 2.2.

$$\begin{aligned}\sum_{n\le x} d(n) &= 2\sum_{d\le\sqrt{x}}\left(\frac{x}{d} + O(1) - d\right) + O(\sqrt{x})\\ &= 2x\sum_{d\le\sqrt{x}}\frac{1}{d} - 2\sum_{d\le\sqrt{x}} d + O(\sqrt{x})\\ &= 2x\left(\log\sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\right) - 2\left(\frac{\sqrt{x}^2}{2} + O(\sqrt{x})\right) + O(\sqrt{x})\\ &= x\log x + (2\gamma - 1)x + O(\sqrt{x})\end{aligned}$$

as desired. ∎

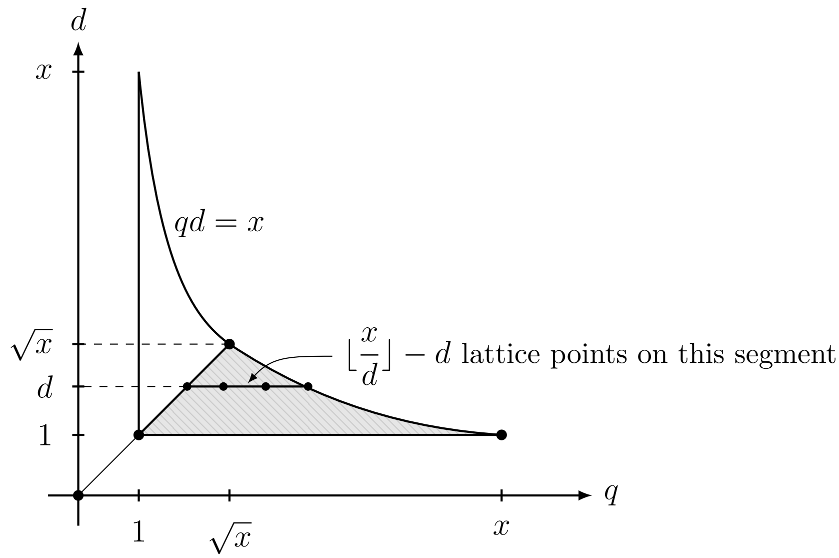

Figure 1. The hyperbola method for counting divisor pairs. Adapted from [Apo76, Chapter 3]

We have now proved the bound of $\theta = \frac{1}{2}$.

4. Average Order of $\phi(n)$

First, let us present some definitions.

Definition 4.1. Euler's *totient function*, denoted $\phi(n)$, counts the number of positive integers not more than n that are relatively prime to n. Equivalently,

$$\phi(n) = \sum_{\substack{k \le n \\ \gcd(k,n)=1}} 1.$$

Definition 4.2. The *Möbius function*, denoted $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^k & n \text{ is the product of } k \text{ distinct primes,} \\ 0 & n \text{ is divisible by the square of a prime.} \end{cases}$$

Using the Möbius inversion formula on the well known identity $n = \sum_{d|n} \phi(d)$, we get

$$\phi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right).$$

A more detailed discussion of this can be found in [Apo76, Chapter 2].

Definition 4.3. The *Riemann zeta function*, denoted $\zeta(s)$, is defined for complex numbers s with $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In the case of $s = 2$, the solution to the famous Basel problem tells us that

$$\zeta(2) = \frac{\pi^2}{6}.$$

We now prove a result about the Möbius function.

Lemma 4.4. *For any complex s where $\Re(s) > 1$,*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Proof. First we prove a well-known identity about the Möbius function:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1, \\ 0 & n > 1. \end{cases}$$

The case where $n = 1$ is trivial. Then, for $n > 1$, we can write $n = p_1^{e_1} \cdots p_k^{e_k}$. In the sum $\sum_{d|n} \mu(d)$, we only care about the non-zero terms which don't have powers of primes larger than 1. So, our sum goes over all 2^k possible cases.

$$\begin{aligned} \sum_{d|n} \mu(d) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_k) + \cdots + \mu(p_1 \cdots p_k) \\ &= \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots + \binom{k}{k}(-1)^k = (1-1)^k = 0 \end{aligned}$$

Now, consider the product of two Dirichlet series which converge since $\Re(s) > 1$.

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)$$

Terms of this product are of the form $\frac{\mu(n)}{(mn)^s}$. If we let $k = mn$, we can write this product as a double sum, where the outer sum goes over all products k, and the inner sum goes over factors $n|k$.

$$\begin{aligned} \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} &= \sum_{k=1}^{\infty} \sum_{n|k} \frac{\mu(n)}{k^s} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k^s} \sum_{n|k} \mu(n)\right) \end{aligned}$$

Here, the inner sum is zero for all $k > 1$ (as we showed earlier in this proof), so the entire sum simplifies to 1. Thus, we get that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

■

Theorem 4.5. *Euler's totient function $\phi(n)$ has average order $\frac{6n}{\pi^2}$.*

Proof. Using the identity

$$\phi(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right),$$

we have

$$\begin{aligned}\sum_{n\leq x} \phi(n) &= \sum_{n\leq x} \sum_{d|n} \mu(d) \left(\frac{n}{d}\right) \\ &= \sum_{d\leq x} \sum_{q\leq \frac{x}{d}} \mu(d) q.\end{aligned}$$

Where $q = \frac{n}{d}$. Then, we have

$$\begin{aligned}\sum_{n\leq x} \phi(n) &= \sum_{d\leq x} \mu(d) \sum_{q\leq \frac{x}{d}} q \\ &= \sum_{d\leq x} \mu(d) \left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right) \\ &= \frac{x^2}{2} \sum_{d\leq x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d\leq x} \frac{|\mu(d)|}{d}\right).\end{aligned}$$

For the second term,

$$O\left(x \sum_{d\leq x} \frac{|\mu(d)|}{d}\right) = O\left(x \sum_{d\leq x} \frac{1}{d}\right) = O(x \log x),$$

since $|\mu(t)| \leq 1$ for any natural t. Now, for the first term, applying Lemma 4.4 and Theorem 2.1, we have

$$\begin{aligned}\frac{x^2}{2} \sum_{d\leq x} \frac{\mu(d)}{d^2} &= \frac{x^2}{2} \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d>x} \frac{\mu(d)}{d^2}\right) \\ &= \frac{x^2}{2} \left(\frac{1}{\zeta(2)} + O\left(\sum_{d>x} \frac{1}{d^2}\right)\right) \\ &= \frac{x^2}{2} \left(\frac{6}{\pi^2} + O\left(\frac{1}{x}\right)\right).\end{aligned}$$

Now, we have

$$\sum_{n\leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x) + O(x \log x) = \frac{3x^2}{\pi^2} + O(x \log x).$$

Thus, we have showed that $\sum_{n\leq x} \phi(n) \sim \frac{3x^2}{\pi^2}$ as $x \to \infty$. However, to actually find the average order of $\phi(n)$, we need to find the function $g(n)$ such that $\sum_{n\leq x} g(n) = \frac{3x^2}{\pi^2}$. Let

$g(n) = c \cdot n$. Then, we have

$$\sum_{n \le x} g(n) = c \sum_{n \le x} n$$
$$= c\left(\frac{x^2}{2} + O(x)\right) = \frac{3x^2}{\pi^2}.$$

Therefore,

$$c = \frac{6}{\pi^2},$$

so we can write

$$\sum_{n \le x} \phi(n) \sim \sum_{n \le x} \frac{6n}{\pi^2}.$$

Thus, $\phi(n)$ has average order $\frac{6n}{\pi^2}$. ■

5. Euclid's Orchard

A nice application of the average order of Euler's totient function lies in Euclid's Orchard, a problem of determining which lattice points are "visible" from the origin.

Definition 5.1. A lattice point P is said to be visible from the origin if the line segment joining $(0, 0)$ to P does not pass through any other lattice points.

Since trees in Euclid's Orchard are planted on lattice points, our definition of visibility makes intuitive sense.

Theorem 5.2. *A lattice point (a, b) is visible from the origin iff $\gcd(a, b) = 1$.*

Proof. For the forward direction, assume for the sake of a contradiction that the point (a, b) is not visible from the origin. Then, there is some other point (c, d) lying on the line segment connecting $(0, 0)$ and (a, b). Therefore, $\frac{c}{d} = \frac{a}{b}$. Equivalently,

$$a = kc \tag{1}$$

and

$$b = kd \tag{2}$$

for some rational $k > 1$. Let $k = \frac{p}{q}$ where $\gcd(p, q) = 1$. From the two equations above, we get

$$aq = pc, \quad bq = pd.$$

Since $\gcd(p, q) = 1$, q divides both c and d. Let $c = mq$ and $d = nq$. Then

$$a = pm, \quad b = pn.$$

Thus a and b share the common factor $p > 1$, contradicting $\gcd(a, b) = 1$.

For the reverse direction, assume for the sake of a contradiction that $\gcd(a, b) = g > 1$. Then, for integers c and d, we have

$$a = gc \quad b = gd.$$

The point (c, d) is a lattice point on the line segment connecting $(0, 0)$ and (a, b) since $\frac{a}{b} = \frac{c}{d}$, so we have a contradiction. ■

Theorem 5.3. *The probability that a randomly selected lattice point is visible from the origin is $\frac{6}{\pi^2}$.*

Proof. Consider squares of side length $2r$ centered at the origin, where r is an integer. We want to find the ratio of visible points to total points contained in this square as $r \to \infty$. To count the number of points visible from the origin, we use symmetry by dividing the coordinate plane into 8 sections using the lines $y = x$, $y = -x$, $x = 0$, and $y = 0$, as shown in Figure 2.

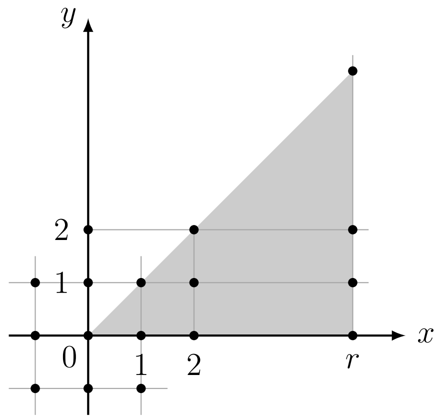

Figure 2. Adapted from [Apo76, Chapter 3]

Writing this as a double sum (we exclude the origin), we have

$$\sum_{1 \le x \le r} \sum_{\substack{0 \le y < x \\ \gcd(x,y)=1}} 1 = \sum_{0 \le x \le r} \phi(x)$$

$$= \frac{3r^2}{\pi^2} + O(r \log r)$$

by Theorem 4.5. Now, we just need to multiply our expression by 8, and divide by the total number of points in the square, $(2r+1)^2 = 4r^2 + O(r)$.

$$\frac{8\left(\frac{3r^2}{\pi^2} + O(r \log r)\right)}{4r^2 + O(r)} = \frac{\frac{6}{\pi^2} + O\left(\frac{\log r}{r}\right)}{1 + O\left(\frac{1}{r}\right)},$$

which approaches $\frac{6}{\pi^2}$ as $r \to \infty$. ■

Then, by Theorem 5.2, the probability that two integers a and b are relatively prime is $\frac{6}{\pi^2}$.

6. Average order of $\sigma(n)$

Definition 6.1. The *sum-of-divisors function*, denoted $\sigma(n)$, is the sum of the positive divisors of n. Equivalently,

$$\sigma(n) = \sum_{d \mid n} d.$$

Theorem 6.2. *The average order of* $\sigma(n)$ *is* $\frac{n\pi^2}{6}$.

Proof. We have

$$\sum_{n \le x} \sigma(n) = \sum_{n \le x} \sum_{d \mid n} d.$$

Letting $q = \frac{n}{d}$, we swap the double sum.

$$\begin{aligned}
\sum_{n\le x}\sigma(n) &= \sum_{d\le x}\sum_{q\le \frac{x}{d}} q \\
&= \sum_{d\le x}\left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right) \\
&= \frac{x^2}{2}\sum_{d\le x}\frac{1}{d^2} + O\left(x\sum_{d\le x}\frac{1}{d}\right) \\
&= \frac{x^2}{2}\left(\zeta(2) - \frac{1}{x} + O\left(\frac{1}{x^2}\right)\right) + O(x\log x) \\
&= \frac{\zeta(2)x^2}{2} + O(x\log x) \\
&= \frac{\pi^2 x^2}{12} + O(x\log x).
\end{aligned}$$

Thus we have shown that

$$\sum_{n\le x}\sigma(n) \sim \frac{\pi^2 x^2}{12}$$

as $x \to \infty$. Now, we find the actual average order $g(n) = cn$ such that $\sum_{n\le x}\sigma(n) \sim \sum_{n\le x} g(n)$.

$$\begin{aligned}
\sum_{n\le x} g(n) &= c\sum_{n\le x} n \\
&= c\left(\frac{x^2}{2} + O(x)\right) = \frac{\pi^2 x^2}{12}
\end{aligned}$$

so $c = \frac{\pi^2}{6}$. Thus, the average order of $\sigma(n)$ is $\frac{\pi^2 n}{6}$.

■

7. Sum Of Squares Function – $r_k(n)$

Definition 7.1. The *sum-of-squares function*, denoted $r_k(n)$, counts the number of integer k-tuples $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ such that

$$x_1^2 + \cdots + x_k^2 = n.$$

We will mainly be concerned with the case when $k = 2$, so we will simply write $r(n) = r_2(n)$. Additionally, $r(n)$ can be geometrically interpreted as counting the number of lattice points on the circle of radius $\sqrt{n}$ centered at the origin.

Theorem 7.2. *The average order of $r(n)$ is π.*

Proof. We consider the partial sums

$$\sum_{n\le x} r(n),$$

which counts all lattice points contained in a circle of radius $\sqrt{x}$ centered at the origin. The area of the circle is $\pi(\sqrt{x})^2 = \pi x$. We can also approximate this area by placing a square of side length one centered at every lattice point in the circle. This is an approximation, so we now need to take into account the magnitude of error.

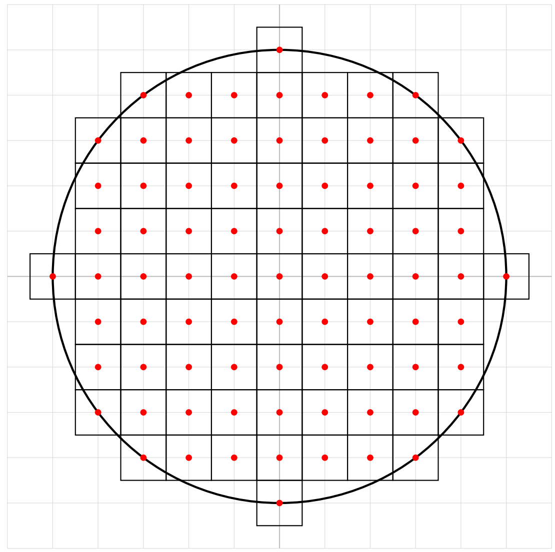

Figure 3. Counting lattice points by approximating the area of a circle with unit squares.

Since the distance from the center of a unit square to its furthest point (any of the corners) is $\frac{\sqrt{2}}{2}$, our approximated area cannot exceed the area of a circle of radius $\sqrt{x}+\frac{\sqrt{2}}{2}$. Similarly, we claim that our approximated area is at least that of a circle of radius $\sqrt{x}-\frac{\sqrt{2}}{2}$. To show this, consider an arbitrary point P in our circle of radius $\sqrt{x}-\frac{\sqrt{2}}{2}$. Now, let Q be the nearest lattice point to P. As we discussed earlier, the distance between P and Q is at most $\frac{\sqrt{2}}{2}$. By the triangle inequality, this means point Q lies within our original circle of radius $\sqrt{x}$ centered at the origin. Therefore, our approximated area includes the unit square centered at Q, which must therefore include P as desired. Now, we have

$$\pi\left(\sqrt{x}-\frac{\sqrt{2}}{2}\right)^2 \leq \sum_{n\leq x} r(n) \leq \pi\left(\sqrt{x}+\frac{\sqrt{2}}{2}\right)^2.$$

After simplifying, we get

$$\sum_{n\leq x} r(n) = \pi x + O(\sqrt{x}).$$

So, the average order of $r(n)$ is π.

■

8. Connection Between $d(n)$ and $r(n)$

The two arithmetic functions $d(n)$ and $r(n)$ are related in terms of the error term on their partial sum formulae. Just like the Dirichlet Divisor Problem mentioned earlier, there is an analogous Gauss Circle Problem, which asks the same question for the error term on the partial sum formula for $r(n)$. Historically, progress made towards the conjectured $\theta = \frac{1}{4}$ has always been identical for both problems. The most recent progress was in 2003, when Huxley proved the bound of $\theta = 131/416$ for the Dirichlet Divisor Problem, and $\theta = 131/208$ for the Gauss Circle Problem using exponential sums. More can be read about this in [Hux03]. Note that the difference by a factor of two comes merely from whether or not we count lattice points in a circle of radius x or $\sqrt{x}$ (the former is the convention for the Gauss Circle Problem). Since we're more concerned with average orders, we will be dealing with circles of radius up to $\sqrt{x}$.

The reason for this apparent coincidence lies in the mathematical tools used in their respective proofs, and the nature of the problems themselves, i.e. counting lattice points under (or inside) curved regions.

9. Dirichlet's Divisor Problem

Dirichlet's divisor problem asks what's the best possible error term for the formula for partial sums of $d(n)$. Specifically, what's the smallest θ in the error term $O(x^\theta)$. We showed in Theorem 3.3 that we can achieve $\theta = \frac{1}{2}$. In 1903, Voronoi showed the even stronger bound of $\theta = \frac{1}{3}$ (up to a logarithmic factor), still using a geometric approach which I will present based on [Vor03]. Because mathematicians still have not reached the conjectured $\theta = \frac{1}{4}$, this remains one of the largest open problems in analytic number theory.

Theorem 9.1. *The error term for partial sums of $d(n)$ can be bounded as $O\left(x^{\frac{1}{3}} \log x\right)$.*

In the following proof, we will omit some of the more technical algebraic manipulations for the sake of clarity and convenience.

Proof. Let $F(x) = \sum_{(S)} f(m,n) = \sum_{n \le x} d(n)$ where $f(m,n) = 1$ and $S = \{(m,n) \in \mathbb{N}^2 : mn \le x\}$.

9.1. Dissecting The Region Under The Hyperbola.

To count lattice points in S, we need to dissect it into many smaller regions. To do this, consider a sequence of k points $P_1, \ldots, P_k = (m_1, n_1), \ldots, (m_k, n_k)$ which all lie on our hyperbola (we will go into more detail later in the proof on how we selected these points). Also, include the two points P_0 and P_{k+1} which are located at positive infinity on the m and n axes respectively. Now, we draw tangent lines to the hyperbola at each P_i, along with segments connecting adjacent points on the hyperbola as shown in Figure 4.

After some quick computation, the equation of the tangent line at P_i is given by

$$n_i m + m_i n = 2x.$$

Thus, we have split the region under the hyperbola into a polygon, and many "triangles" which lie between our polygon and the hyperbola. Each of the $k+1$ triangles S_i is enclosed by two adjacent tangent lines and the hyperbola, so each can be represented by the set of inequalities

$$S_i = \begin{cases} 2x < mn_i + nm_i \le m_{i+1}n_i + n_{i+1}m_i, \\ 2x < mn_{i+1} + nm_{i+1} \le m_i n_{i+1} + n_i m_{i+1}, \\ mn \le x. \end{cases}$$

Now, we will discuss how to select our points P_i using Farey sequences. Let $t \ge 1$ be some real number, and consider all coprime pairs (a,b) such that $ab \le t$. We can arrange the

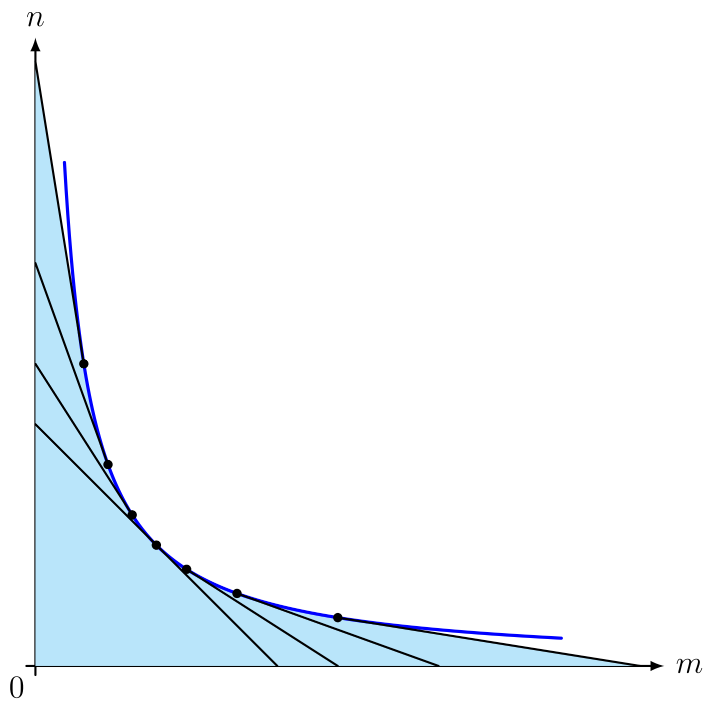

Figure 4. Voronoï's tangent dissection of the region under the hyperbola.

following fractions in order:

$$\frac{a_1}{b_1} > \cdots > \frac{a_k}{b_k}.$$

We also define $a_0 = 1$ and $b_0 = 0$, and $a_{k+1} = 0$ and $b_{k+1} = 1$. We will now show that (by construction) for two adjacent fractions $\frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1}}$,

$$a_i b_{i+1} - a_{i+1} b_i = 1.$$

To construct our sequence of fractions to preserve the above property, we take mediants of fractions. For the initial case between our two outermost fractions, we have

$$a_0 b_{k+1} - a_{k+1} b_0 = (1 \cdot 1) - (0 \cdot 0) = 1.$$

as desired. Next, define the mediant between two fractions $\frac{\alpha'}{\beta'} > \frac{\alpha''}{\beta''}$ to be $\frac{\alpha}{\beta}$ where $\alpha = \alpha' + \alpha''$ and $\beta = \beta' + \beta''$. Our new fraction lies between our two original fractions, and we want to show that our desired property holds on both sides

$$\frac{\alpha'}{\beta'} > \frac{\alpha}{\beta} > \frac{\alpha''}{\beta''}.$$

For the left side,

$$\begin{aligned} \alpha'\beta - \alpha\beta' &= \alpha'(\beta' + \beta'') - (\alpha' + \alpha'')\beta' \\ &= \alpha'\beta'' - \alpha''\beta' = 1, \end{aligned}$$

and similarly for the right side. Thus, we take mediants as long as $\alpha_i \beta_i \leq t$ until we achieve our desired sequence.

Now we want to convert each fraction to a point $P_i = (m_i, n_i)$ on the hyperbola. We can do this by letting

$$m_i = \sqrt{\frac{a_i}{b_i}x}$$

and

$$n_i = \sqrt{\frac{b_i}{a_i}x}$$

which multiply to x as desired. On our boundary fractions, we get the desired points $P_{k+1} = (0, \infty)$ and $P_0 = (\infty, 0)$. With these substitutions, we can rewrite our inequalities for S_i. We start with the lower bound on our first inequality

$$\begin{aligned}
2x &< mn_i + nm_i \\
2x &< m\sqrt{\frac{b_i}{a_i}x} + n\sqrt{\frac{a_i}{b_i}x} \\
2x &< \sqrt{\frac{x}{a_ib_i}}\,(mb_i + na_i) \\
2x\sqrt{\frac{a_ib_i}{x}} &< \sqrt{\frac{x}{a_ib_i}}\,(mb_i + na_i) \\
2\sqrt{a_ib_ix} &< mb_i + na_i.
\end{aligned}$$

Now for the upper bound of $m_{i+1}n_i + n_{i+1}m_i$, we have

$$\begin{aligned}
m_{i+1}n_i + n_{i+1}m_i &= \left(\sqrt{\frac{a_{i+1}}{b_{i+1}}x}\right)\left(\sqrt{\frac{b_i}{a_i}x}\right) + \left(\sqrt{\frac{b_{i+1}}{a_{i+1}}x}\right)\left(\sqrt{\frac{a_i}{b_i}x}\right) \\
&= x\left(\sqrt{\frac{a_{i+1}b_i}{a_ib_{i+1}}} + \sqrt{\frac{a_ib_{i+1}}{a_{i+1}b_i}}\right) \\
&= x\left(\frac{a_ib_{i+1} + a_{i+1}b_i}{\sqrt{a_ib_ia_{i+1}b_{i+1}}}\right).
\end{aligned}$$

Putting this back into our inequality yields

$$\sqrt{\frac{x}{a_ib_i}}(mb_i + na_i) \le x\left(\frac{a_ib_{i+1} + a_{i+1}b_i}{\sqrt{a_ib_i}\cdot\sqrt{a_{i+1}b_{i+1}}}\right).$$

Multiplying both sides by $\sqrt{\frac{a_ib_i}{x}}$ yields

$$\begin{aligned}
mb_i + na_i &\le \frac{x}{\sqrt{x}}\left(\frac{a_ib_{i+1} + a_{i+1}b_i}{\sqrt{a_{i+1}b_{i+1}}}\right) \\
&= (a_ib_{i+1} + a_{i+1}b_i)\sqrt{\frac{x}{a_{i+1}b_{i+1}}}.
\end{aligned}$$

The same algebraic steps can be used on the second inequality. Now, we can rewrite S_i as

$$S_i = \begin{cases} 2\sqrt{a_i b_i x} < mb_i + na_i \leq (a_i b_{i+1} + a_{i+1} b_i)\sqrt{\frac{x}{a_{i+1}b_{i+1}}}, \\ 2\sqrt{a_{i+1} b_{i+1} x} < mb_{i+1} + na_{i+1} \leq (a_i b_{i+1} + a_{i+1} b_i)\sqrt{\frac{x}{a_i b_i}}, \\ mn \leq x. \end{cases}$$

Since our points P_i are constructed in this way, we can draw tangent lines at each point in the same order of the points which will continually split the region under the polygon into triangles. When constructing the points, each triangle T_i is formed by the tangents of two adjacent points and the tangent to the point corresponding to the mediant fraction. For T_i the three fractions corresponding to the three tangent lines are $\frac{\alpha_i'}{\beta_i'} > \frac{\alpha_i}{\beta_i} > \frac{\alpha_i''}{\beta_i''}$. Note that we are now using Greek letters when dealing with the inner triangles, since they are formed by using the mediant, rather than simply two adjacent points for S_i. We define T_i with the following inequalities detailed above.

$$T_i = \begin{cases} m\beta_i' + n\alpha_i' > 2\sqrt{\alpha_i'\beta_i' x}, \\ m\beta_i'' + n\alpha_i'' > 2\sqrt{\alpha_i''\beta_i'' x}, \\ m\beta_i + n\alpha_i \leq 2\sqrt{\alpha_i\beta_i x}. \end{cases}$$

Now that we have inequalities for both the curvilinear triangles S_i and main triangles T_i, we want to make it easier to count lattice points. We do this via a change of variable, to make our inequalities nicer. We define

$$mb_i + na_i = m' \quad \text{and} \quad mb_{i+1} + na_{i+1} = n'.$$

Recall from earlier that we have

$$a_i b_{i+1} - a_{i+1} b_i = 1,$$

We also use our earlier system to isolate m and n.

$$m = -m'a_{i+1} + n'a_i \quad \text{and} \quad n = m'b_{i+1} - n'b_i.$$

We have a one-to-one correspondence between points (m, n) and transformed points (m', n'). If we let the corresponding S_i' be the set of points (m', n'), we can write

$$S_i' = \begin{cases} 2\sqrt{a_i b_i x} < m' \leq (a_i b_{i+1} + a_{i+1} b_i)\sqrt{\frac{x}{a_{i+1}b_{i+1}}}, \\ 2\sqrt{a_{i+1} b_{i+1} x} < n' \leq (a_i b_{i+1} + a_{i+1} b_i)\sqrt{\frac{x}{a_i b_i}}, \\ (-m'a_{i+1} + n'a_i)(m'b_{i+1} - n'b_i) \leq x. \end{cases}$$

We can also substitute into our earlier inequalities for T_i, calling our corresponding set of points here T_i'.

$$T_i' = \begin{cases} m' > 2\sqrt{\alpha_i'\beta_i' x}, \\ n' > 2\sqrt{\alpha_i''\beta_i'' x}, \\ m' + n' \leq 2\sqrt{\alpha_i\beta_i x}. \end{cases}$$

We now have transformed inequalities S_i' and T_i'. Now, we have

$$\sum_{(S)} f(m,n) = \sum_{i=0}^{k} \sum_{(S_i')} f(-m'a_{i+1} + n'a_i,\, m'b_{i+1} - n'b_i) + \sum_{i=1}^{k} \sum_{(T_i')} f(-m'\alpha_i'' + n'\alpha_i',\, m'\beta_i'' - n'\beta_i'),$$

Now, we will further dissect S_i', as shown in Figure 5. N.B. I've slightly changed the lettering here to have $P = (\mu', \nu')$.

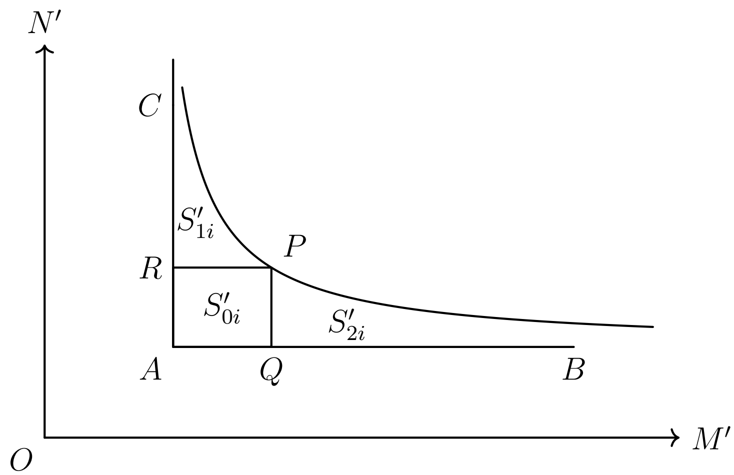

Figure 5. The decomposition of S_i' into S_{0i}', S_{1i}', and S_{2i}'.

To be clear, S_{0i}' consists solely of the rectangle $ARPQ$, while both S_{1i}' and S_{2i}' include their curved sections and the rectangle. Our three new regions are defined by the following inequalities:

$$(S_{0i}') = \quad 2\sqrt{a_ib_ix} < m' \le \mu', \quad 2\sqrt{a_{i+1}b_{i+1}x} < n' \le \nu',$$

$$(S_{1i}') = \quad \begin{cases} 2\sqrt{a_ib_ix} < m' \le \mu', \\ 2\sqrt{a_{i+1}b_{i+1}x} < n' \le (a_ib_{i+1} + a_{i+1}b_i)\sqrt{\dfrac{x}{a_ib_i}}, \\ (-m'a_{i+1} + n'a_i)(m'b_{i+1} - n'b_i) \le x, \end{cases}$$

$$(S_{2i}') = \quad \begin{cases} 2\sqrt{a_ib_ix} < m' \le (a_ib_{i+1} + a_{i+1}b_i)\sqrt{\dfrac{x}{a_{i+1}b_{i+1}}}, \\ 2\sqrt{a_{i+1}b_{i+1}x} < n' \le \nu', \\ (-m'a_{i+1} + n'a_i)(m'b_{i+1} - n'b_i) \le x. \end{cases}$$

We can further simplify our inequalities for S_{1i}' and S_{2i}'. Assume $a_ib_i > 0$. We multiply

$$(-m'a_{i+1} + n'a_i)(m'b_{i+1} - n'b_i) \le x$$

by $-4a_ib_i$, which yields

$$(-2a_ib_in' + m'(a_ib_{i+1} + a_{i+1}b_i))^2 \ge m'^2 - 4a_ib_ix. \tag{3}$$

We claim that the following expression is positive

$$-2a_ib_in' + m'(a_ib_{i+1} + a_{i+1}b_i).$$

From our inequalities for S'_{1i}, we have

$$-2a_ib_in' \geq -(a_ib_{i+1} + a_{i+1}b_i)\, 2\sqrt{a_ib_ix},$$

$$(a_ib_{i+1} + a_{i+1}b_i)m' > (a_ib_{i+1} + a_{i+1}b_i)\, 2\sqrt{a_ib_ix}.$$

Thus,

$$-2a_ib_in' + (a_ib_{i+1} + a_{i+1}b_i)m' > 0.$$

Therefore, we can simplify (3) to

$$-2a_ib_in' + (a_ib_{i+1} + a_{i+1}b_i)m' \geq \sqrt{m'^2 - 4a_ib_ix},$$

yielding

$$n' \leq \frac{(a_ib_{i+1} + a_{i+1}b_i)m' - \sqrt{m'^2 - 4a_ib_ix}}{2a_ib_i}.$$

Now, we can define S'_{1i} as

$$(S'_{1i}) = \begin{cases} 2\sqrt{a_ib_ix} < m' \leq \mu', \\ 2\sqrt{a_{i+1}b_{i+1}x} < n' \leq \dfrac{(a_ib_{i+1} + a_{i+1}b_i)m' - \sqrt{m'^2 - 4a_ib_ix}}{2a_ib_i}. \end{cases}$$

Similarly, we define S'_{2i} as follows.

$$(S'_{2i}) = \begin{cases} 2\sqrt{a_ib_ix} < m' \leq \dfrac{(a_ib_{i+1} + a_{i+1}b_i)n' - \sqrt{n'^2 - 4a_{i+1}b_{i+1}x}}{2a_{i+1}b_{i+1}}, \\ 2\sqrt{a_{i+1}b_{i+1}x} < n' \leq \nu'. \end{cases}$$

Finally, we can write

$$\begin{aligned} &\sum_{(S'_i)} f(\,-m'a_{i+1} + n'a_i,\, m'b_{i+1} - n'b_i) \\ &\quad = \sum_{(S'_{1i})} f(-m'a_{i+1} + n'a_i,\, m'b_{i+1} - n'b_i) \\ &\qquad + \sum_{(S'_{2i})} f(-m'a_{i+1} + n'a_i,\, m'b_{i+1} - n'b_i) \\ &\qquad - \sum_{(S'_{0i})} f(-m'a_{i+1} + n'a_i,\, m'b_{i+1} - n'b_i). \end{aligned}$$

because S'_{1i} and S'_{2i} both include S'_{0i}.

9.2. **Completing the Error Bound.**
Now that we've split the region under the hyperbola into smaller regions defined by nice inequalities, we can start to count lattice points.

Recall that for two adjacent Farey fractions

$$\frac{a}{b} > \frac{a'}{b'}, \quad ab' - a'b = 1.$$

Let $c = a + a'$ and $d = b + b'$. Now let

$$s\begin{pmatrix} a, b \\ a', b' \end{pmatrix}$$

denote the number of integer pairs (m, n) in S_i'. Equivalently, it counts the number of integer pairs (m, n) satisfying.

$$\begin{cases} 2\sqrt{abx} < m \leq (ab' + a'b)\sqrt{\dfrac{x}{a'b'}}, \\ 2\sqrt{a'b'x} < n \leq (ab' + a'b)\sqrt{\dfrac{x}{ab}}, \\ (-ma' + na)(mb' - nb) \leq x. \end{cases}$$

Also let

$$\sigma\begin{pmatrix} a, b \\ a', b' \end{pmatrix}$$

denote the number of integer (m, n) in T_i', namely. That is,

$$m > 2\sqrt{abx}, \quad n > 2\sqrt{a'b'x}, \quad m + n \leq 2\sqrt{cdx}.$$

With these definitions, we now have

$$F(x) = \sum_{i=0}^{k} s\begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix} + \sum_{j=1}^{k} \sigma\begin{pmatrix} \alpha_j', \beta_j' \\ \alpha_j'', \beta_j'' \end{pmatrix}. \tag{4}$$

To estimate

$$s\begin{pmatrix} a, b \\ a', b' \end{pmatrix},$$

Let

$$\eta = 2\sqrt{abx}, \quad \zeta = 2\sqrt{a'b'x},$$

Next, if we define (u, v) to be the point $P = (\mu', \nu')$ from Figure 5, we have

$$u = (ad + bc)\sqrt{\frac{x}{cd}}, \quad v = (cb' + da')\sqrt{\frac{x}{cd}}.$$

Recall that (u, v) is chosen on the transformed hyperbola so that $u+v$ is as small as possible. For the upper boundary of S_{1i}' define

$$\varphi(M) = \frac{(ab' + a'b)M - \sqrt{M^2 - 4abx}}{2ab},$$

and for the upper boundary of S_{2i}' define

$$\psi(N) = \frac{(ab' + a'b)N - \sqrt{N^2 - 4a'b'x}}{2a'b'}.$$

These formulas are obtained by isolating m' and n' respectively in the hyperbola inequality

$$(-m'a' + n'a)(m'b' - n'b) \leq x.$$

which defined one of the boundaries for S_i'. Intuitively, $\varphi(m')$ returns how high each vertical column of S_{1i}' will go (up until the hyperbola), and $\psi(n')$ does the same for S_{2i}' but horizontally.

Now we can write,

$$S'_0 = \begin{cases} \eta < m \le u \\ \zeta < n \le v \end{cases}$$

$$S'_1 = \begin{cases} \eta < m \le u \\ \zeta < n \le \varphi(m) \end{cases}$$

$$S'_2 = \begin{cases} \zeta < n \le v \\ \eta < m \le \psi(n) \end{cases}$$

Counting column by column in S'_{1i} and row by row in S'_{2i} gives

$$\begin{aligned} s\begin{pmatrix} a, b \\ a', b' \end{pmatrix} &= |S'_{1i}| + |S'_{2i}| - |S'_{0i}| \\ &= \sum_{\eta < m \le u} (\lfloor \varphi \rfloor(m) - \lfloor \zeta \rfloor) + \sum_{\zeta < n \le v} (\lfloor \psi(n) \rfloor - \lfloor \eta \rfloor) - (\lfloor u \rfloor - \lfloor \eta \rfloor)(\lfloor v \rfloor - \lfloor \zeta \rfloor) \quad (5) \\ &= \sum_{\eta < m \le u} \lfloor \varphi(m) \rfloor + \sum_{\zeta < n \le v} \lfloor \psi(n) \rfloor - \lfloor u \rfloor \lfloor v \rfloor + \lfloor \eta \rfloor \lfloor \zeta \rfloor \end{aligned}$$

Now we will convert our discrete sums into continuous integrals, but first, let

$$r_0(y) = \lfloor y \rfloor - y + \frac{1}{2}, \quad r_0(y+1) = r_0(y).$$

Then

$$|r_0(y)| \le \frac{1}{2}.$$

Also define

$$r_1(y) = \int_0^y r_0(w)\ dw - \frac{1}{12}, \quad r_1(y+1) = r_1(y).$$

Then,

$$|r_1(y)| \le \frac{1}{12}, \quad |r_1(y) - r_1(z)| \le \frac{1}{8}.$$

The function r_0 measures the error made when replacing a floor by its argument, and r_1 is the antiderivative of this error.

Since $\lfloor y \rfloor = y - \frac{1}{2} + r_0(y)$, we have

$$\sum_{\eta < m \le u} \lfloor \varphi(m) \rfloor = \sum_{\eta < m \le u} \varphi(m) - \frac{1}{2}(\lfloor u \rfloor - \lfloor \eta \rfloor) + \sum_{\eta < m \le u} r_0(\varphi(m)).$$

Now, we will use Theorem 2.1. We have

$$\sum_{\eta < m \le u} \varphi(m) = \int_\eta^u \varphi(t)\,dt + \int_\eta^u \varphi'(t)(t - \lfloor t \rfloor)\,dt + \varphi(\eta)(\eta - \lfloor \eta \rfloor) - \varphi(u)(u - \lfloor u \rfloor).$$

Since

$$r_0(t) = \lfloor t \rfloor - t + \frac{1}{2},$$

we have

$$t - \lfloor t \rfloor = \frac{1}{2} - r_0(t).$$

Thus,

$$\begin{aligned}\sum_{\eta<m\le u}\varphi(m) &= \int_\eta^u \varphi(t)\,dt + \int_\eta^u \varphi'(t)\left(\frac{1}{2}-r_0(t)\right)dt + \varphi(\eta)\left(\frac{1}{2}-r_0(\eta)\right) - \varphi(u)\left(\frac{1}{2}-r_0(u)\right)\\ &= \int_\eta^u \varphi(t)\,dt + \frac{1}{2}\int_\eta^u \varphi'(t)\,dt - \int_\eta^u r_0(t)\varphi'(t)\,dt + \frac{1}{2}\varphi(\eta) - r_0(\eta)\varphi(\eta) - \frac{1}{2}\varphi(u) + r_0(u)\varphi(u).\end{aligned}$$

But

$$\frac{1}{2}\int_\eta^u \varphi'(t)\,dt = \frac{1}{2}\left(\varphi(u)-\varphi(\eta)\right),$$

so the $\frac{1}{2}$ terms cancel. Therefore,

$$\sum_{\eta<m\le u}\varphi(m) = \int_\eta^u \varphi(t)\,dt + r_0(u)\varphi(u) - r_0(\eta)\varphi(\eta) - \int_\eta^u r_0(t)\varphi'(t)\,dt.$$

Substituting this formula gives

$$\sum_{\eta<m\le u}\lfloor\varphi(m)\rfloor = \int_\eta^u \varphi(t)\,dt + r_0(u)\varphi(u) - r_0(\eta)\varphi(\eta) - \int_\eta^u r_0(t)\varphi'(t)\,dt - \frac{1}{2}(\lfloor u\rfloor - \lfloor\eta\rfloor) + \sum_{\eta<m\le u} r_0(\varphi(m)).$$

Similarly,

$$\begin{aligned}\sum_{\zeta<n\le v}\lfloor\psi(n)\rfloor = &\int_\zeta^v \psi(t)\,dt + r_0(v)\psi(v) - r_0(\zeta)\psi(\zeta) - \int_\zeta^v r_0(t)\psi'(t)\,dt\\ &- \frac{1}{2}(\lfloor v\rfloor - \lfloor\zeta\rfloor) + \sum_{\zeta<n\le v} r_0(\psi(n)).\end{aligned}$$

Now, we substitute back into (5). Substituting these two formulae back into our expression for $|S_i'|$, we get

$$\begin{aligned}|S_i'| = &\int_\eta^u \varphi(t)\,dt + \int_\zeta^v \psi(t)\,dt\\ &+ r_0(u)\varphi(u) - r_0(\eta)\varphi(\eta) - \int_\eta^u r_0(t)\varphi'(t)\,dt\\ &+ r_0(v)\psi(v) - r_0(\zeta)\psi(\zeta) - \int_\zeta^v r_0(t)\psi'(t)\,dt\\ &- \frac{1}{2}(\lfloor u\rfloor - \lfloor\eta\rfloor) - \frac{1}{2}(\lfloor v\rfloor - \lfloor\zeta\rfloor)\\ &+ \sum_{\eta<m\le u} r_0(\varphi(m)) + \sum_{\zeta<n\le v} r_0(\psi(n))\\ &- \lfloor u\rfloor\lfloor v\rfloor + \lfloor\eta\rfloor\lfloor\zeta\rfloor.\end{aligned}$$

Recall that φ and ψ were derived by isolating one variable in the hyperbola inequality, so we have $\varphi(u) = v$ and $\psi(v) = u$ since (u, v) lies on the hyperbola. Also, for the two sums involving r_0, recall that $|r_0(y)| \le \frac{1}{2}$. Thus, those two sums can be written as

$$\sum_{\eta<m\le u} r_0(\varphi(m)) + \sum_{\zeta<n\le v} r_0(\psi(n)) = \frac{\vartheta}{2}(u-\eta+1+v-\zeta+1) = \frac{\vartheta}{2}(u+v-\eta-\zeta+2)$$

The constant ϑ satisfies $|\vartheta| < 1$, though its value may change from time to time.

After simplifying, our formula becomes

$$\begin{aligned} s\left(\begin{array}{c} a, b \\ a', b' \end{array}\right) = & \int_\eta^u \varphi(w)\ dw + \int_\zeta^v \psi(w)\ dw + \eta\zeta - uv \\ & - r_0(\eta)\big(\varphi(\eta) - \zeta\big) - r_0(\zeta)\big(\psi(\zeta) - \eta\big) \\ & + r_0(\eta)r_0(\zeta) - r_0(u)r_0(v) \\ & - \int_\eta^u r_0(w)\ d\varphi(w) - \int_\zeta^v r_0(w)\ d\psi(w) + \frac{\vartheta}{2}\,(u + v - \eta - \zeta + 2). \end{aligned} \tag{6}$$

The area part of (6) simplifies nicely.

$$\begin{aligned} \int_\eta^u \varphi(w)\ dw + \int_\zeta^v \psi(w)\ dw = & \frac{ab' + a'b - ad - bc}{4ab} \cdot \frac{x}{cd} + \frac{ab' + a'b - cb' - da'}{4a'b'} \cdot \frac{x}{cd} \\ & + x \log\left(\frac{ad + bc + 1}{2\sqrt{abcd}}\right) + x \log\left(\frac{cb' + da' + 1}{2\sqrt{a'b'cd}}\right), \end{aligned}$$

which becomes

$$\int_\eta^u \varphi(w)\, dw + \int_\zeta^v \psi(w)\, dw = -\frac{x}{cd} + \frac{x}{2}\log\left(\frac{ab'}{a'b}\right).$$

We also want to simplify the other terms in (6).

$$\begin{aligned} \eta\zeta - uv &= -2x\left(\sqrt{ab'} - \sqrt{a'b}\right)^2 + \frac{x}{cd}, \\ \varphi(\eta) - \zeta &= \left(\sqrt{ab'} - \sqrt{a'b}\right)^2 \sqrt{\frac{x}{ab}}, \\ \psi(\zeta) - \eta &= \left(\sqrt{ab'} - \sqrt{a'b}\right)^2 \sqrt{\frac{x}{a'b'}}. \end{aligned}$$

Now, substituting back into (6) yields

$$\begin{aligned} s\left(\begin{array}{c} a, b \\ a', b' \end{array}\right) = & \frac{x}{2}\log\left(\frac{ab'}{a'b}\right) - 2x\left(\sqrt{ab'} - \sqrt{a'b}\right)^2 - r_0(\eta)\left(\sqrt{ab'} - \sqrt{a'b}\right)^2 \sqrt{\frac{x}{ab}} \\ & - r_0(\zeta)\left(\sqrt{ab'} - \sqrt{a'b}\right)^2 \sqrt{\frac{x}{a'b'}} + r_0(\eta)r_0(\zeta) - r_0(u)r_0(v) \\ & - \int_\eta^u r_0(w)\, d\varphi(w) - \int_\zeta^v r_0(w)\, d\psi(w) + \frac{\vartheta}{2}\left(u + v - \eta - \zeta + 2\right). \end{aligned}$$

The terms involving $\int r_0\, d\varphi$ and $\int r_0\, d\psi$ are the oscillating boundary errors. Voronoi estimates them by introducing r_1 and integrating by parts. The details are slightly involved, so we won't discuss all of it here. The resulting error is

$$R(y) = \frac{19}{24} + \left(\frac{\sqrt{x}}{8t} + \frac{\sqrt{t}}{12}\right)\frac{1}{\sqrt{y}} + \frac{3}{8}\sqrt[4]{\frac{x}{y^3}}. \tag{7}$$

Thus, we have

$$s\left(\begin{array}{c} a, b \\ a', b' \end{array}\right) = \frac{x}{2}\log\frac{ab'}{a'b} + \tau\left(\begin{array}{c} a, b \\ a', b' \end{array}\right) + \vartheta\big(R(ab) + R(a'b')\big), \tag{8}$$

where we define τ as

$$\tau\left(\begin{array}{c} a,b \\ a',b' \end{array}\right) = -\left(\sqrt{ab'}-\sqrt{a'b}\right)^2\left(2x + r_0(\eta)\sqrt{\frac{x}{ab}} + r_0(\zeta)\sqrt{\frac{x}{a'b'}}\right) \\ + \frac{ab'+a'b}{2ab}r_1(\eta) + \frac{ab'+a'b}{2a'b'}r_1(\zeta). \tag{9}$$

The τ term is specifically chosen for some very nice telescoping later.

Remark 9.2. Note that the above formulas were written for the interior Farey fractions, where $ab, a'b' > 0$. The two outside fractions, namely $\frac{1}{0}$ and $\frac{0}{1}$, must be considered separately because some of the expressions above have denominators ab or $a'b'$. The details aren't very interesting, but just know that we can handle these also with the Euler Summation Formula, and their total contribution is bounded by the same type of error term from above.

Those two formulae are listed below.

$$s\begin{pmatrix}1,0\\a',1\end{pmatrix} - \tau\begin{pmatrix}1,0\\a',1\end{pmatrix} = \frac{x}{2}\log\left(\frac{x}{a'}\right) + \gamma x + \frac{x}{2} \\ + \vartheta\left(R\left(a'\right) + \frac{3}{4} + \frac{t}{12} + \frac{1}{2}\sqrt{\frac{x}{t}}\right), \tag{10}$$

and

$$s\begin{pmatrix}1,b\\0,1\end{pmatrix} - \tau\begin{pmatrix}1,b\\0,1\end{pmatrix} = \frac{x}{2}\log\left(\frac{x}{b}\right) + \gamma x + \frac{x}{2} \\ + \vartheta\left(R\left(b\right) + \frac{3}{4} + \frac{t}{12} + \frac{1}{2}\sqrt{\frac{x}{t}}\right), \tag{11}$$

where γ is the Euler-Mascheroni constant.

We also find that

$$\tau\begin{pmatrix}1,0\\0,1\end{pmatrix} = -2x + \frac{1}{4}$$

Now we handle the triangles T_i', using the symbol σ defined earlier. For a pair $\frac{a}{b} > \frac{a'}{b'}$ with mediant $\frac{c}{d} = \frac{a+a'}{b+b'}$, we can prove the following identity.

$$\sigma\left(\begin{array}{c} a,b \\ a',b' \end{array}\right) = \tau\left(\begin{array}{c} a,b \\ a',b' \end{array}\right) - \tau\left(\begin{array}{c} c,d \\ a',b' \end{array}\right) - \tau\left(\begin{array}{c} a,b \\ c,d \end{array}\right). \tag{12}$$

This identity is a direct substitution of the formula for τ and the formula for the lattice count in the triangle. The algebra is long so I won't include it here, but the main identities used are

$$(\sqrt{ad}-\sqrt{bc})^2 + (\sqrt{cb'}-\sqrt{da'})^2 - (\sqrt{ab'}-\sqrt{a'b})^2 \\ = (\sqrt{cd}-\sqrt{ab}-\sqrt{a'b'})^2,$$

along with these relations:

$$\frac{ad+bc}{2cd} + \frac{cb'+da'}{2cd} = 1, \quad \frac{ad+bc}{2ab} - \frac{ab'+a'b}{2ab} = 1,$$

$$\frac{cb'+da'}{2a'b'} - \frac{ab'+a'b}{2a'b'} = 1.$$

Now we sum (12) over all mediants inserted in the Farey sequence, which will telescope and cancel all the middle terms. Only the original outside pair and the final adjacent Farey fractions survive:

$$\sum_{j=1}^{k} \sigma \begin{pmatrix} \alpha'_j, \beta'_j \\ \alpha''_j, \beta''_j \end{pmatrix} = \tau \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} - \sum_{i=0}^{k} \tau \begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix}. \tag{13}$$

Substituting this into (4), we get the simplification

$$F(x) = \tau \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} + \sum_{i=0}^{k} \left(s \begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix} - \tau \begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix} \right). \tag{14}$$

Now, we can plug back in our formulae for s and τ. For the interior indices $1 \le i < k$, we have by (8),

$$s \begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix} - \tau \begin{pmatrix} a_i, b_i \\ a_{i+1}, b_{i+1} \end{pmatrix} = \frac{x}{2} \log \frac{a_i b_{i+1}}{a_{i+1} b_i} + \vartheta\big(R(a_i b_i) + R(a_{i+1} b_{i+1})\big). \tag{15}$$

When we sum these logarithms, they telescope.

$$\begin{aligned} \sum_{i=1}^{k-1} \log \frac{a_i b_{i+1}}{a_{i+1} b_i} &= \log \frac{a_1 b_2}{a_2 b_1} + \log \frac{a_2 b_3}{a_3 b_2} + \cdots + \log \frac{a_{k-1} b_k}{a_k b_{k-1}} \\ &= \log(a_1 b_k), \end{aligned}$$

since $b_1 = 1$ and $a_k = 1$.

Now we add the logarithms from (10) and (11) to the interior logarithms (which are multiplied by $\frac{x}{2}$),

$$\begin{aligned} &\frac{x}{2} \log \frac{x}{a_1} + \frac{x}{2} \log(a_1 b_k) + \frac{x}{2} \log \left(\frac{x}{b_k} \right) \\ &= x \log x. \end{aligned}$$

Using (14) and $\tau \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} = -2x + 1/4$, we obtain

$$F(x) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \vartheta \left(2 \sum_{i=1}^{k} R(a_i b_i) + \frac{3}{2} + \frac{t}{6} + \sqrt{\frac{x}{t}} \right). \tag{16}$$

Now, we will bound the error term. Recall that the reduced fractions $\frac{a_i}{b_i}$ used in the Farey sequence were chosen so that $a_i b_i < t$. We have

$$\sum_{i=1}^{k} R(a_i b_i) \le \sum_{\substack{m,n \ge 1 \\ mn < t}} R(mn).$$

Using the definition from (7), we have

$$\sum_{i=1}^{k} R(a_i b_i) \le \sum_{mn < t} \left(\frac{19}{24} + \left(\frac{\sqrt{x}}{8t} + \frac{\sqrt{t}}{12} \right) \frac{1}{\sqrt{mn}} + \frac{3}{8} \sqrt[4]{\frac{x}{(mn)^3}} \right). \tag{17}$$

For $0 \le s < 1$, we claim

$$\sum_{\substack{m,n \ge 1 \\ mn < t}} \frac{1}{(mn)^s} \le \frac{t^{1-s}}{1-s}(\log t + 1). \tag{18}$$

To show this, write this as a double sum.

$$\sum_{m<t} \frac{1}{m^s} \sum_{n<t/m} \frac{1}{n^s}.$$

The function inside the inner sum is decreasing, so we can bound it by its integral, which gives

$$\sum_{n<Y} \frac{1}{n^s} \le \int_0^Y \frac{dy}{y^s} = \frac{Y^{1-s}}{1-s}.$$

Taking $Y = t/m$, this gives

$$\sum_{n<t/m} \frac{1}{n^s} \le \frac{1}{1-s}\left(\frac{t}{m}\right)^{1-s}.$$

Therefore

$$\sum_{m<t} \frac{1}{m^s} \sum_{n<t/m} \frac{1}{n^s} \le \sum_{m<t} \frac{1}{m^s} \cdot \frac{1}{1-s}\left(\frac{t}{m}\right)^{1-s} = \frac{t^{1-s}}{1-s} \sum_{m<t} \frac{1}{m}.$$

Also,

$$\sum_{m<t} \frac{1}{m} \le 1 + \int_1^t \frac{dy}{y} = 1 + \log t,$$

so we have

$$\sum_{\substack{m,n \ge 1 \\ mn < t}} \frac{1}{(mn)^s} \le \frac{t^{1-s}}{1-s}\left(\log t + 1\right),$$

as desired.

Applying (18) with $s = \frac{1}{2}$ and $s = \frac{3}{4}$(because those are the values that show up in (17)), and also using $\sum_{mn<t} 1 \le t(\log t + 1)$(the case when $s = 0$), there is ϑ such that

$$\sum_{i=1}^{k} R(a_i b_i) = \vartheta(\log t + 1)\left(\frac{23}{24}t + \frac{1}{4}\sqrt{\frac{x}{t}} + \frac{3}{2}\sqrt[4]{xt}\right). \tag{19}$$

Substituting this into (16) gives

$$\begin{aligned} F(x) &= x(\log x + 2\gamma - 1) + \frac{1}{4} \\ &\quad + \vartheta\left((\log t + 1)\left(\frac{23}{12}t + \frac{1}{2}\sqrt{\frac{x}{t}} + 3\sqrt[4]{xt}\right) + \frac{3}{2} + \frac{t}{6} + \sqrt{\frac{x}{t}}\right). \end{aligned} \tag{20}$$

Now we just need to choose the best t to minimize error. The three main error terms in (20) are

$$t, \quad \sqrt{\frac{x}{t}}, \quad \sqrt[4]{xt}.$$

We want them to be approximately the same size, and setting

$$t = x^{1/3}$$

works perfectly:

$$t = x^{1/3}, \quad \sqrt{\frac{x}{t}} = x^{1/3}, \quad \sqrt[4]{xt} = x^{1/3}.$$

Therefore (20) becomes

$$F(x) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \vartheta\left(\frac{65}{36}x^{1/3}\log x + \frac{79}{12}x^{1/3} + \frac{3}{2}\right).$$

Since $|\vartheta| < 1$, this implies

$$F(x) = x(\log x + 2\gamma - 1) + O\left(x^{\frac{1}{3}}\log x\right).$$

as desired. ■

References

[Apo76] Tom M. Apostol. *Introduction to Analytic Number Theory.* Springer-Verlag, New York-Heidelberg, 1976. Undergraduate Texts in Mathematics.

[Hux03] M. N. Huxley. Exponential sums and lattice points iii. *Proceedings of the London Mathematical Society*, 87(3):591–609, 2003.

[Vor03] Georges Voronoï. Sur un problème du calcul des fonctions asymptotiques. *Journal für die reine und angewandte Mathematik*, 126:241–282, 1903.